Gauss Curvature Estimates for Surfaces whose Mean Curvature is Constant or has Special Properties

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Abstract

Let $z = w(x, y)$ represent an embedded (not necessarily simply-connected), compact nonparametric surface in R^3 with mean curvature H , nonpositive Gauss curvature K . Set $(-K)_{\max}, |H|_{\max}$ to be the global maximum of $-K$ and H and set K_0, H_0 to be the maximum of $-K, H$ on the boundary. If $(\log H)_{xx} + (\log H)_{yy} \geq 0$, $K_0 > 0$, $(-K)_{\max} \geq (|H|_{\max})$, $\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} \geq 0$ and $|\nabla \omega|$ is bounded, then either $(|H|_{\max})^2 + (-K)_{\max} = (H_0)^2 + K_0$ or $((|H|_{\max})^2 + (-K)_{\max})(H_2\{x \in M : (H(x))^2 - K(x) > (H_0)^2 + K_0\}) < \frac{44}{15}\pi$. In particular, if $H \equiv$ constant, $K_0 < 0$, $(-K)_{\max} \geq H^2$, and if $|\nabla \omega|$ is bounded, then either $|K|_{\max} = K_0$ or $(H^2 + |K|_{\max})(H_2\{x \in M : -K(x) > K_0\}) < \frac{44}{15}\pi$.

In Part II, this result is extended to a special class of doubly-connected surfaces of constant mean curvature, whose boundary consists of two components, on one of which $K = K_0$ and on the other of which $K = (-K)_{\max}$.

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For embedded surfaces M in R^3 with mean curvature H and Gauss curvature K , we set

$$(-K)_{\max} = \ell.u.b.\{-K(x), x \in M\},$$

the global maximum of $-K$ and

$$K_0 = \ell.u.b.\{-K(x), x \in \partial M\},$$

the maximum of $-K$ on the boundary. Throughout this paper, we consider compact, (not necessarily simply-connected) surfaces in R^3 of constant mean curvature H such that

$$(-K)_{\max} > K_0 > 0, \quad \text{and} \quad (-K)_{\max} > H^2.$$

Our main concern is to relate $|K|_{\max}$ with K_0 and the first main result is stated as follows.

Main Theorem 1 *Let M be an embedded (not necessarily simply-connected), compact surface in R^3 of constant mean curvature H and nonpositive Gauss curvature K . Suppose that a normal vector field N makes an angle larger than β with a fixed vector $-V$ and $\beta > \frac{\pi}{2}$. Then if $K_0 > 0$, and $(-K)_{\max} > H^2$ the following alternative holds. Either*

$$(-K)_{\max} = K_0$$

or

$$(H^2 + (-K)_{\max})(H_2\{x \in M : -K(x) > K_0\}) < \frac{44}{15}\pi.$$

Here H_2 denotes the two-dimensional Hausdorff measure.

An important feature of this result is its independence of the distance of a point to the boundary, in contrast with the results in Heinz[8], L. Simon[14], Choi-Schoen[4], Schoen-Simon-Yau[13], Anderson[1] and B. White[17]. This feature is to be of importance in its application to compactness theorems for surfaces of constant mean curvature or surfaces whose mean curvature has special properties.

The crucial step is to estimate

$$\int_E (H^2 - K)H_2(dx)$$

for measurable $E \subset M$, in consideration of **Theorem I.6** below. We notice that the value of this integral is invariant after rescaling. We aim at deriving an inequality (19) in **Theorem I.7** below which and (13) in **Theorem I.6** will lead to a contradiction after rescaling for some number $p > 1$. Hence, for this value of p , the inequality (12) in **Theorem I.6** must hold. This then yields **Main Theorem 1**.

For this purpose, suggested by Spruck [15], we make the further assumption that a unit normal vector field N on M makes an angle larger than β with a fixed vector $-V$. We will obtain the integral inequality (29) in Section **I.2.0** for measurable $E \subset M$, in which E^δ is a subdomain of E with $H_2(E^\delta) = \delta H_2(E)$ and the test function η is chosen to be a modified version of the warping function in the torsion problem for the multiply-connected region $E \setminus E^\delta$. Let

c be the boundary value of η on ∂E^δ . One of the main difficulties encountered in this work is to obtain a lower bound for the value c in terms of δ and $H_2(E)$. The other difficulty is to construct a specific solution η to the modified version of the torsion problem so that $0 < \eta < c$ holds in $E \setminus E^\delta$. This will be done in **I.2.1.2**.

In **Part II**, this result is extended to a special class of doubly-connected surfaces of constant mean curvature, whose boundary consists of two components, on one of which $K = K_0$ and on the other of which $K = K_{\max}$. **Main Theorem 2** and Corollary 1 are presented in the beginning of **Part II**.

In **Part III**, we extend the results in **Main Theorem 1** to surfaces of suitably prescribed mean curvature with the help of some results in Kenmotsu[10]. **Main Theorem 3** is presented in the end of **Part III**.

I Constant Mean Curvature Surfaces.

We start with a partial differential equation satisfied by the Gauss curvature of a surface of constant mean curvature, which has appeared in page 549 of Spruck[15].

Lemma I.1 *Let M be a surface in R^3 of constant mean curvature H . Then at non-umbilical points, the Gauss curvature K of M satisfies the partial differential equation*

$$\Delta \log(H^2 - K) = 4K, \quad (1)$$

or equivalently

$$(K - H^2)\Delta K - |DK|^2 = 4K(K - H^2)^2, \quad (2)$$

where D and Δ are the covariant derivative and the Laplace-Beltrami operator on M , respectively.

Proof of Lemma I.1 Let M be an oriented two-dimensional connected surface and $x : M \rightarrow R^3$ be an isometric immersion of M into R^3 . At a neighborhood of any point of M , we may adopt an isothermal coordinate $z = \xi_1 + i\xi_2$ such that the first fundamental form is, for some constant $\lambda > 0$, $ds^2 = \lambda^2|dz|^2$. Let us set $e_i = \left(\frac{1}{\lambda}\right) \frac{\partial x}{\partial \xi_i}$, $i = 1, 2$, and define the unit normal vector field by $e_3 = e_1 \times e_2$. We may define 1-forms w_i, w_{12}, w_{i3} by the following formulae

$$dx = w_1 e_2 + w_2 e_3, \quad de_1 = w_{12} e_2 + w_{13} e_3, \quad de_2 = -w_{12} e_1 + w_{23} e_3.$$

Then, writing, for $i = 1, 2$,

$$w_{i3} = h_{i1} w_1 + h_{i2} w_2,$$

we have $h_{12} = h_{21}$ and the mean curvature of M is given by $H = \frac{1}{2}(h_{11} + h_{22})$. Furthermore, setting $\phi = \frac{1}{2}(h_{11} - h_{22}) - ih_{12}$, there holds

$$K = H^2 - |\phi|^2, \quad (3)$$

where K is the Gauss curvature of M . We note that, as a consequence of the Codazzi equation, the following result holds.

Lemma I.2 (Hopf [9], page 137). *There holds $\frac{\partial(\lambda^2\phi)}{\partial\bar{z}} = \lambda^2\frac{\partial H}{\partial\bar{z}}$.*

From Lemma 2, the following result readily follows.

Lemma I.3 (Hopf [9], page 138). *M has constant mean curvature if and only if the function $\lambda^2\phi$ is an analytic function of z .*

Hence, at a non-umbilical point of M ,

$$0 = \Delta \log(\lambda^4|\phi|^2) = 4\Delta \log \lambda + \Delta \log(H^2 - K),$$

by (3). But, by the Gauss equation

$$K = -\Delta \log \lambda.$$

A combination of the last two identities thus gives (1). $\diamond$

Since the umbilical points of a surface of constant mean curvature are isolated unless it is a piece of a plane or sphere, the analogy between equations (61) of Talenti [16] and (2) above suggests us to adapt the framework of [16] to surfaces of constant mean curvature. The following result generalizes Theorem 3 of [16], which will be proved in **I.2**.

Theorem I.4 *Let M be an embedded (not necessarily simply-connected) surface in R^3 of constant mean curvature H . Set, for $s \in R, s > 0$,*

$$\overline{K}(s) = \frac{1}{s} \ell.u.b. \left\{ \int_E (-K(x)) H_2(dx) : E \subset M, H_2(E) = s \right\}.$$

If

$$(-K)_{\max} > K_0 > 0,$$

then

$$\frac{1}{H^2 + \overline{K}(s)} \leq \frac{1}{H^2 + (-K)_{\max}} + \frac{s}{2\pi} \quad (4)$$

for every s such that

$$0 < s \leq H_2\{x \in M : -K(x) > K_0\}. \quad (5)$$

I.1. Proof of Theorem I.4

In this section, we give a brief proof of **Theorem I.4**, which is adapted from that in [16] without essential modification. We briefly repeat it mainly for the convenience of adopting its framework in **Part II**.

For any number t such that

$$K_0 < t < (-K)_{\max}, \quad (6)$$

we may consider the set

$$E_0(t) = \{x \in M : -K(x) > t\}$$

and observe that its boundary coincides with the level set

$$\Gamma_0(t) = \{x \in M : -K(x) = t\},$$

provided that no critical point of K lies on $\Gamma_0(t)$. We may note that, if t satisfies (6) and no critical point of K lies on $\Gamma_0(t)$, $\Gamma_0(t)$ consists of a finite number of simple closed curves.

Following [16], we start with an integration of both sides of equation (2) on the set $E_0(t)$, which gives

$$(-H^2 - t) \oint_{\Gamma_0(t)} |DK| H_1(dx) - 2 \int_{E_0(t)} |DK|^2 H_2(dx) = 4 \int_{E_0(t)} K(H^2 - K)^2 H_2(dx), \quad (7)$$

where H_1 denotes the one dimensional Hausdorff measures.

With essentially the same method as in [16], we see that (7) can be written as

$$\oint_{\Gamma_0(t)} |DK| H_1(dx) = 4(H^2 + t) \int_{E_0(t)} (-K) H_2(dx). \quad (8)$$

Introduce the distribution function

$$a_0(t) = H_2(E_0(t)).$$

Since $t > K_0 > 0$, we have $K < 0$ in $E_0(t)$ and we may apply Proposition 1 of [6] to obtain

$$\oint_{\Gamma_0(t)} |DK| H_1(dx) \geq 4\pi \frac{a_0(t)}{(-a'_0(t))}. \quad (9)$$

Inserting (9) into (8), we obtain, for almost every t obeying (6),

$$1 \leq \frac{-a'_0(t)}{\pi a_0(t)} (H^2 + t) \int_{E_0(t)} (-K) H_2(dx). \quad (10)$$

In order to handle the right hand side of (10), we introduce the decreasing rearrangement K^* of $-K$; namely,

Definition I.5 K^* is the decreasing function from $[0, H_2(M)]$ into $[0, (-K)_{\max}]$ such that the sets $\{s \geq 0; K^*(s) > t\}$ are intervals with end points 0 and $a_0(t)$.

Then, obviously, $\int_{E_0(t)} (-K) H_2(dx) = \int_0^{a_0(t)} K^*(s) ds$, and for every smooth function ϕ with compact support in $(0, H_2(M))$,

$$\int_0^{a_0(K_0)} K^*(s) \phi'(s) ds = \int_{K_0}^{(-K)_{\max}} \phi(a_0(t)) dt.$$

If ϕ is nonnegative, this and (10) yield

$$\begin{aligned} \int_0^{a_0(K_0)} K^*(s) \phi'(s) ds &\leq \int_{K_0}^{(-K)_{\max}} \phi(a_0(t)) \frac{(H^2 + t)}{\pi a_0(t)} \left[\int_0^{a_0(t)} K^*(s) ds \right] (-a_0'(t)) dt \\ &\leq \int_0^{a_0(K_0)} \phi(s) \frac{H^2 + K^*(s)}{\pi s} \left[\int_0^s K^*(s') ds' \right] ds. \end{aligned}$$

This yields

$$-\pi s \frac{dK^*(s)}{ds} \leq (H^2 + K^*(s)) \left(\int_0^s K^*(s') ds' \right) \quad (11)$$

for every s such that $0 < s < a_0(K_0)$. As in [16], we note that, for the function $\overline{K}$ defined in the statement of **Theorem I.4**, there holds

$$\overline{K}(s) = \frac{1}{s} \int_0^s K^*(s') ds'.$$

Inequality (11) can be rewritten as

$$-\frac{d}{ds} \left(\pi s^2 \frac{d\overline{K}(s)}{ds} \right) \leq \frac{1}{2} \frac{d}{ds} (s(\overline{K}(s))^2 + H^2(s\overline{K}(s))).$$

Hence

$$-\frac{d}{ds} \left(\pi s^2 \frac{d\overline{K}(s)}{ds} \right) \leq \frac{1}{2} \frac{d}{ds} (s(H^2 + \overline{K}(s)))^2.$$

This means that $\pi s^2 \frac{d\overline{K}(s)}{ds} + \frac{1}{2} (s(H^2 + \overline{K}(s)))^2$ is an increasing function of s . Since $\overline{K}(s) \rightarrow K^*(0) = (-K)_{\max}$ as $s \rightarrow 0$ and $s \frac{d\overline{K}}{ds} = K^*(s) - \overline{K}(s) \rightarrow 0$ as $s \rightarrow 0$, we have

$$\pi \frac{d\overline{K}(s)}{ds} + \frac{1}{2} (H^2 + \overline{K}(s))^2 \geq 0$$

for every s such that $0 < s < a_0(K_0)$, i.e. $\frac{d}{ds} \frac{1}{H^2 + \overline{K}(s)} \leq \frac{1}{2\pi}$. An integration gives

$$\frac{1}{H^2 + \overline{K}(s)} \leq \frac{1}{H^2 + (-K)_{\max}} + \frac{s}{2\pi},$$

since $\overline{K}(s) \rightarrow K^*(0) = (-K)_{\max}$ as $s \rightarrow 0$. This is precisely the inequality (4) in the statement of **Theorem I.4**. $\diamond$

Accordingly, Theorem 1 of [16] can be generalized as follows.

