On Nonconvex Subdifferential Calculus in Binormed Spaces

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Abstract

We give in this paper some useful calculus results related to the limiting subdifferential in binormed spaces (generalized limiting subdifferential) which is a generalization of the limiting subdifferential in Banach spaces [5, 6].

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1 Introduction

The limiting subdifferential was first studied by Mordukhovich in [10], followed by joint work with Kruger in [5], and by work of Ioffe [1, 2].

In the beginning Ioffe, Jourani, kruger, Loewen, Mordukhovich, Shao and Thibault used the limiting subdifferential to establish full principal calculus rules and applications in Banach spaces which possess some geometric assumptions (we can see [2, 3, 4, 9, 12, 13]). Later, Mordukhovich and Shao

in [11] extend some of those results to arbitrary Banach spaces including sum rules, chain rules, product and quotient rules, however most of those results must involve the strict differentiability notion.

In this paper, we generalize some interesting calculus rules of [11] for non necessarily Fréchet strictly differentiable mappings using the notion of limiting subdifferential in binormed spaces (generalized limiting subdifferential). Thus, our results are closely related to (extending in some sense) those of [11]. The rest of this paper is organized as follows. Section 2 contains some definitions and preliminary material, while in section 3, we establish the result related to sum rule for the generalized limiting subdifferential and in section 4 we prove some other calculus results: chain rule, products and quotients rules. Finally, we give an example of a non Fréchet strictly differentiable mapping for which we can apply our sum rule theorem, while the classical sum rule theorem of [11] cannot be used.

2 Basic definitions and properties

Throughout this paper we consider two binormed spaces $(X, \|\cdot\|_1, \|\cdot\|_2)$ and $(Y, \|\cdot\|_3, \|\cdot\|_4)$ such that $(X, \|\cdot\|_2)$ and $(Y, \|\cdot\|_4)$ are two separable Banach spaces, and for some $c > 0$, $c' > 0$ $\|\cdot\|_1 \leq c \|\cdot\|_2, \|\cdot\|_3 \leq c' \|\cdot\|_4$.

For $(x, y) \in X \times Y$ we set $\|(x, y)\|_5 = \|x\|_1 + \|y\|_3$ and $\|(x, y)\|_6 = \|x\|_2 + \|y\|_4$.

Hence, $X \times Y$ becomes a binormed space under the pair of norms $(\|\cdot\|_5, \|\cdot\|_6)$. Moreover, $\|\cdot\|_5 \leq \max(c, c') \|\cdot\|_6$.

By $Cl^1 \Omega$ we denote the closure of Ω in $(X, \|\cdot\|_1)$, while $x \xrightarrow{\varphi, \|\cdot\|_1} \bar{x}$ (respectively $x \xrightarrow{\Omega, \|\cdot\|_1} \bar{x}$) means that $x \xrightarrow{\|\cdot\|_1} \bar{x}$ with $\varphi(x) \rightarrow \varphi(\bar{x})$ (respectively, $x \rightarrow \bar{x}$ with respect to the norm $\|\cdot\|_1$ and $x, \bar{x} \in \Omega$).

Let $B_\delta^1(\bar{x})$ denotes the open ball centered at $\bar{x}$ with radius δ in $(X, \|\cdot\|_1)$, where $\bar{x} \in X$ and $\delta > 0$.

For any multifunction Φ acting from X into its topological dual $(X, \|\cdot\|_2)^*$, we define the $(\|\cdot\|_1, \|\cdot\|_2)$ -sequential Kuratowski-Painlevé upper limit

by: $\limsup_{x \rightarrow \bar{x}}^{(1,2)} \Phi(x) \equiv (\|\cdot\|_1, \|\cdot\|_2) - \limsup_{x \rightarrow \bar{x}} \Phi(x) = \{x^* \in (X, \|\cdot\|_2)^* / \exists x_k^* \xrightarrow{\omega^*}$

$x^*, \exists x_k \xrightarrow{\|\cdot\|_1} \bar{x} \text{ with } x_k^* \in \Phi(x_k) \quad \forall k = 1, 2, \dots\}$, where $\xrightarrow{\omega^*}$ denotes the convergence for the weak-star topology $*\sigma((X, \|\cdot\|_2)^*, X)$ of $(X, \|\cdot\|_2)^*$.

Recall that if $\varphi: X \rightarrow \bar{R}$ is an extended-real-valued function, then the domain $\text{dom} \varphi$ of φ and the epigraph $\text{epi} \varphi$ of φ are respectively defined by:

$$\text{dom} \varphi = \{x \in X / |\varphi(x)| < +\infty\}, \text{epi} \varphi = \{(x, \mu) \in X \times R / \mu \geq \varphi(x)\}.$$

Definition 2.1 Let U be an open subset of $(X, \|\cdot\|_1)$ and let $\varphi: U \rightarrow R$ be a real-valued function. Assume that $\bar{x} \in U$.

i) We say that φ is $(\|\cdot\|_1, \|\cdot\|_2)$ -strictly Fréchet differentiable at $\bar{x}$ if there exists a continuous linear operator L from $(X, \|\cdot\|_2)$ into R such that

$$\lim_{\substack{u, x \xrightarrow{\|\cdot\|_1} \bar{x}}} \frac{\varphi(u) - \varphi(x) - L(u - x)}{\|u - x\|_2} = 0. \quad (1)$$

In this case, we set $\nabla^{1,2}\varphi(x) \equiv (\|\cdot\|_1, \|\cdot\|_2) - \nabla\varphi(x) = L$.

ii) We say that φ is $(\|\cdot\|_1, \|\cdot\|_2)$ -locally Lipschitzian around $\bar{x}$ if for some positive scalar k , we have:

$$|\varphi(x) - \varphi(x')| \leq k \|x - x'\|_2 \quad (2)$$

for all x, x' close to $\bar{x}$ with respect to the norm $\|\cdot\|_1$.