Theorem I.6 *Let M be an embedded (not necessarily simply-connected) surface in R^3 of constant mean curvature H . If*

$$(-K)_{\max} > K_0 > 0,$$

then the following alternative holds for each number $p > 1$. Either

$$(H^2 + (-K)_{\max})(H_2\{x \in M : -K(x) > K_0\}) < \frac{2\pi}{p-1}, \quad (12)$$

or

$$H^2 + (-K)_{\max} \leq \frac{2\pi}{p-1} \left[\frac{p}{2\pi} \|H^2 - K\|_{p,\infty} \right]^{\frac{p}{p-1}}. \quad (13)$$

Here

$$\begin{aligned} \|H^2 - K\|_{p,\infty} = \ell.u.b. \quad \{ \quad & c : \int_E (H^2 - K) H_2(dx) < c(H_2(E))^{1-\frac{1}{p}}, \\ & \forall \quad E \subset M, \quad E : \text{measurable} \}. \end{aligned} \quad (14)$$

Moreover, if (13) holds, then

$$H_2\{x \in M : -K(x) > K_0\} \geq \left[\frac{p}{2\pi} \|H^2 - K\|_{p,\infty} \right]^{-\frac{p}{p-1}}. \quad (15)$$

We note that **Theorem I.6** follows from **Theorem I.4** essentially in the same way as Theorem 1 of [11] follows from Theorem 2 of [16], and we omit its proof.

We are thus led by **Theorem I.6** to estimate $\int_E (-K + H^2) H_2(dx)$ for measurable $E \subset M$. For this, suggested by Spruck [15], we make the further assumption that a unit normal vector field N on M makes an angle larger than β with a fixed vector $-V$. We proceed to prove the following.

Theorem I.7 *Let M be an embedded (not necessarily simply-connected) surface in R^3 of constant mean curvature H and non-positive Gauss curvature K . Suppose that a unit normal vector field N on M makes an angle larger than β and $\beta > \frac{\pi}{2}$ with a fixed vector $-V$. Then, for every subset E of M whose boundary ∂E is twice differentiable and every δ sufficiently close to 1, there exists a set E^δ , $E^\delta \subset\subset E$ with $H_2(E^\delta) = \delta H_2(E)$ such that there holds*

$$\int_{E^\delta} (H^2 - K) H_2(dx) \leq \alpha^{-\frac{15}{16}b - \frac{11}{16}} C'_1 (H_2(E))^{\frac{5}{16}} + \alpha^{3b-2} C'_2 (H_2(E))^{-1}, \quad (16)$$

for every real number α , $0 < \alpha < 1$, and every positive real number b , where C'_1 and C'_2 are constants depending on δ such that

$$\begin{aligned} C'_1 &\leq \frac{64}{27} (16\pi(1-\delta)^{-\frac{11}{16}} \delta_2^{-\frac{11}{16}} - \frac{(1-\delta)^{\frac{5}{16}} \delta_2^{-1}}{4} \delta_1^{\frac{5}{16}}) \left(1 - \frac{1+\cos\beta}{1-\cos\beta}\right)^{-1}, \\ C'_2 &\leq \frac{7}{5} \delta_2 \left(\frac{32\delta_1^{-3}\pi^2}{(1-\delta)^2} - \frac{4\pi^2\delta_2^{-2}}{(1-\delta)} \right) \left(1 - \frac{1+\cos\beta}{1-\cos\beta}\right)^{-1}, \end{aligned}$$

with

$$\delta_1 = \delta_1(E) = \left(\log \left(1 + \frac{K_0 H_2(E - E^\delta)}{4\pi} \right) \right) \left(1 + \frac{4\pi}{K_0 H_2(E - E^\delta)} \right) - 1, \quad (17)$$

$$\delta_2 = \delta_2(E) = \frac{19}{16} + \frac{9}{8} \frac{H_2(E)(-K)_{\max}}{\pi}. \quad (18)$$

In particular, there holds

$$\int_{E^\delta} (H^2 - K) H_2(dx) \leq \alpha^{-\frac{15}{16}b - \frac{11}{16}} C_1 (H_2(E))^{\frac{5}{16}} + \alpha^{3b-2} C_2 (H_2(E))^{-1}, \quad (19)$$

where

$$C_1 = \frac{64}{27} \left(16\pi(1 - \delta)^{-\frac{11}{16}} \delta_2^{-\frac{11}{16}} \right) \left(1 - \frac{1 + \cos \beta}{1 - \cos \beta} \right)^{-1}, \quad (20)$$

and

$$C_2 = \frac{7}{5} \delta_2 \left(\frac{32\delta_1^{-3}\pi^2}{(1 - \delta)^2} \right) \left(1 - \frac{1 + \cos \beta}{1 - \cos \beta} \right)^{-1}. \quad (21)$$

Here we may disregard the complicated expression of the constants and simply notice that $\delta_1(E)$ and $\delta_2(E)$ remain invariant after rescaling, so do the constants C'_1 , C'_2 , C_1 , and C_2 . Besides the invariance of these constants, another crucial feature of the inequalities (16) and (19) is the appearance of the power of the area $H_2(E)$, which, combining with suitable choice of p and α will lead to some useful results after rescaling. We will show this in **I.2.** below and then proves **Main Theorem 1.**

The motivation for the designation of these scaling invariant constants will be made clear in the course of proving **Theorem I.7.** We make the remark that throughout this paper we place special emphasis on the *invariance after rescaling* of each designated constant and may disregard its complicated expressions.

I.2. Proof of Theorem I.7.

We now proceed to estimate $\int_E (H^2 - K) H_2(dx)$ for measurable $E \subset M$.

I.2.0. Following Spruck [15], we resort to the following result which can be found out, e.g., in page 14 of [3] and page 549 of [15].

Lemma I.8 *Let M be a surface of constant mean curvature H and let V be a fixed unit vector. Then, a unit normal vector field N on M obeys*

$$\Delta(N \cdot V) + 2(2H^2 - K)(N \cdot V) = 0. \quad (22)$$

Choosing $\psi(u) = \log(1 + u)$ in

$$\Delta\psi(u) = \psi'(u)\Delta u + \psi''(u)|Du|^2,$$

this lemma yields

$$\Delta \log(1 + N \cdot V) = -2(2H^2 - K) \frac{N \cdot V}{1 + N \cdot V} - \frac{|DN \cdot V|^2}{(1 + N \cdot V)^2}. \quad (23)$$

For an arbitrary Lipschitz set $E \subset M$, we may choose $\eta \in W_0^{1,\infty}(E)$, $\nabla\eta \in BV(E)$. Multiplying (23) by $\eta^3(x)$ and integrating over E , we obtain

$$\begin{aligned} & \int_E \eta^3(x) \left\{ 2(2H^2 - K) \frac{N \cdot V}{1 + N \cdot V} + \frac{|DN \cdot V|^2}{(1 + N \cdot V)^2} \right\} H_2(dx) \\ &= - \int_E \eta^3(x) \Delta \log(1 + N \cdot V) H_2(dx) \\ &= -3 \int_E \eta^2(x) \langle D(\log(1 + N \cdot V)), D\eta \rangle, \text{ using Green's identity,} \\ &\leq 3 \int_E \eta^2 |D\eta| \frac{|DN \cdot V|}{(1 + N \cdot V)} H_2(dx) \\ &= 3^{\frac{5}{14} + \frac{9}{14}} \int_E \eta^{\frac{5}{7} + \frac{9}{7}} |D\eta|^{\frac{5}{21} + \frac{16}{21}} \left(\frac{|DN \cdot V|}{1 + N \cdot V} \right)^{\frac{10}{21} + \frac{11}{21}} H_2(dx) \\ &\leq \frac{15}{14} \alpha^b \int_E \eta^2 |D\eta|^{\frac{2}{3}} \left(\frac{|DN \cdot V|}{1 + N \cdot V} \right)^{\frac{4}{3}} H_2(dx) \\ &\quad + \frac{27}{14} \alpha^{-\frac{5}{9}b} \int_E \eta^2 |D\eta|^{\frac{32}{27}} \left(\frac{|DN \cdot V|}{1 + N \cdot V} \right)^{\frac{22}{27}} H_2(dx), \end{aligned} \quad (24)$$

for each $\alpha \in R$, $\alpha > 0$ and each $b \in R$, by applying Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

with $p = \frac{14}{5}$, $q = \frac{14}{9}$ and obvious choice of a , b . For the first term of the right hand side, we observe that

$$\begin{aligned} & \frac{15}{14} \alpha^b \int_E \eta^2 |D\eta|^{\frac{2}{3}} \left(\frac{|DN \cdot V|}{1 + N \cdot V} \right)^{\frac{4}{3}} H_2(dx) \\ &\leq \frac{15}{14} \alpha^b \left(\frac{1}{3\alpha^{b-1}} \int_E \eta^3 \left(\frac{|DN \cdot V|}{1 + N \cdot V} \right)^2 H_2(dx) + \frac{2}{3} \alpha^{2b-2} \int_E |D\eta|^2 H_2(dx) \right), \end{aligned} \quad (25)$$

by applying again Young's inequality with $p = 3$ and $q = \frac{3}{2}$. As far as the second term is concerned, we have

$$\frac{27}{14} \alpha^{-\frac{5}{9}b} \int_E \eta^2 |D\eta|^{\frac{32}{27}} \left(\frac{|DN \cdot V|}{1 + N \cdot V} \right)^{\frac{22}{27}} H_2(dx)$$

$$\begin{aligned}
&= \left(\frac{27}{14}\right)^{0+1} \alpha^{-\frac{5}{9}b} \int_E \eta^{\frac{11}{9}+\frac{7}{9}} |D\eta|^{\frac{32}{27}} \left(\frac{|DN \cdot V|}{1+N \cdot V}\right)^{\frac{22}{27}} H_2(dx) \\
&\leq \alpha^{-\frac{5}{9}b} \left(\frac{11}{27} \alpha^{(\frac{5}{9}b+1)} \int_E \eta^3 \left(\frac{|DN \cdot V|}{1+N \cdot V}\right)^2 H_2(dx) \right. \\
&\quad \left. + \left(\frac{16}{27}\right) 2^{\frac{27}{16}} \alpha^{-(\frac{5}{9}b+1)\frac{11}{16}} \int_E \eta^{\frac{21}{16}} |D\eta|^2 H_2(dx) \right) \\
&< \alpha^{-\frac{5}{9}b} \left(\frac{11}{27} \alpha^{(\frac{5}{9}b+1)} \int_E \eta^3 \left(\frac{|DN \cdot V|}{1+N \cdot V}\right)^2 H_2(dx) \right. \\
&\quad \left. + \frac{64}{27} \alpha^{-(\frac{5}{9}b+1)\frac{11}{16}} \int_E \eta^{\frac{21}{16}} |D\eta|^2 H_2(dx) \right), \tag{26}
\end{aligned}$$

applying Young's inequality with $p = \frac{27}{11}$ and $q = \frac{27}{16}$. Inserting (25) and (26) into (24), we obtain

$$\begin{aligned}
&\int_E \eta^3(x) \left\{ 2(2H^2 - K) \frac{N \cdot V}{(1+N \cdot V)} + (1-\alpha) \frac{|DN \cdot V|^2}{(1+N \cdot V)^2} \right\} H_2(dx) \\
&\leq \frac{64}{27} \alpha^{-\frac{15}{16}b-\frac{11}{16}} \int_E \eta^{\frac{21}{16}} |D\eta|^2 H_2(dx) + \frac{5}{7} \alpha^{3b-2} \int_E |D\eta|^2 H_2(dx). \tag{27}
\end{aligned}$$

Recalling the following estimate in Lemma 2.5 of Spruck [15]:

$$|DN \cdot V|^2 \geq (2H^2 - K - 2H\sqrt{H^2 - K}) (1 - (N \cdot V)^2) \tag{28}$$

we have, as in [15], for $0 < \alpha < 1$,

$$\begin{aligned}
&2(2H^2 - K) \frac{N \cdot V}{1+N \cdot V} + (1-\alpha) \frac{|DN \cdot V|^2}{(1+N \cdot V)^2} \\
&\geq 2(2H^2 - K) \frac{N \cdot V}{1+N \cdot V} + (1-\alpha) (2H^2 - K - 2H\sqrt{H^2 - K}) \frac{1 - (N \cdot V)^2}{(1+N \cdot V)^2} \\
&= (2H^2 - K) - \alpha(2H^2 - K) \frac{1 - N \cdot V}{1+N \cdot V} - 2(1-\alpha)H\sqrt{H^2 - K} \frac{1 - (N \cdot V)}{(1+N \cdot V)} \\
&\geq (2H^2 - K) - H^2 \frac{1 - N \cdot V}{1+N \cdot V} - (H^2 - K) \frac{1 - N \cdot V}{1+N \cdot V},
\end{aligned}$$

since $2(1-\alpha)H\sqrt{H^2 - K} \leq (1-\alpha)H^2 + (1-\alpha)(H^2 - K)$. Thus, by (27),

$$\begin{aligned}
&\int_E \left(1 - \frac{1 - N \cdot V}{1+N \cdot V}\right) \eta^3(x) (H^2 - K) H_2(dx) \\
&\leq \frac{64}{27} \alpha^{-\frac{15}{16}b-\frac{11}{16}} \int_E \eta^{\frac{21}{16}} |D\eta|^2 H_2(dx) + \frac{5}{7} \alpha^{3b-2} \int_E |D\eta|^2 H_2(dx), \tag{29}
\end{aligned}$$

if $\frac{1-N \cdot V}{1+N \cdot V} \leq 1$, that is, N makes an angle larger than β with a fixed vector $-V$ and $\beta \geq \frac{\pi}{2}$. This suggests us to choose a constant c such that, for some δ , $\frac{1}{2} \leq \delta < 1$, there holds

$$\eta \equiv c \text{ in } E^\delta \subset\subset E, \quad (30)$$

where E^δ is some subdomain E^δ of E , $E^\delta \subset\subset E$ and $H_2(E^\delta) = \delta H_2(E)$.

I.2.1. Thus, for C^2 subset E of M and Lipschitz subdomain E^δ of E , we are led to construct $\eta(x)$ in $E - E^\delta$ so that $\eta \in C^\infty(E - E^\delta)$ and, recalling the designation of δ_2 in (18),

$$\begin{cases} 2\delta_2 = \frac{19}{8} + \frac{9}{4} \frac{H_2(E)(-K)_{\max}}{\pi} \geq \Delta\eta \geq 2 & \text{in } E - E^\delta, \\ \eta = 0 & \text{on } \partial E, \\ \eta = c & \text{on } \partial E^\delta, \end{cases} \quad (31)$$

in which the positive constant c is not known apriori but is determined from the condition

$$-\oint_{\partial E^\delta} \frac{\partial \eta}{\partial n} H_1(dx) \geq 2H_2(E^\delta), \quad (32)$$

where n is the unit outward normal of ∂E^δ with respect to E^δ ; moreover, we require that

$$0 < \eta < c \quad (33)$$

in $E - E^\delta$. Set

$$\begin{aligned} E_1(\mu) &= \{x \in E, \eta(x) \geq \mu\}, \\ \Gamma_1(\mu) &= \{x \in E, \eta(x) = \mu\}, \\ a_1(\mu) &= H_2(E_1(\mu)). \end{aligned}$$

Moreover, $\mu(a_1)$ is defined to be the inverse of $a_1(\mu)$. To proceed further, as mentioned before, we adapt the procedure of Payne [12], and *make that hypothesis of the nonpositivity of K on M* which enables us to apply Proposition 1 of [11]. We obtain, analogously to (10),

$$-\frac{1}{d\mu/da_1} \geq \frac{4\pi a_1(\mu)}{\oint_{\Gamma_1(\mu)} |D\eta|}. \quad (34)$$

On the other hand, by the first inequality in (59), (61) and Green's identity, we have,

$$\oint_{\Gamma_1(\mu)} |D\eta| H_1(dx) \leq 2\delta_2 \int_{(E-E^\delta) \cap E_1(\mu)} H_2(dx) + 2H_2(E^\delta) \leq 2a_1(\mu)\delta_2. \quad (35)$$

From (34) and (35) it follows that $-\frac{1}{d\mu/da_1} \geq 2\pi\delta_2^{-1}$ for almost every μ , $\mu \neq c$. Thus we obtain

$$2\pi\delta_2^{-1}\mu(a_1) \leq H_2(E) - a_1(\mu),$$

for almost every μ , $0 < \mu < c$. From this, two important estimates follow, i.e.

$$c \leq \frac{\delta_2}{2\pi}(H_2(E) - H_2(E^\delta)) = \frac{\delta_2}{2\pi}(1 - \delta)H_2(E) \quad (36)$$

and

$$a_1(\mu) \leq H_2(E) - 2\pi\delta_2^{-1}\mu(a_1), \quad (37)$$

for almost every μ , $0 < \mu < c$.

I.2.1.1. We now proceed to construct, for δ sufficiently close to 1, a subdomain $E^\delta \subset\subset E$ with $H_2(E^\delta) = \delta H_2(E)$ and a function η satisfying (31) and (32) for E and E^δ such that (33) holds in $E \setminus E^\delta$.

Suppose that the boundary of E is made up by k simple closed curves, each of which, by a standard procedure of approximation, is assumed without loss of generality to be *twice differentiable*. We may represent the boundary of E by $x(s)$, s being the arc length with respect to the metric of M . Let $x_\rho(s)$ be a point in E at a distance ρ from ∂E such that the geodesic arc from $x_\rho(s)$ to ∂E intersects ∂E at $x(s)$. If ρ is small, the set

$$\Gamma^-(\rho) = \{x_\rho(s) : 0 \leq s \leq H_1(\partial E)\}$$

is also made up by k twice continuously differentiable, simple closed curves. If ε_0 is sufficiently small, then in $\cup_{\rho \leq \varepsilon_0} \Gamma^-(\rho)$, we may adapt the geodesic parallel coordinates

$$d\sigma^2 = d\rho^2 + G(\rho, s)ds^2,$$

where the function $G(\rho, s)$ has a square root of class C^1 satisfying

$$G(0, s) = 1, \quad (38)$$

and

$$\frac{\partial \sqrt{G}}{\partial \rho}(0, s) = k_g(s) = \text{the geodesic curvature of } \partial E \text{ at } x(s), \quad (39)$$

(cf. page 709 of [7]). Then the function

$$\eta(s, \rho) = \frac{9}{8}\rho^2$$

satisfies

$$\Delta \eta = \frac{1}{\sqrt{G}} \frac{\partial}{\partial \rho} \left(\sqrt{G} \frac{\partial \eta}{\partial \rho} \right) = \frac{\partial^2 \eta}{\partial \rho^2} + \frac{1}{\sqrt{G}} \frac{\partial \eta}{\partial \rho} \frac{\partial \sqrt{G}}{\partial \rho} \frac{9}{4} + \frac{9}{4} \frac{\rho}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho}. \quad (40)$$

Since $0 \leq K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial \rho^2}$, we have $\frac{\partial^2 \sqrt{G}}{\partial \rho^2} > 0$ in $\cup_{\rho \leq \varepsilon_0} \Gamma^-(\rho)$, which yields, together with (39) and (40),

$$\frac{\partial \sqrt{G}}{\partial \rho}(\rho, s) > \frac{\partial \sqrt{G}}{\partial \rho}(0, s) = k_g(s) \quad (41)$$

and

$$\sqrt{G}(\rho, s) > \sqrt{G}(0, s) + \rho \frac{\partial \sqrt{G}}{\partial \rho}(0, s) = 1 + \rho k_g(s), \quad (42)$$

in $\bigcup_{\rho \leq \varepsilon_0} \Gamma^-(\rho)$. We may choose ε_0 so small that

$$\rho |k_g(s)| < \frac{1}{18} \quad (43)$$

and hence, by (42),

$$\sqrt{G} > \frac{17}{18}, \quad (44)$$

in $\bigcup_{\rho \leq \varepsilon_0} \Gamma^-(\rho)$. (41), (43) and (44) yield

$$\frac{\rho}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho}(\rho, s) > -\frac{18}{17} \rho |k_g(s)| > -\frac{1}{17}$$

in $\bigcup_{\rho \leq \varepsilon_0} \Gamma^-(\rho)$. Whence, by (40), we obtain, in $\bigcup_{\rho \leq \varepsilon_0} \Gamma^-(\rho)$,

$$\Delta \eta > \frac{9}{4} - \left(\frac{9}{4}\right) \left(\frac{1}{17}\right) > 2.$$

We may denote

$$E \setminus E^\delta = \bigcup_{\rho \leq \varepsilon_0} \Gamma^-(\rho).$$

Also, we observe that η can be extended to E by defining

$$\hat{\eta}(x) = \left(\frac{9}{4}\right) (\text{dist}(x, \partial E))^2$$

for $x \in E$. We note that there exists a (possibly empty) set F of exceptional p -values which does not have a finite cluster point such that if $\rho \notin F$, $\Gamma(\rho)$ is the union of a finite number of simple closed curves of class C^2 , while if $\rho \in F$, $\Gamma(\rho)$ is a set of simple closed curves of class C^2 except for possible finitely many corners. Furthermore, the length $L(\rho)$ of $\Gamma(\rho)$ can be defined at the points of $\rho \in F$ so as to become a continuous function, (cf. [5], [7]). Consider a small neighborhood $E_{\rho\varepsilon}$ of $\Gamma(\rho)$ for each $\rho \in F$, such that $\partial E_{\rho\varepsilon}$ is of class C^2 and $\bigcup_{\rho \in F} E_{\rho\varepsilon} = E_\varepsilon$ has $H_2(E_\varepsilon) < \frac{\varepsilon}{2}$. By the previous argument, we know that

$$\Delta \hat{\eta}(x) > 2$$

for every point in $E - E_\varepsilon$. Setting $E_\varepsilon^\delta = E^\delta \cap (E \setminus E_\varepsilon)$, Green's theorem then yields

$$\oint_{\partial E_\varepsilon^\delta} \frac{\partial \hat{\eta}}{\partial n} H_1(dx) = \int_{E_\varepsilon^\delta} \Delta \hat{\eta} H_2(dx) > H_2(E_\varepsilon^\delta) > 2H_2(E) - \varepsilon,$$

in which $\partial E_\varepsilon^\delta = \partial E^\delta \cup (\cup_{\rho \in F} (\partial E_{\rho\varepsilon}))$. Since $L(\rho)$ is continuous, $\hat{\eta}$ is regular on $\partial E_{\rho\varepsilon}$ and $\partial E_{\rho\varepsilon}$ is of class C^2 , $\oint_{\partial E_{\rho\varepsilon}} \frac{\partial \hat{\eta}}{\partial n} H_1(dx)$ can be made arbitrarily small for each $\rho \in F$ by choosing ε small enough. Hence, we obtain

$$\oint_{\partial E^\delta} \frac{\partial \eta}{\partial n} H_1(dx) = \oint_{\partial E^\delta} \frac{\partial \hat{\eta}}{\partial n} H_1(dx) \geq 2H_2(E^\delta),$$

which is (32).

It remains to verify the first inequality of (31). For this, we observe that

$$\begin{aligned} \frac{\partial}{\partial \rho} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho} \right) &= \frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial \rho^2} - \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho} \right)^2 \\ &= -K(\rho, s) - \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho} \right)^2 \\ &\leq (-K)_{\max} - \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho} \right)^2, \end{aligned}$$

which yields, for $f(\rho, s) = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho}(\rho, s)$,

$$\begin{aligned} \rho &\geq \frac{1}{\sqrt{(-K)_{\max}}} \operatorname{arctanh} \left(\frac{f(\rho, s)}{\sqrt{(-K)_{\max}}} \right) - \frac{1}{\sqrt{(-K)_{\max}}} \operatorname{arctanh} \left(\frac{k_g(s)}{\sqrt{(-K)_{\max}}} \right) \\ &\geq \frac{1}{\sqrt{(-K)_{\max}}} \operatorname{arctanh} \left(\frac{f(\rho, s) - k_g(s)}{\sqrt{(-K)_{\max}}} \right), \end{aligned}$$

i.e.

$$\begin{aligned} \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho}(\rho, s) &\leq \sqrt{(-K)_{\max}} \tanh \left(\rho \sqrt{(-K)_{\max}} \right) + k_g(s) \\ &\leq \rho(-K)_{\max} + k_g(s), \end{aligned}$$

whence, by (43) and the estimate $\operatorname{dist}(x, \partial E) < \sqrt{\frac{H_2(E)}{\pi}}$, derived in Theorem 2 of [11], we have

$$\frac{\rho}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial \rho}(\rho, s) \leq \rho^2(-K)_{\max} + \frac{1}{18} \leq \frac{(-K)_{\max} H_2(E)}{\pi} + \frac{1}{18}.$$

Inserting this into (40) yields the first inequality (31) in $E - E^\delta$.

I.2.1.2. We note that the conditions

$$\begin{cases} \Delta \eta \geq 2 & \text{in } E - E^\delta \\ \eta = c & \text{on } \partial E^\delta, \end{cases}$$

together with (32) yield

$$\eta \geq \eta_1, \quad (45)$$

where η_1 is modified from the so called warping function in the torsion problem; namely, $\eta_1(x)$ satisfies

$$\begin{cases} \Delta \eta_1 = 2 & \text{in } E - E^\delta, \\ \eta_1 = 0 & \text{on } \partial E, \\ \eta_1 = c_1 & \text{on } \partial E^\delta, \end{cases}$$

in which the constant c_1 is not known apriori but is determined from the condition

$$-\oint_{\partial E^\delta} \frac{\partial \eta_1}{\partial n} H_1(dx) = 2H_2(E^\delta).$$

This follows from, e.g., a straightforward modification of the proof of Theorem 5.1 of [6]. Hence, by (45) and an obvious modification of the reasoning leading to Theorem 4 of [11],

$$c = \sup_{x \in E - E^\delta} \eta(x) \geq \sup_{x \in E - E^\delta} \eta_1(x) \geq \frac{\delta_1}{4\pi} H_2(E - E^\delta), \quad (46)$$

where δ_1 is as indicated in (17).

I.2.1.3. Setting

$$S_0(\mu) = \int_{(E - E^\delta) \cap E_1(\mu)} |D\eta|^2 H_2(dx),$$

the coarea formula and (35) yield

$$\frac{dS_0(\mu)}{d\mu} = - \oint_{\Gamma_1(\mu)} |D\eta| H_1(dx) \geq -2\delta_2 a_1(\mu).$$

Thus, by (33) and (37),

$$\int_{E - E^\delta} |D\eta|^2 H_2(dx) = S_0(0) - S_0(c) \leq (2cH_2(E) - 2\pi\delta_2^{-1}c^2) \delta_2.$$

Hence, by (36), we have

$$\int_{E - E^\delta} |D\eta|^2 H_2(dx) \leq \left(\frac{1 - \delta}{2\pi} (H_2(E))^2 - \pi\delta_2^{-1}c^2 \right) 2\delta_2. \quad (47)$$

Likewise, setting

$$S_1(\mu) = \int_{(E - E^\delta) \cap E_1(\mu)} \eta^{\frac{21}{16}} |D\eta|^2 H_2(dx),$$

the coarea formula and (35) yield

$$\frac{dS_1}{d\mu} = -\mu^{\frac{21}{16}} \oint_{\Gamma_1(\mu)} |D\eta| H_1(dx) \geq -2\delta_2 \mu^{\frac{21}{16}} (a_1(\mu)).$$

Thus, by (33) and (37), we have

$$\int_{E-E^\delta} \eta^{\frac{21}{16}} |D\eta|^2 H_2(dx) \leq 2 \int_0^c \mu^{\frac{21}{16}} (H_2(E) - 2\pi\delta_2^{-1}\mu) d\mu = \frac{32}{37} c^{\frac{37}{16}} H_2(E) - \frac{64}{53} \pi \delta_2^{-1} c^{\frac{53}{16}}.$$

Hence, by (36), we have

$$\int_{E-E^\delta} \eta^{\frac{21}{16}} |D\eta|^2 \leq \delta_2^{\frac{37}{16}} \frac{(1-\delta)^{\frac{37}{16}}}{4\pi^2} (H_2(E))^{\frac{53}{16}} - \pi \delta_2^{-1} c^{\frac{53}{16}}. \quad (48)$$

Thus, by (30), (33), (47) and (48), we have, for all $\alpha \in R$, $0 < \alpha < 1$, if $N \cdot V > -\cos \beta > 0$ everywhere on M ,

$$\begin{aligned} c^3 \int_{E^\delta} (H^2 - K) H_2(dx) &\leq \int_E \eta^3(x) (H^2 - K) H_2(dx) \\ &\leq \left(1 - \frac{1 + \cos \beta}{1 - \cos \beta}\right)^{-1} \left[\frac{64}{27} \alpha^{-\frac{15}{16}b - \frac{11}{16}} \left(\delta_2^{\frac{37}{16}} \frac{(1-\delta)^{\frac{37}{16}}}{4\pi^2} (H_2(E))^{\frac{53}{16}} - \pi \delta_2^{-1} c^{\frac{53}{16}} \right) \right. \\ &\quad \left. + \frac{7}{5} \alpha^{3b-2} \delta_2 \left(\frac{(1-\delta)}{2\pi} (H_2(E))^2 - \pi \delta_2^{-1} c^2 \right) \right]. \end{aligned}$$

Inserting (36) and (46) into this, we obtain

$$\begin{aligned} &\int_{E^\delta} (H^2 - K) H_2(dx) \quad (49) \\ &\leq \frac{64}{27} \alpha^{-\frac{15}{16}b - \frac{11}{16}} \left(\frac{64\pi(1-\delta)^{-\frac{11}{16}}}{4} \delta_2^{\frac{11}{16}} (H_2(E))^{\frac{5}{16}} - \frac{(1-\delta)^{\frac{5}{16}} \delta_2^{-1} \delta_1^{\frac{5}{16}}}{4} (H_2(E))^{\frac{5}{16}} \right) \\ &\quad + \frac{7}{5} \alpha^{3b-2} \delta_2 \left(\frac{32\delta_1^{-3}\pi^2}{(1-\delta)^2 H_2(E)} - \frac{4\pi^2 \delta_2^{-2}}{(1-\delta) H_2(E)} \right) \left(1 - \frac{1 + \cos \beta}{1 - \cos \beta}\right)^{-1}, \end{aligned}$$

under the assumption that N makes an angle large than β with a fixed vector $-V$ and $\beta > \frac{\pi}{2}$. This completes the proof of **Theorem I.7**. $\diamond$

I.3. A Proof of Main Theorem 1.

In this section, we prove **Main Theorem 1** with the aid of **Theorem I.7**.

For this, we suppose $p > 1$ and consider a sequence of measurable subsets E_{ip} of M for which we have

$$\frac{\int_{E_{ip}} (H^2 - K) H_2(dx)}{H_2((E_{ip}))^{1-\frac{1}{p}}} \rightarrow \|H^2 - K\|_{p,\infty},$$

as $i \rightarrow \infty$. By a standard approximation procedure, we assume, without loss of generality, that the boundary ∂E_{ip} of E_{ip} is twice differentiable for each i .

I.3.1. Preliminary Results.

We need the following result, which follows from some simple algebraic manipulation of the results in **Theorem I.6**.

Proposition I.9 *Let M be an embedded surface in R^3 of constant mean curvature H . For p and E_{ip} indicated above, choose $E_{ip}^{\delta^i} \subset\subset E_{ip}$ for $0 < \delta^i < 1$, so that $H_2(E_{ip}^{\delta^i}) = \delta^i H_2(E_{ip})$. Suppose there exists a positive number C such that*

$$\frac{\int_{E_{ip}^{\delta^i}} (H^2 - K) H_2(dx)}{(H_2(E_{ip}))^{1-\frac{1}{p}}} < C,$$

for i sufficiently large. Set

$$A_{ip} = \frac{2\pi}{p-1} \left(\frac{p}{2\pi} \right)^{\frac{p}{p-1}} \left[(H^2 + (-K)_{\max})^{\frac{1}{p-1}} (H_2(E_{ip}))^{\frac{1}{p-1}} \right]. \quad (50)$$

If we choose δ^i sufficiently close to 1 so that

$$(1 - \delta^i)^{\frac{p}{p-1}} A_{ip} < \frac{1}{2} \quad (51)$$

for i sufficiently large and if $(-K)_{\max} > K_0$, then the following alternative holds. Either

$$(H^2 + (-K)_{\max})(H_2\{x \in M : -K(x) > K_0\}) < \frac{2\pi}{p-1}$$

or

$$H^2 + (-K)_{\max} < (1 - 2(1 - \delta^i)^{\frac{p}{p-1}} A_{ip})^{-1} \frac{4\pi}{p-1} \left(\frac{p}{2\pi} C \right)^{\frac{p}{p-1}}, \quad (52)$$

for i sufficiently large.

Proof of Proposition I.9

Suppose

$$\frac{\int_{E_i} (H^2 - K) H_2(dx)}{H_2(E_i)} \rightarrow \|H^2 - K\|_{p,\infty},$$

for a sequence of measurable subsets E_i of M . Choose $E_i^{\delta^i} \subset\subset E_i$, ($0 < \delta^i < 1$), so that (8) and (11) hold. Suppose there exists a positive number C such that

$$\frac{\int_{E_i^{\delta^i}} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1-\frac{1}{p}}} < C, \quad (53)$$

for i sufficiently large. We have

$$\frac{\int_{E_i} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1-\frac{1}{p}}} = \frac{\int_{E_i^{\delta^i}} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1-\frac{1}{p}}} + \frac{\int_{E_i - E_i^{\delta^i}} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1-\frac{1}{p}}}, \quad (54)$$

in which

$$\frac{\int_{E_i - E_i^{\delta^i}} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1 - \frac{1}{p}}} < (H^2 + (-K)_{\max})(1 - \delta^i)(H_2(E_i))^{\frac{1}{p}}. \quad (55)$$

Inserting (53) and (55) into (54), we obtain

$$\frac{\int_{E_i} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1 - \frac{1}{p}}} \leq C + (H^2 + (-K)_{\max})(1 - \delta^i)(H_2(E_i))^{\frac{1}{p}}. \quad (56)$$

Suppose (14) holds. (56) and (14) then yield

$$H^2 + (-K)_{\max} \leq \frac{2\pi}{p-1} \left[\frac{p}{2\pi} (1 - \delta^i)(H^2 + (-K)_{\max})(H_2(E_i))^{\frac{1}{p}} + \frac{pC}{2\pi} \right]^{\frac{p}{p-1}}. \quad (57)$$

As is easy to verify, for $a, b \in R$, there holds

$$(a + b)^{\frac{p}{p-1}} \leq (a + b)(a^{\frac{1}{p-1}} + b^{\frac{1}{p-1}}) \leq 2a^{\frac{p}{p-1}} + 2b^{\frac{p}{p-1}}.$$

Inserting this into (57), we obtain

$$\begin{aligned} H^2 + (-K)_{\max} &\leq \frac{4\pi}{p-1} \left(\frac{p}{2\pi} C \right)^{\frac{p}{p-1}} \\ &+ \frac{4\pi}{p-1} \left(\frac{p(1 - \delta^i)}{2\pi} \right)^{\frac{p}{p-1}} \left[(H^2 + (-K)_{\max}) H_2(E_i) \right]^{\frac{1}{p-1}} (H^2 + (-K)_{\max}) \\ &= \frac{4\pi}{p-1} \left(\frac{p}{2\pi} C \right)^{\frac{p}{p-1}} + 2(1 - \delta^i)^{\frac{p}{p-1}} A_{ip} (H^2 + (-K)_{\max}), \end{aligned}$$

by (50). From the choice of δ^i made in (51) follows (52). $\diamond$

The following result is also needed.

Proposition I.10 *Let M be an embedded surface in R^3 of constant mean curvature H and nonpositive Gauss curvature K . Suppose (14) holds for some number $p > 1$ and the sequence of measurable subsets E_{ip} of M with twice differentiable boundary is as indicated above. If $(-K)_{\max} < K_0$, then*

$$(H^2 + (-K)_{\max}) H_2(E_{ip}) \geq \left(\frac{2\pi}{p-1} \right)^{-p+1} \left[\frac{p}{2\pi} \right]^{-p} = C_p \quad (58)$$

and

$$\int_{E_{ip}} (H^2 - K) H_2(dx) \geq \left(\frac{2\pi}{p-1} \right)^{-p+1} \left[\frac{p+1}{2\pi} \right]^{-p} = \hat{C}_p \quad (59)$$

for i sufficiently large.

Proof of Proposition I.10 Given ε , there holds, for i sufficiently large,

$$\begin{aligned}\|H^2 - K\|_{p,\infty} &< \frac{(H^2 + (-K)_{\max})(H_2(E_{ip}))}{(H_2(E_{ip}))^{1-\frac{1}{p}}} + \varepsilon \\ &= (H^2 + (-K)_{\max})(H_2(E_{ip}))^{\frac{1}{p}} + \varepsilon,\end{aligned}$$

by the definition of $\|H^2 - K\|_{p,\infty}$ and the choice of the sequence E_{ip} . Hence, if (14) holds,

$$H^2 + (-K)_{\max} \leq \frac{2\pi}{p-1} \left(\frac{p}{2\pi}\right)^{\frac{p}{p-1}} (H^2 + (-K)_{\max})^{\frac{p}{p-1}} (H_2(E_{ip}) + \varepsilon)^{\frac{1}{p-1}},$$

that is,

$$(H^2 + (-K)_{\max})^{-\frac{1}{p-1}} \leq \frac{2\pi}{p-1} \left(\frac{p}{2\pi}\right)^{\frac{p}{p-1}} (H_2(E_{ip}) + \varepsilon)^{\frac{1}{p-1}},$$

which proves (58), since ε is arbitrary.

Also, by (14) and the definition of $\| -K + H^2 \|_{p,\infty}$

$$\begin{aligned}\int_{E_{ip}} (H^2 - K)H_2(dx) &< (H^2 + (-K)_{\max})H_2(E_{ip}) \\ &< \frac{2\pi}{p-1} \left[\frac{p+\varepsilon}{2\pi} \|H^2 - K\|_{p,\infty} \right]^{\frac{p}{p-1}} H_2(E_{ip}) \\ &< \frac{2\pi}{p-1} \left[\frac{p+\varepsilon}{2\pi} \int_{E_{ip}} (H^2 - K)H_2(dx) \right]^{\frac{p}{p-1}}\end{aligned}$$

for given $\varepsilon > 0$ and i sufficiently large. This proves (59). $\diamond$

We may observe that, for the numbers A_{ip} and C_p defined respectively in (50) and (58), there holds

$$A_{ip} = C_p^{-\frac{1}{p-1}} \left[(H^2 + (-K)_{\max})H_2(E_{ip}) \right]^{\frac{1}{p-1}}, \quad (60)$$

Hence, from **Proposition I.10**, the following estimates follow.

Lemma I.11 *Let M , H , K , p and E_{ip} be as indicated in Proposition 1. Suppose (14) holds and $(-K)_{\max} > K_0$, then*

$$A_{ip} > 1 \quad (61)$$

for i sufficiently large. Under the hypothesis that $(-K)_{\max} \geq H^2$, for the number $\delta_2(E_{ip})$ designated in (18), there holds

$$\delta_2(E_{ip}) < \left(\frac{19}{8} C_p^{-1} + \frac{9}{8} \frac{1}{\pi} \right) (H_2(E_{ip}))(-K)_{\max} = C_p^*(H_2(E_{ip}))(-K)_{\max}, \quad (62)$$

for i sufficiently large, where

$$C_p^* = \frac{19}{8}C_p^{-1} + \frac{9}{8}\frac{1}{\pi}. \quad (63)$$

Proof of Lemma I.11 The inequality (61) follows from (58) and (60). The inequality (62) follows from (18), (58) and the assumption $(-K)_{\max} \geq H^2$. $\diamond$

I.3.2. Proof of Main Theorem 1. Assume henceforth that

$$H^2 \leq (-K)_{\max}, \quad \text{and} \quad 0 > K_0 \leq (-K)_{\max}, \quad (64)$$

For some fixed p , denote E_{ip} abbreviatedly as E_i . Under some fixed scaling, we suppose

$$H_2(E_i) < 1 \quad (65)$$

for each i sufficiently large.

I.3.2.1. We first proceed to make a suitable choice of the number α in (16) and (19). We can choose δ^i sufficiently close to 1, say $\delta^i > \delta^0$, such that the set $E_i^{\delta^i}$ with $H_2(E_i^{\delta^i}) = \delta^i H_2(E_i)$ satisfies (16) and, by (59), has

$$\int_{E_i^{\delta^i}} (H^2 - K) H_2(dx) > \frac{1}{2} \hat{C}_p, \quad (66)$$

for i sufficiently large. As is easily verified by (17) and (18), there exists a constant $C_{\delta_1 \delta_2}$,

$$C_{\delta_1 \delta_2} > 2, \quad (67)$$

such that

$$\begin{aligned} C_{\delta_1 \delta_2} \delta_2(E_i) &> C_{\delta_1 \delta_2} K_0 H_2(E_i) \\ &> \left(\left(\log \left(1 + \frac{K_0 H_2(E_i)}{4\pi} \right) \right) \left(1 + \frac{4\pi}{K_0 H_2(E_i)} \right) - 1 \right)^3 \\ &= (\delta_1(E_i))^3. \end{aligned}$$

Since this yields $(C_{\delta_1 \delta_2})^{-1} (\delta_2(E_i))^{-1} (\delta_1(E_i))^3 < 1$, and $\hat{C}_p$ defined in (59) is less than 1, the condition (65) permits us to set, in (19),

$$\alpha_i = \left(\left(\frac{5}{224 C_{\delta_1 \delta_2} \pi^2} \right) \left(1 - \frac{1 + \cos \beta}{1 - \cos \beta} \right) (1 - \delta_i)^2 \frac{(\delta_1(E_i))^3}{(\delta_2(E_i))} \hat{C}_p (H_2(E_i))^2 \right)^{\frac{1}{3b-2}},$$

for these i , δ_i and some positive real number b , $b > \frac{2}{3}$. Then, by (21) and (66),

$$\alpha_i^{3b-2} C_2(H_2(E_i))^{-1} = (C_{\delta_1 \delta_2})^{-1} \hat{C}_p(H_2(E_i)) \leq 2 \frac{H_2(E_i)}{C_{\delta_1 \delta_2}} \int_{E_i^{\delta^i}} (H^2 - K) H_2(dx),$$

And hence, by (19), (20) and (67),

$$\begin{aligned} (1 - H_2(E_i)) \int_{E_i^{\delta^i}} (H^2 - K) H_2(dx) &\leq \alpha_i^{-\left(\frac{15}{16}b + \frac{11}{16}\right)} C_1(H_2(E_i))^{\frac{5}{16}} \quad (68) \\ &= \frac{64}{27} (16\pi) \left(\frac{224\pi^2}{5}\right)^{\hat{b}} (C_{\delta_1\delta_2})^{-\hat{b}} (\hat{C}_p)^{\hat{b}} \left(1 - \frac{1 + \cos\beta}{1 - \cos\beta}\right)^{-\hat{b}-1} (1 - \delta_i)^{-2\hat{b} - \frac{11}{16}} \\ &\quad (\delta_1(E_i))^{-3\hat{b}} (\delta_2(E_i))^{\hat{b} - \frac{11}{16}} (H_2(E_i))^{-2\hat{b} + \frac{5}{16}}, \end{aligned}$$

where $\hat{b} = \left(\frac{1}{3b-2}\right) \left(\frac{15}{16}b + \frac{11}{16}\right)$.

I.3.2.2. By (18), we have $(\delta_2(E_i))^{-1} < \left(\frac{(-K)_{\max} H_2(E_i)}{\pi}\right)^{-1}$, which yields

$$(\delta_2(E_i))^{-\frac{11}{16}} (H_2(E_i))^{\frac{11}{16}} < (\pi(-K)_{\max})^{-\frac{11}{16}}. \quad (69)$$

Suppose (14) holds with some $p < \frac{37}{22}$. (The reason for this choice of the number p will be made clear in (78) and (79) below). Then, (62) yields

$$(\delta_2(E_i))^{\hat{b}} ((H_2(E_i))^{-\hat{b}} < (C_p^*((-K)_{\max}))^{\hat{b}} \quad (70)$$

for this p . Inserting (69) and (70) into (68), we obtain, for this p ,

$$\begin{aligned} &\frac{\int_{E_i^{\delta^i}} (H^2 - K) H_2(dx)}{(H_2(E_i))^{1 - \frac{1}{p}}} \quad (71) \\ &\leq C_{\hat{b}} (1 - \delta^i)^{-2\hat{b} - \frac{11}{16}} (\delta_1(E_i))^{-3\hat{b}} ((-K)_{\max})^{-\frac{11}{16} + \hat{b}} (H_2(E_i))^{-\frac{3}{8} - (1 - \frac{1}{p}) + \hat{b}}, \end{aligned}$$

where

$$C_{\hat{b}} = \frac{64}{27} (16) \left(\frac{5}{224} \pi^2\right)^{\hat{b}} \left(1 - \frac{1 + \cos\beta}{1 - \cos\beta}\right)^{-\hat{b}-1} (C_p^*)^{\hat{b}} (\hat{C}_p)^{\hat{b}} (C_{\delta_1\delta_2})^{-\hat{b}}. \quad (72)$$

Since we suppose (14) holds for this p , inequality (61) permits us to take

$$1 - \delta^i > \frac{1}{4} (A_{ip})^{-\frac{p-1}{p}} = \frac{1}{4} (C_p)^{-\frac{1}{p}} [(H_2 + (-K)_{\max}) H_2(E_i)]^{-\frac{1}{p}} \quad (73)$$

for i sufficiently large. With such a choice of δ^i , (52), (71) and (73) yield

$$\begin{aligned} &H^2 + (-K)_{\max} \\ &< \frac{8\pi}{p-1} \left[\frac{p}{2\pi} C_{\hat{b}} (1 - \delta^i)^{-2\hat{b} - \frac{11}{16}} (\delta_1(E_i))^{-3\hat{b}} ((-K)_{\max})^{-\frac{11}{16} + \hat{b}} (H_2(E_i))^{-\frac{3}{8} - (\frac{p-1}{p}) + \hat{b}} \right]^{\frac{p}{p-1}} \\ &\leq \hat{C}_{\hat{b}p} [(H^2 + (-K)_{\max}) H_2(E_i)]^{(2\hat{b} + \frac{11}{16}) \frac{1}{p-1}} (\delta_1(E_i))^{-\frac{3p}{p-1} \hat{b}} \\ &\quad [(-K)_{\max}]^{(-\frac{11}{16} + \hat{b}) \frac{p}{p-1}} (H_2(E_i))^{-\frac{3}{8} + \hat{b} \frac{p}{p-1} - 1}, \quad (74) \end{aligned}$$

where

$$\hat{C}_{\hat{b}p} = \frac{8\pi}{p-1} \left(\frac{p}{2\pi} C_{\hat{b}} \right)^{\frac{p}{p-1}} 2^{2(2\hat{b} + \frac{11}{16}) \frac{p}{p-1}} (C_p)^{(2\hat{b} + \frac{11}{16}) \frac{1}{p-1}}. \quad (75)$$

and $C_{\hat{b}}$ is defined in (72). (74) and (64) yield

$$((-K)_{\max})^{p^*} \leq \hat{C}_{\hat{b}p}^* (\delta_1(E_i))^{-3\hat{b} \frac{p}{p-1}} (H_2(E_i))^{p^{**}} \quad (76)$$

where $\hat{C}_{\hat{b}p}^* = 2^{(2\hat{b} + \frac{11}{16}) \frac{1}{p-1}} \hat{C}_{\hat{b}p}$ (with $\hat{C}_{\hat{b}p}$ being defined in (75)),

$$p^* = 1 + \left(\frac{11}{16} - \hat{b} \right) \frac{p}{p-1} - \left(2\hat{b} + \frac{11}{16} \right) \frac{1}{p-1} \quad (77)$$

and

$$p^{**} = -1 + \left(-\frac{3}{8} + \hat{b} \right) \left(\frac{p}{p-1} \right) + \left(2\hat{b} + \frac{11}{16} \right) \frac{1}{p-1}.$$

I.3.2.3. Observe that we can choose b arbitrarily large to make $\hat{b}$ larger than and arbitrarily close to $\frac{5}{16}$. Also observe that the right hand side of (76) becomes zero if we set $\hat{b} = \frac{5}{16}$ and $p = \frac{37}{22}$ in it. If $p < \frac{37}{22}$ and $\hat{b} > \frac{5}{16}$, we have

$$p^* < 0, \quad p^{**} > 0 \quad (78)$$

and

$$p^* + p^{**} = \frac{5}{16} \left(\frac{p}{p-1} \right). \quad (79)$$

This, together with (76) and (78), yields

$$((-K)_{\max} H_2(E_i))^{p^*} \leq \hat{C}_{\hat{b}p}^* (\delta_1(E_i))^{-3\hat{b} \frac{p}{p-1}} (H_2(E_i))^{\frac{5}{16} \frac{p}{p-1}}. \quad (80)$$

Rescale the surface to increase H^2 , $(-K)_{\max}$, H^2 and decrease $H_2(E_i)$ proportionally. In view of (17), (58), and (61), $\delta_1(E_i)$ and C_p^* remain unchanged and hence (80) must fail to hold after a rescaling which makes $H_2(E_i)$ sufficiently small. Hence, the second alternative of **Theorem I.6** cannot hold for any $p < \frac{37}{22}$. Whence, there must hold, for each $p < \frac{37}{22}$,

$$(-K)_{\max} (H_2\{x \in M : -K(x) > K_0\}) < \frac{2\pi}{p-1}.$$

This completes the proof of **Main Theorem 1**. $\diamond$

II On a Special Class of Doubly-connected Surfaces of Constant Mean Curvature.

In this part, we consider a doubly-connected surface M of constant mean curvature H such that, setting

$$\Gamma(t) = \{x : x \in M \cup \partial M, -K(x) = t\}$$

for $0 < K_0 \leq t \leq (-K)_{\max}$, we have $M = \bigcup_{K_0 < t < (-K)_{\max}} \Gamma(t)$, and $\partial M = \Gamma(K_0) \cup \Gamma((-K)_{\max})$. Here we assume $\Gamma(t)$ is a simple closed curve for $K_0 \leq t \leq (-K)_{\max}$.

We assume furthermore that a number A exists such that, for each point x_0 on $\Gamma((-K)_{\max})$ and each point x_1 on $\Gamma(K_0)$,

$$\limsup_{x_j \rightarrow x_0} |DK(x_j)| \rightarrow 0, \quad (81)$$

$$\liminf_{x_j \rightarrow x_1} |DK(x_j)| \geq A, \quad (82)$$

if a sequence x_j in M approaches x_0 or x_1 as $j \rightarrow \infty$. Moreover, we assume

$$\sup_{x \in \Gamma((-K)_{\max} - \varepsilon)} |DK(x)| = O \left(\inf_{x \in \Gamma((-K)_{\max} - \varepsilon)} |DK(x)| \right) = o(\varepsilon^q)$$

for some positive number q . That is, we assume the existence of a function $f(\varepsilon)$, $f(\varepsilon) = O(\varepsilon^q)$ for some $q > 0$, and positive constants ε_1^* , A_0 , A_1 such that

$$A_0 f(\varepsilon) \leq |DK(x)| \leq A_1 f(\varepsilon) \quad (83)$$

wherever $x \in \Gamma((-K)_{\max} - \varepsilon)$ and $\varepsilon < \varepsilon_1^*$. (For example, we may consider a portion of a complete catenoid which has the waist-circle of the catenoid as a component of boundary and a level curve of K as the other component of boundary. We note that we may set $f(\varepsilon) = \varepsilon^{\frac{1}{2}}$ in (79) for such a portion of catenoid.) We shall prove

Main Theorem 2 *Let M be an embedded doubly-connected surface in R^3 of constant mean curvature H and with finite area, whose Gauss curvature K satisfies the conditions indicated in the beginning of this part. Suppose that a normal vector field N makes an angle larger than β with a fixed vector $-V$ and $\beta > \frac{\pi}{2}$. Suppose $(-K)_{\max} > H^2$. Then*

$$(H^2 + (-K)_{\max})H_2(M) < \frac{33}{5}\pi,$$

and hence $H_2(M) < \frac{33\pi}{5} \left(\frac{1}{H^2 + (-K)_{\max}} \right)$.

This result enables us to formulate the following

Corollary II.1 *Let $\{M_j\}$ be a sequence of embedded doubly-connected surface in R^3 of constant mean curvature H and with finite area, whose Gauss curvature K_j has respective boundary values $(-K_j)_{\max}$ and $(K_j)_0$ on two components of boundary and satisfies the conditions indicated in the beginning of this part. Suppose that, for each j , a normal vector field N_j makes an angle larger than β with a fixed vector $-V_j$ and $\beta > \frac{\pi}{2}$. Suppose $(-K_j)_{\max} > H^2$ for each j , and as $j \rightarrow \infty$,*

$$(-K_j)_{\max} \rightarrow \infty, \quad \text{and} \quad (K_j)_0 \rightarrow K_0 \in R$$

such that $H^2 + K_0 > 0$. Then $H_2(M_j)$ is uniformly bounded and for each $\varepsilon > 0$, $H_2(M_j) < \varepsilon$ if j is sufficiently large.

II.1. Consider the subset

$$M_\varepsilon = \{x \in M : K_0 < -K(x) < (-K)_{\max} - \varepsilon\}$$

of M and let

$$E_\varepsilon(t) = \{x \in M : t < -K(x) < (-K)_{\max} - \varepsilon\}$$

for t in the range

$$K_0 < t < (-K)_{\max} - \varepsilon, \quad (84)$$

where ε is a small number, $\varepsilon < \varepsilon_1^*$, to be determined below. An integration of both sides of equation (2) on the set $E_\varepsilon(t)$ gives

$$\begin{aligned} & (H^2 + (-K)_{\max} - \varepsilon) \oint_{\Gamma((-K)_{\max} - \varepsilon)} |DK| H_1(dx) - (t + H^2) \oint_{\Gamma(t)} |DK| H_1(dx) \\ & - 2 \int_{E_\varepsilon(t)} |DK|^2 H_2(dx) = 4 \int_{E_\varepsilon(t)} K(H^2 - K)^2 H_2(dx). \end{aligned} \quad (85)$$

Following [16], we set

$$\Phi(t) = \int_{E_\varepsilon(t)} |DK|^2 H_2(dx).$$

In terms of Φ , the identity (85) takes the form

$$\begin{aligned} & -\frac{d}{dt}[(H^2 + t)^{-2} \Phi(t)] \\ & = 4(t + H^2)^{-3} \int_{E_\varepsilon(t)} K(H^2 - K)^2 H_2(dx) - (t + H^2)^{-3} \int_{\Gamma((-K)_{\max} - \varepsilon)} |DK| H_1(dx); \end{aligned}$$

hence an integration gives

$$\begin{aligned} \Phi(t) &= 4 \int_{E_\varepsilon(t)} (-K)((H^2 - K)^2 - (H^2 + t)^2) H_2(dx) \\ &+ \frac{H^2 + t}{2} \left(\frac{H^2 + t}{(H^2 + (-K)_{\max} - \varepsilon)} - \frac{H^2 + (-K)_{\max} - \varepsilon}{H^2 + t} \right) \\ &\quad \int_{\Gamma((-K)_{\max} - \varepsilon)} |DK| H_1(dx). \end{aligned} \quad (86)$$

Inserting (86) into (85) gives

$$\begin{aligned} \frac{1}{2} \left(\frac{H^2+t}{(H^2+(-K)_{\max}-\varepsilon)} - 3 \frac{H^2+(-K)_{\max}-\varepsilon}{H^2+t} \right) \int_{\Gamma((-K)_{\max}-\varepsilon)} |DK| H_1(dx) \\ + \int_{\Gamma(t)} |DK| H_1(dx) = 4(H^2+t) \int_{E_\varepsilon(t)} (-K) H_2(dx). \end{aligned} \quad (87)$$

II.2. Introduce the distribution function

$$a_\varepsilon(t) = H_2(E_\varepsilon(t)).$$

Then, as in (37) of [16], we have

$$(H_1(\Gamma(t)))^2 \leq -(a'_\varepsilon(t)) \int_{\Gamma(t)} |DK| H_1(dx) \quad (88)$$

for almost every t satisfying (80). To apply *isoperimetric inequality* to handle the left hand side of (84), we need the following result.

Proposition II.2 *Let M be an embedded simply-connected or doubly connected surface of constant mean curvature in R^3 . Let $\Gamma(k) = \{x : x \in M \cup \partial M, -K(x) = k\}$. Suppose $\Gamma(k)$ is either empty or a simple closed curve for each k which is not a critical value of K . If $k_1 \geq k_2 \geq 0$, and if both $\Gamma(k_1)$ and $\Gamma(k_2)$ are nonempty, then $H_1(\Gamma(k_1)) \leq H_1(\Gamma(k_2))$.*

Proof of Proposition II.2.

Let M be an embedded surface in R^3 with constant mean curvature H and $x : M \rightarrow R^3$ be an isometric immersion of M into R^3 . At a neighborhood of any point of M , we may adopt an isothermal coordinate $z = \xi_1 + i\xi_2 = \gamma e^{i\theta}$ such that the first fundamental form is, for some constant $\lambda > 0$, $ds^2 = \lambda^2 |dz|^2 = \lambda^2 \gamma^2 (d\theta)^2$.

Consider the sets $\Gamma(k) = \{x \in M \cup \partial M : -K(x) = k\}$. Suppose M is *simply-connected* or *doubly-connected* such that $\Gamma(k)$ is either empty or a simple closed curve for each k which is not a critical value of K . Then the length $H_1(\Gamma(k))$ of $\Gamma(k)$ is $H_1(\Gamma(k)) = \int_0^{2\pi} \lambda \gamma_k(\theta) |d\theta|$, in which we may assume without loss of generality that $\gamma_{k_1}(\theta) \leq \gamma_{k_2}(\theta)$ for $0 \leq \theta < 2\pi$ and $k_1 \geq k_2 \geq 0$. To prove **Proposition 3**, it suffices to show that

$$\int_0^{2\pi} \frac{\partial \lambda}{\partial n} \gamma_{k_0}(\theta) |d\theta| < 0, \quad (89)$$

if n is the unit normal of $\gamma_{k_0}(\theta)$ pointing into $\bigcup_{k > k_0} \gamma_k(\theta)$ and $k_0 > 0$ is not a critical value of K .

To do so, we assume that $\phi \neq 0$ on $\Gamma(k_0)$, i.e., no umbilical point is on $\Gamma(k_0)$. By **Lemma 2** and the fact that H is constant,

$$\frac{(\lambda^2 \phi)_{\bar{z}}}{\lambda^2 \phi} = \frac{H_z}{\phi} \equiv 0,$$

and hence, there holds

$$\frac{(\lambda^2)_{\bar{z}}}{\lambda^2} = -\frac{\phi_{\bar{z}}}{\phi}, \quad \text{and} \quad \frac{(\lambda^2)_z}{\lambda^2} = -\frac{\bar{\phi}_z}{\bar{\phi}},$$

Henceforth, dot denotes partial differentiation with respect to the arc length $\tilde{s}$ of $\gamma_{k_0}(\theta)$. The last two identities yield

$$-\dot{z}\frac{(\lambda^2)_z}{\lambda^2} + \dot{\bar{z}}\frac{(\lambda^2)_{\bar{z}}}{\lambda^2} = \dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} - \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi}, \quad (90)$$

in which

$$-\dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} + \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi} = \left(-\dot{z}\frac{(|\phi|^2)_z}{|\phi|^2} + \dot{\bar{z}}\frac{(|\phi|^2)_{\bar{z}}}{|\phi|^2}\right) - \left(-\dot{z}\frac{\phi_z}{\phi} + \dot{\bar{z}}\frac{(\bar{\phi})_{\bar{z}}}{\bar{\phi}}\right). \quad (91)$$

But

$$\begin{aligned} -\dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} + \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi} &= -\frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \bar{\phi} \right) - \frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \bar{\phi} \right) \\ &\quad - \frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \phi \right) + \frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \phi \right), \end{aligned}$$

and

$$\begin{aligned} -\dot{z}\frac{\phi_z}{\phi} + \dot{\bar{z}}\frac{(\bar{\phi})_{\bar{z}}}{\bar{\phi}} &= -\frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \phi \right) - \frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \phi \right) \\ &\quad - \frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \bar{\phi} \right) + \frac{1}{2} \left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \bar{\phi} \right). \end{aligned}$$

Hence

$$\begin{aligned} -\dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} + \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi} &= -\dot{z}\frac{\phi_z}{\phi} + \dot{\bar{z}}\frac{(\bar{\phi})_{\bar{z}}}{\bar{\phi}} - \left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}} \right) \log \frac{\bar{\phi}}{\phi} \right) \\ &= -\dot{z}\frac{\phi_z}{\phi} + \dot{\bar{z}}\frac{(\bar{\phi})_{\bar{z}}}{\bar{\phi}} - \frac{\partial}{\partial \tilde{s}} \log \frac{\bar{\phi}}{\phi}. \end{aligned}$$

Thus, (91) yields

$$-\dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} + \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi} = \frac{1}{2} \left(-\dot{z}\frac{(|\phi|^2)_z}{|\phi|^2} + \dot{\bar{z}}\frac{(|\phi|^2)_{\bar{z}}}{|\phi|^2} \right) - \frac{1}{2} \left(\frac{\partial}{\partial \tilde{s}} \log \frac{\bar{\phi}}{\phi} \right),$$

and then (90) yields

$$-\dot{z}\frac{(\lambda^2)_z}{\lambda^2} + \dot{\bar{z}}\frac{(\lambda^2)_{\bar{z}}}{\lambda^2} = -\frac{1}{2} \left(-\dot{z}\frac{(|\phi|^2)_z}{|\phi|^2} + \dot{\bar{z}}\frac{(|\phi|^2)_{\bar{z}}}{|\phi|^2} \right) + \frac{1}{2} \left(\frac{\partial}{\partial \tilde{s}} \log \frac{\bar{\phi}}{\phi} \right).$$

Since $-\dot{z}\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial n}$ and

$$\int_0^{2\pi} \left(\frac{\partial}{\partial \bar{s}} \log \frac{\bar{\phi}}{\phi} \right) \gamma_{k_0}(\theta) |d\theta| = -2 \int_0^{H_1(\gamma_{k_0})} \left(\frac{\partial}{\partial \bar{s}} \arg \phi \right) d\bar{s} = -4\pi,$$

this last identity yields (89) and completes the proof of **Proposition II.2**. $\diamond$

II.3 Proposition II.2 yields

$$(H_1(\Gamma(t)))^2 \geq \frac{1}{4}(H_1(\Gamma(t)) + H_1(\Gamma((-K)_{\max} - \varepsilon)))^2.$$

The *isoperimetric inequality* then yields

$$(H_1(\Gamma(t)))^2 \geq \pi(a_\varepsilon(t)). \quad (92)$$

Also, from **Proposition II.2** follows

$$\frac{H^2 + ((-K)_{\max} - \varepsilon)}{H^2 + t} \int_{\Gamma((-K)_{\max} - \varepsilon)} |DK| H_1(dx) \leq C_\varepsilon (H_1(\Gamma(t)))^2,$$

where

$$C_\varepsilon = \frac{H^2 + (-K)_{\max}}{H^2 + K_0} (H_1(\Gamma((-K)_{\max})))^{-1} \left(\sup_{x \in \Gamma((-K)_{\max} - \varepsilon)} |DK(x)| \right). \quad (93)$$

This, (87), (88) and (92) yield

$$\begin{aligned} 4(t + H^2) \int_{E_\varepsilon(t)} (-K) H_2(dx) &\geq \left(\frac{-1}{a'_\varepsilon(t)} - \frac{3}{2} C_\varepsilon \right) (H_1(\Gamma(t)))^2 \\ &\geq \left(\frac{-1}{a'_\varepsilon(t)} - \frac{3}{2} C_\varepsilon \right) \pi a(t). \end{aligned} \quad (94)$$

By the coarea formula,

$$-a'_\varepsilon(t) = \int_{\Gamma(t)} \frac{1}{|DK|} H_1(dx) \quad (95)$$

for almost every t . By (81) and (82), we can choose ε_1 , so small that $\varepsilon < \varepsilon_1 < \varepsilon_1^*$ and

$$\inf_{x \in M_{\varepsilon_1}} |DK(x)| = \inf_{x \in \Gamma((-K)_{\max} - \varepsilon_1)} |DK(x)| \quad (96)$$

where $M_{\varepsilon_1} = \{x : x \in M, K_0 < -K(x) < (-K)_{\max} - \varepsilon_1\}$. Hence

$$-a'_\varepsilon(t) \leq \left(\inf_{x \in \Gamma((-K)_{\max} - \varepsilon_1)} |DK(x)| \right)^{-1} H_1(\Gamma(K_0)), \quad (97)$$

if $K_0 < t < (-K)_{\max} - \varepsilon_1$. Given $\varepsilon_0 > 0$, by (81) and (83), we can choose ε so small that $C_\varepsilon \leq \frac{2}{3}\varepsilon_0 \left(\inf_{x \in \Gamma((-K)_{\max} - \varepsilon_1)} |DK| \right) (H_1(\Gamma(K_0)))^{-1}$, i.e.

$$\frac{A_1}{A_0} \frac{\varepsilon^q}{(\varepsilon_1)^q} \leq \frac{\sup_{x \in \Gamma((-K)_{\max} - \varepsilon)} |DK|}{\inf_{x \in \Gamma((-K)_{\max} - \varepsilon_1)} |DK|} \leq \frac{2}{3}\varepsilon_0 \frac{H^2 + K_0}{H^2 + (-K)_{\max}} \frac{H_1(\Gamma((-K)_{\max}))}{H_1(\Gamma(K_0))}, \quad (98)$$

so that, by (97), $C_\varepsilon \leq \frac{2}{3}\varepsilon_0 \left(\frac{1}{-a'_\varepsilon(t)} \right)$, if $K_0 < t < (-K)_{\max} - \varepsilon_1$. Inserting this into (94), we obtain

$$4(H^2 + t) \int_{E_\varepsilon(t)} (-K) H_2(dx) \geq (1 - \varepsilon_0) \pi \frac{a_\varepsilon(t)}{-a'_\varepsilon(t)},$$

if $K_0 < t < (-K)_{\max} - \varepsilon_1$.

II.4. As in [16] and **I.1**, we introduce the decreasing rearrangement K^* of K and obtain

$$\int_0^{a_\varepsilon(K_0)} K^*(s) \phi'(s) ds \leq \frac{4}{1 - \varepsilon_0} \int_0^{a_\varepsilon(K_0)} \phi(s) \frac{H^2 + K^*(x)}{\pi s} \left[\int_0^s K^*(s') ds' \right] ds,$$

for every smooth, nonnegative function ϕ with compact support in $(a_\varepsilon((-K)_{\max} - \varepsilon_1), H_2(M_\varepsilon))$. This yields

$$\frac{\pi s}{4} \frac{dK^*(s)}{ds} \leq \frac{1}{1 - \varepsilon_0} (H^2 + K^*(s)) \left(\int_0^s K^*(s') ds' \right),$$

that is,

$$-\frac{d}{ds} \left(\frac{\pi s^2}{4} \frac{d\overline{K}(s)}{ds} \right) \leq \frac{1}{2} \frac{1}{1 - \varepsilon_0} \frac{d}{ds} (s\overline{K}(s))^2 + \frac{1}{1 - \varepsilon_0} H^2 (s\overline{K}(s)),$$

if

$$a_\varepsilon((-K)_{\max} - \varepsilon_1) < s < H_2(M_\varepsilon), \quad (99)$$

where $\overline{K}$ is the maximal function of $-K$ defined in **Theorem I.4**. And hence

$$-\frac{d}{ds} \left(\frac{\pi s^2}{4} \frac{d\overline{K}(s)}{ds} \right) \leq \frac{1}{2} \frac{1}{1 - \varepsilon_0} \frac{d}{ds} (s\overline{K}(s) + H^2)^2,$$

that is,

$$\pi s^2 \frac{d\overline{K}(s)}{ds} + \frac{2}{1 - \varepsilon_0} (s(H^2 + \overline{K}(s)))^2 \quad (100)$$

is an increasing function of s in the interval (99).

II.5. We intend to claim that (100) is nonnegative at $s = a_\varepsilon((-K)_{\max} - \varepsilon_1)$. To do so, we turn our attention from the interval (99) of s to the interval

$$0 < s < a_\varepsilon((-K)_{\max} - \varepsilon_1).$$

We observe that (87), (88) and (92) yield

$$\begin{aligned} -\pi \frac{a_\varepsilon(t)}{a'_\varepsilon(t)} &\leq \int_{\Gamma(t)} |DK| H_1(dx) \leq 4(H^2 + t) \int_{E_\varepsilon(t)} (-K) H_2(dx) \\ &+ 3 \frac{H^2 + (-K)_{\max} - \varepsilon}{H^2 + t} \int_{\Gamma((-K)_{\max} - \varepsilon)} |DK| H_1(dx). \end{aligned}$$

Hence, by **Proposition II.2**,

$$1 \leq \frac{-a'_\varepsilon(t)}{\pi a_\varepsilon(t)} (H^2 + t) \left[4 \int_{E_\varepsilon(t)} (-K) H_2(dx) + C_{\varepsilon\varepsilon_1} \right],$$

if

$$(-K)_{\max} - \varepsilon_1 < t < (-K)_{\max}, \quad (101)$$

where we set

$$C_{\varepsilon\varepsilon_1} = \frac{3}{4} \frac{H^2 + (-K)_{\max}}{(H^2 + (-K)_{\max} - \varepsilon_1)^2} H_1(\Gamma((-K)_{\max})) \left(\sup_{x \in \Gamma((-K)_{\max} - \varepsilon)} |DK(x)| \right). \quad (102)$$

Hence for every nonnegative smooth function ϕ with compact support in $(0, a_\varepsilon((-K)_{\max} - \varepsilon_1))$

$$\begin{aligned} &\int_0^{a_\varepsilon((-K)_{\max} - \varepsilon_1)} K^*(s) \phi'(s) ds \\ &\leq \int_0^{a_\varepsilon((-K)_{\max} - \varepsilon_1)} \phi(s) \frac{H^2 + K^*(s)}{\pi s} [-4(\int_0^s K^*(s') ds' + C_{\varepsilon\varepsilon_1})] ds. \end{aligned}$$

This yields

$$-\pi s \frac{dK^*(s)}{ds} \leq (H^2 + K^*(s)) \left(-4(\int_0^s K^*(s') ds' + C_{\varepsilon\varepsilon_1}) \right), \quad (103)$$

if $0 < s < a_\varepsilon((-K)_{\max} - \varepsilon_1)$.

II.5.1. By (92) and **Proposition II.2**, we have

$$\frac{H_1(\Gamma((-K)_{\max} - \varepsilon_1))}{\inf_{x \in \Gamma(t)} |DK(x)|} \geq -a'(t) \geq \frac{H_1(\Gamma(-K)_{\max})}{\sup_{x \in \Gamma(t)} |DK(x)|},$$

if t satisfies (98). An integration of this gives

$$\frac{H_1(\Gamma(-K)_{\max})}{a(t)} \leq \left(\int_{(-K)_{\max} - \varepsilon}^t \left(\sup_{x \in \Gamma(t)} |DK(x)| \right)^{-1} dt \right)^{-1}, \quad (104)$$

and

$$a_\varepsilon(t) \leq \left(\int_{(-K)_{\max} - \varepsilon}^t \left(\inf_{x \in \Gamma(t)} |DK(x)| \right)^{-1} dt \right) H_1(\Gamma((-K)_{\max} - \varepsilon_1)). \quad (105)$$

Setting

$$C_{\varepsilon\varepsilon_1 t}^* = \frac{3}{4} \frac{(H^2 + (-K)_{\max})}{H^2 + (-K)_{\max} - \varepsilon_1)^2} \frac{\sup_{x \in \Gamma(-K)_{\max} - \varepsilon} |DK(x)|}{\int_{(-K)_{\max} - \varepsilon}^t \left(\sup_{x \in \Gamma(t)} |DK| \right)^{-1} dt} \quad (106)$$

and

$$\overline{\overline{K}}(s) = \overline{K}(s) + C_{\varepsilon\varepsilon_1 t}^*,$$

we have, by (102), (104) and (106), $\frac{C_{\varepsilon\varepsilon_1}}{a(t)} \leq C_{\varepsilon\varepsilon_1 t}^*$ if t satisfies (101). This and (103) yields

$$-\frac{1}{4} \frac{d}{ds} \left(\pi s^2 \frac{d\overline{\overline{K}}(s)}{ds} \right) \leq \frac{1}{2} \frac{d}{ds} (s^2 (H^2 + \overline{\overline{K}}(s))^2),$$

if $0 < s < a_\varepsilon((-K)_{\max} - \varepsilon_1)$.

II.5.2. This means that

$$\pi s^2 \frac{d\overline{\overline{K}}(s)}{ds} + 2s^2 (H^2 + \overline{\overline{K}}(s))^2$$

is an increasing function of s in $(0, a_\varepsilon((-K)_{\max} - \varepsilon_1))$.

Since $\overline{\overline{K}}(s) \rightarrow (-K)_{\max} + C_{\varepsilon\varepsilon_1 t}^*$ as $s \rightarrow 0$ and

$$s \frac{d\overline{\overline{K}}(s)}{ds} = s \frac{d\overline{K}(s)}{ds} = K^*(s) - \overline{K}(s) \rightarrow 0$$

as $s \rightarrow 0$, we have

$$\pi s^2 \frac{d\overline{\overline{K}}(s)}{ds} + 2s^2 (H^2 + \overline{\overline{K}}(s))^2 \geq 0,$$

i.e.

$$\pi s^2 \frac{d\overline{\overline{K}}(s)}{ds} + 2s^2 (H^2 + \overline{\overline{K}}(s))^2 \geq -2s^2 ((C_{\varepsilon\varepsilon_1 t}^*)^2 + 2C_{\varepsilon\varepsilon_1 t}^* (H^2 + \overline{\overline{K}}(s))),$$

if $0 < s < a_\varepsilon((-K)_{\max} - \varepsilon_1)$. To handle the right hand side, we derive from (105), (106) and (83) that

$$\frac{4}{3} \frac{A_0}{A_1} \frac{a_\varepsilon(t) C_{\varepsilon\varepsilon_1 t}^*}{\sup_{x \in \Gamma((-K)_{\max} - \varepsilon)} |DK(x)|} \leq \frac{H^2 + (-K)_{\max}}{(H^2 + (-K)_{\max} - \varepsilon_1)^2} H_1(\Gamma((-K)_{\max} - \varepsilon_1)),$$

if t satisfies (101). Hence, given ε_0 , we can choose ε so small that

$$s^2 \pi \frac{d\overline{\overline{K}}(s)}{ds} + 2s^2 (H^2 + \overline{\overline{K}}(s))^2 \geq -2\varepsilon_0$$

if $0 < s < a_\varepsilon((-K)_{\max} - \varepsilon_1)$. This yields that in (100)

$$s^2 \pi \frac{d\overline{K}(s)}{ds} + \frac{2s^2}{1 - \varepsilon_0} (H^2 + \overline{K}(s))^2 > -2\varepsilon_0$$

if $s = a_\varepsilon((-K)_{\max} - \varepsilon_1)$. Since (100) is an increasing function of s in the interval (99), we conclude that

$$s^2 \pi \frac{d\overline{K}(s)}{ds} + \frac{2s^2}{1 - \varepsilon_0} (H^2 + \overline{K}(s))^2 > -2\varepsilon_0$$

if $0 < s < a_\varepsilon(K_0)$. We observe that ε_0 can be chosen arbitrarily small by taking ε small enough. Hence, manipulating as in [16] and **Part 1**, we obtain

Theorem II.3 *Let M be an embedded doubly-connected surface in R^3 of constant mean curvature H whose Gauss curvature K satisfies the conditions indicated in the beginning of this part. Then*

$$\frac{1}{H^2 + \overline{K}(s)} \leq \frac{1}{H^2 + (-K)_{\max}} + \frac{2s}{\pi}$$

for every s such that

$$0 \leq s \leq H_2\{x \in M : -K(x) > K_0\}.$$

Furthermore, the following alternative holds for each number $p > 1$. Either

$$(H^2 + (-K)_{\max})(H_2\{x \in M : -K(x) > K_0\}) < \frac{2\pi}{(p-1)} \left(1 + \frac{2\pi(H^2 + (-K)_{\max})}{H^2 + K_0}\right)$$

or

$$H^2 + (-K)_{\max} - \varepsilon \leq \frac{2\pi}{(p-1)} \left[\frac{p}{2\pi} \left(1 + \frac{2\pi(H^2 + (-K)_{\max})}{H^2 + K_0}\right) \|H^2 - K\|_{\varepsilon, p, \infty} \right]^{\frac{p}{p-1}},$$

for each sufficiently small ε . Here

$$\|H^2 - K\|_{\varepsilon, p, \infty} = l.u.b.\{c : \int_E (H^2 - K) H_2(dx) < c(H_2(E))^{1-\frac{1}{p}} \\ \text{for } E \subset M_\varepsilon, E : \text{measurable}\}$$

and

$$M_\varepsilon = \{x : x \in M, (-K)_{\max} - \varepsilon > -K(x) > K_0\}.$$

We note that C_ε and $C_{\varepsilon\varepsilon_1}$ can become arbitrarily large for each fixed ε and ε_1 after a rescaling performed in **I.3.2**. However, we can adjust ε to keep C_ε and $C_{\varepsilon\varepsilon_1}$ small in the rescaling process.

Main Theorem 2 follows from an obvious modification of the reasoning in **I.3.1** and **I.3.2**.

III Part III. On Surfaces of Prescribed Mean Curvature.

In this part, we shall find out suitable conditions on mean curvature under which the results in Part I and Part II can be extended.

III.1. The proof of Lemma 1 makes it clear that the results in **Theorem I.4** and hence **Theorem I.6** remain true for surfaces of prescribed mean curvature H with $\Delta(\lambda^4|\phi|^2) \geq 0$, i.e.

$$(K - H^2)\Delta(K - H^2) - |D(K - H^2)|^2 \geq 4K(K - H^2)^2. \quad (107)$$

Indeed, (107) yields

$$(K - H^2)\Delta(K - H^2) - |D(K - H^2)|^2 \geq 4(K - H^2)^3,$$

and hence, instead of (45), we have

$$\begin{aligned} & t \oint_{K-H^2=-t} |D(K - H^2)| H_1(dx) - 2 \int_{K-H^2<-t} |D(K - H^2)|^2 H_2(dx) \\ & \geq 4 \int_{K-H^2<-t} (K - H^2)^3 H_2(dx). \end{aligned}$$

And then, setting $\Phi(t) = \int_{K-H^2<-t} |D(K - H^2)|^2 H_2(dx)$, we obtain, instead of (30a) of [11],

$$\int_{K-H^2<-t} |D(K - H^2)| H_1(dx) \leq 4t \int_{K-H^2<-t} (-K) H_2(dx).$$

The remaining part of the proof of **Theorem I.4** can then proceed in our present setting.

This suggests us to express $\Delta(\lambda^4|\phi|^2)$ in terms of λ and H . For this, we adopt notations used in [10] and recall some results derived there.

III.2. We let S^2 be the unit sphere $\{(y_1, y_2, y_3) : y_1^2 + y_2^2 + y_3^2 < 1, y_1, y_2, y_3 \in \mathbb{R}\}$ in $\mathbb{R}^3$ and consider it as the standard Riemann sphere. We may cover S^2 by the union of two open sets $U_1 = S^2 - \{\text{north pole}\} = S^2 - \{(0, 0, 1)\}$ and $U_2 = S^2 - \{\text{south pole}\} = S^2 - \{(0, 0, -1)\}$. The stereographic projection of U_1 and U_2 onto the plane where $w = y_1 + iy_2$ is given by

$$\Psi_1(y_1, y_2, y_3) = \frac{y_1 + iy_2}{1 - y_3}, \quad \text{for } (y_1, y_2, y_3) \in U_1.$$

and

$$\Psi_2(y_1, y_2, y_3) = \frac{y_1 - iy_2}{1 + y_3}, \quad \text{for } (y_1, y_2, y_3) \in U_2.$$

Consider the following sequence of mappings

$$M \xrightarrow{x} x(M) \xrightarrow{\text{Gauss map}} S^2 \xrightarrow{\Psi_i} w\text{-plane}.$$

We denote the composed map as

$$\Psi : M \rightarrow w\text{-plane}.$$

It is shown as in Theorem 1 of [10] that

$$H \frac{\partial \Psi}{\partial z} = \phi \frac{\partial \Psi}{\partial \bar{z}}. \quad (108)$$

Furthermore, it is shown in Theorem 3 of [10] that the Gauss map Ψ satisfies

$$H \left(\frac{\partial^2 \Psi}{\partial z \partial \bar{z}} - \frac{2\bar{\Psi}}{1 + |\Psi|^2} \frac{\partial \Psi}{\partial z} \frac{\partial \Psi}{\partial \bar{z}} \right) = \frac{\partial H}{\partial z} \frac{\partial \Psi}{\partial \bar{z}}. \quad (109)$$

Moreover, (2.11) and (2.12) on page 98 of [10] give us

$$\lambda^2 = \frac{4}{H^2} \frac{|\Psi_z|^2}{(1 + |\Psi|^2)^2}, \quad \text{if } H \neq 0, \quad (110)$$

and

$$\lambda^2 = \frac{4}{|\phi|^2} \frac{|\Psi_z|^2}{(1 + |\Psi|^2)^2}. \quad (111)$$

III.3. By (111), we have $\lambda^2 |\phi|^2 = 4 \frac{|\Psi_z|^2}{(1 + |\Psi|^2)^2}$, and then

$$\begin{aligned} \frac{(\lambda^2 |\phi|^2)_{\bar{z}}}{\lambda^2 |\phi|^2} &= (\log(\lambda^2 |\phi|^2))_{\bar{z}} = (\log |\Psi_z|^2)_{\bar{z}} - 2(\log(1 + |\Psi|^2))_{\bar{z}} \\ &= \frac{\Psi_{z\bar{z}}}{\Psi_z} + \frac{(\bar{\Psi})_{\bar{z}\bar{z}}}{(\bar{\Psi})_{\bar{z}}} - 2 \frac{\bar{\Psi} \Psi_z + \Psi (\bar{\Psi})_{\bar{z}}}{1 + |\Psi|^2}. \end{aligned} \quad (112)$$

Inserting (109) into (111) gives

$$\frac{(\lambda^2 |\phi|^2)_{\bar{z}}}{\lambda^2 |\phi|^2} = \frac{(\bar{\Psi})_{\bar{z}\bar{z}}}{(\bar{\Psi})_{\bar{z}}} + \frac{H_z}{H} - 2 \frac{\Psi (\bar{\Psi})_{\bar{z}}}{1 + |\Psi|^2}. \quad (113)$$

But, by (108) if $H \neq 0$,

$$(\bar{\Psi})_{\bar{z}\bar{z}} = \left(\frac{\bar{\phi}}{H} (\bar{\Psi}_z) \right)_{\bar{z}} = \left(\frac{\bar{\phi}}{H} \right)_{\bar{z}} \bar{\Psi}_{z\bar{z}} + \left(\frac{\bar{\phi}}{H} \right)_{\bar{z}} (\bar{\Psi})_z.$$

Hence, by (109) and (107), if $H \neq 0$,

$$\begin{aligned} (\bar{\Psi})_{\bar{z}\bar{z}} &= \left(\frac{\bar{\phi}}{H} \right) \frac{2\Psi \bar{\Psi}_z (\bar{\Psi})_{\bar{z}}}{1 + |\Psi|^2} + \left(\frac{\bar{\phi} H_z}{H^2} \right) (\bar{\Psi})_z + \left(\frac{\bar{\phi}}{H} \right)_{\bar{z}} (\bar{\Psi})_z \\ &= \frac{2\Psi ((\bar{\Psi})_{\bar{z}})^2}{1 + |\Psi|^2} + \frac{H_z}{H} (\bar{\Psi})_{\bar{z}} + \left(\frac{\bar{\phi}}{H} \right)_{\bar{z}} (\bar{\Psi})_z \end{aligned}$$

and

$$\begin{aligned}\frac{(\bar{\Psi})_{\bar{z}\bar{z}}}{(\bar{\Psi})_{\bar{z}}} &= \frac{2\Psi(\bar{\Psi})_{\bar{z}}}{1+|\Psi|^2} + \frac{H_{\bar{z}}}{H} + \left(\frac{\bar{\phi}}{H}\right)_{\bar{z}} \frac{(\bar{\Psi})_z}{(\bar{\Psi})_{\bar{z}}} \\ &= \frac{2\Psi(\bar{\Psi})_{\bar{z}}}{1+|\Psi|^2} + \frac{H_{\bar{z}}}{H} + \left(\frac{\bar{\phi}}{H}\right)_{\bar{z}} \frac{H}{\bar{\phi}},\end{aligned}$$

by (108). Inserting this into (113), we obtain, if $H \neq 0$,

$$\frac{(\lambda^2|\phi|^2)_{\bar{z}}}{\lambda^2|\phi|^2} = 2\frac{H_{\bar{z}}}{H} + \left(\frac{\bar{\phi}}{H}\right)_{\bar{z}} \frac{H}{\bar{\phi}}$$

and

$$\frac{(\lambda^2|\phi|^2)_z}{\lambda^2|\phi|^2} = \overline{\left(\frac{(\lambda^2|\phi|^2)_{\bar{z}}}{\lambda^2|\phi|^2}\right)} = 2\frac{H_z}{H} + \left(\frac{\phi}{H}\right)_z \frac{H}{\phi}.$$

Hence, if $H \neq 0$,

$$\begin{aligned}(\log(\lambda^4|\phi|^2))_{z\bar{z}} &= \left(\frac{(\lambda^2)_z}{\lambda^2}\right)_{\bar{z}} + \frac{1}{2}\left(\frac{(\lambda^2|\phi|^2)_{\bar{z}}}{\lambda^2|\phi|^2}\right)_z + \frac{1}{2}\left(\frac{(\lambda^2|\phi|^2)_z}{\lambda^2|\phi|^2}\right)_{\bar{z}} \\ &= \left(\frac{(\lambda^2)_z}{\lambda^2}\right)_{\bar{z}} + \frac{1}{2}((-\log H)_{z\bar{z}} + (\log \bar{\phi})_{z\bar{z}}) \\ &\quad + \frac{1}{2}((-\log H)_{z\bar{z}} + (\log \phi)_{z\bar{z}}) + (\log H)_{\bar{z}z} + (\log H)_{z\bar{z}} \\ &= \left(\frac{(\lambda^2)_z}{\lambda^2}\right)_{\bar{z}} - (\log H)_{z\bar{z}} + \frac{1}{2}(\log |\phi|^2)_{z\bar{z}} + 2(\log H)_{z\bar{z}} \\ &= (\log H)_{z\bar{z}} + \frac{1}{2}(\log \lambda^4|\phi|^2)_{z\bar{z}}\end{aligned}$$

i.e.

$$\frac{1}{2}(\log \lambda^4|\phi|^2)_{z\bar{z}} = (\log H)_{z\bar{z}},$$

or equivalently,

$$\Delta(\log \lambda^4|\phi|^2) = 2(\Delta \log H).$$

Thus, we arrive at

Theorem III.1 *Let M be an embedded surface in R^3 of prescribed mean curvature H . Then, there holds, for an isothermal coordinate $z = \xi_1 + i\xi_2$,*

$$\Delta(\log \lambda^4|H^2 - K|^2) = 2(\Delta \log H).$$

Thus, denoting

$$|H|_{\max} = \ell.u.b.\{H(x), x \in M\},$$

and

$$H_0 = \ell.u.b.\{H(x), x \in \partial M\},$$

we have

Main Theorem 3 *Let M be an embedded surface in R^3 of prescribed mean curvature H . Suppose an isothermal coordinate $z = \xi_1 + i\xi_2$ can be so chosen that*

$$(\Delta \log H) \geq 0, \quad H \neq 0. \quad (114)$$

If $(|H|_{\max})^2 + (-K)_{\max} > (H_0)^2 + K_0$, then the inequality (5) holds in the range of $0 \leq s \leq H_2\{x \in M \mid (H(x))^2 - K(x) > (H_0)^2 + K_0\}$. Moreover, if $(|H|_{\max})^2 + (-K)_{\max} > (H_0)^2 + K_0$, then either (13) holds for some number $p > 1$, or inequalities (14) and (15) hold with H replaced by $|H|_{\max}$ and $H_2\{x \in M \mid -K(x) > K_0\}$ replaced by $H_2\{x \in M \mid (H(x))^2 - K(x) > (H_0)^2 + K_0\}$.

Although (22) does not necessarily hold in the general setting, our derivation of (29) and the validity of (28) in the general setting tell us that, for Theorem 3 to hold, it suffices to have

$$\Delta e_3 \cdot V + 2(2H^2 - K)e_3 \cdot V \geq 0$$

under the condition that the normal vector field e_3 makes an angle larger than $\beta > \frac{\pi}{2}$ with a fixed vector $-V$. According to page 14 of [3] this is equivalent to

$$\frac{\partial H}{\partial \xi_1} V \cdot e_1 + \frac{\partial H}{\partial \xi_2} V \cdot e_2 \geq 0. \quad (115)$$

Thus, we may formulate

Corollary III.2 *Let M be an embedded surface in R^3 of prescribed mean curvature H and Gauss curvature K . If (114) holds, and, moreover, the isothermal coordinates of M can be so chosen that (115) holds at each point of M , then, if the normal vector field e_3 makes an angle larger than $\beta > \frac{\pi}{2}$ with a fixed vector $-V$, either*

$$(|H|_{\max})^2 + (-K)_{\max} < (H_0)^2 + K_0,$$

or

$$((|H|_{\max})^2 - (-K)_{\max})(H_2\{x \in M : (H(x))^2 - K(x) > (H_0)^2 + K_0\}) < \frac{44}{23}\pi.$$

In particular, if M is representable in the nonparametric form $z = w(x, y)$, we may take V to be the unit vector in the positive z direction. Then (114), (115) reduce respectively to

$$(\log H)_{xx} + (\log H)_{yy} \geq 0, \quad \frac{\partial H}{\partial x} + \frac{\partial H}{\partial y} \geq 0. \quad (116)$$

We thus have

Corollary III.3 *Let $z = w(x, y)$ represent a surface of prescribed mean curvature $H \neq 0$. If (116) holds and if $|\nabla w|$ is uniformly bounded in the domain of definition of w , then either*

$$(|H|_{\max})^2 + (-K)_{\max} < (H_0)^2 + K_0,$$

or

$$((|H|_{\max})^2 + (-K)_{\max})(H_2\{x \in M : (H(x))^2 - K(x) > (H_0)^2\}) + K_0 < \frac{44}{23}\pi.$$

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