Let us remark that if φ is $(\|\cdot\|_1, \|\cdot\|_2)$ -strictly Fréchet differentiable at $\bar{x}$, then it is necessarily $\|\cdot\|_2$ -strictly Fréchet differentiable at $\bar{x}$ and therefore, continuous with respect to the norm $\|\cdot\|_2$. But in general, even if φ is $(\|\cdot\|_1, \|\cdot\|_2)$ -strictly Fréchet differentiable at $\bar{x}$, it is not necessarily $\|\cdot\|_1$ -strictly Fréchet differentiable at $\bar{x}$. But the converse is true, that is, if φ is $\|\cdot\|_1$ -strictly Fréchet differentiable at $\bar{x}$, then it is $(\|\cdot\|_1, \|\cdot\|_2)$ -strictly Fréchet differentiable at $\bar{x}$.

Definition 2.2 i) Let Ω be a nonempty subset of X and let $\varepsilon \geq 0$. Assume that $x \in Cl^1\Omega$.

The set of Fréchet ε -normals vectors to Ω at x with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$ denoted $\hat{N}_\varepsilon^{1,2}(x, \Omega)$ is given by :

$$\hat{N}_\varepsilon^{1,2}(x, \Omega) = \{x^* \in (X, \|\cdot\|_2)^* / \limsup_{\substack{u \xrightarrow{\|\cdot\|_1} x \\ \Omega, \|\cdot\|_1}} \frac{\langle x^*, u - x \rangle}{\|u - x\|_2} \leq \varepsilon\}. \quad (3)$$

When $\varepsilon=0$ (3) is a cone which is called prenormal cone or Fréchet normal cone to Ω at x with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$ and is denoted by $\hat{N}^{1,2}(x, \Omega)$. If $x \notin Cl^1\Omega$, we set $\hat{N}_\varepsilon^{1,2}(x, \Omega) = \emptyset$ for all $\varepsilon \geq 0$.

ii) Let $\bar{x} \in Cl^1\Omega$, the nonempty cone :

$$N^{1,2}(\bar{x}, \Omega) = \limsup_{x \xrightarrow{\|\cdot\|_1} \bar{x}, \varepsilon \searrow 0} \hat{N}_\varepsilon^{1,2}(x, \Omega). \quad (4)$$

is called the normal cone to Ω at $\bar{x}$ with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$. We set $N^{1,2}(\bar{x}, \Omega) = \emptyset$ for $\bar{x} \notin Cl^1\Omega$.

Denote by $N^2(\bar{x}, \Omega)$ the normal cone to Ω at $\bar{x}$ with respect to the norm $\|\cdot\|_2$. In general, if $x \in Cl^2\Omega$, then $N^{1,2}(\bar{x}, \Omega) \not\subset N^2(\bar{x}, \Omega)$ and $N^2(\bar{x}, \Omega) \not\subset N^{1,2}(\bar{x}, \Omega)$. But, if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, then $N^{1,2}(\bar{x}, \Omega) = N^1(\bar{x}, \Omega) = N^2(\bar{x}, \Omega)$.

So, our normal cone $N^{1,2}(\bar{x}, \Omega)$ defined in binormed space $(X, \|\cdot\|_1, \|\cdot\|_2)$ generalizes in some sens the notion of classical normal cone defined in normed space.

Let $\varepsilon \geq 0$. We define the generalized Fréchet ε -subdifferential of φ at $x \in \text{dom}\varphi$ with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$ by:

$$\hat{\partial}_\varepsilon^{1,2}\varphi(x) = \{x^* \in (X, \|\cdot\|_2)^* / \liminf_{u \xrightarrow{\|\cdot\|_1} x} \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x \rangle}{\|u - x\|_2} \geq -\varepsilon\}. \quad (5)$$

If $\varepsilon=0$, then (5) is called the generalized presubdifferential or generalized Fréchet subdifferential of φ at x with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$ and we denoted it by $\hat{\partial}^{1,2}\varphi(x)$.

Definition 2.3 Let $\varphi : X \longrightarrow \bar{R}$ be an extended-real-valued function and $\bar{x} \in \text{dom}\varphi$. Assume that φ is l.s.c. around $\bar{x}$ with respect to the norm $\|\cdot\|_1$. The set

$$\partial^{1,2}\varphi(\bar{x}) = \limsup_{x \xrightarrow{\varphi, \|\cdot\|_1} \bar{x}, \varepsilon \searrow 0} \hat{\partial}_\varepsilon^{1,2}\varphi(x) \quad (6)$$

is called the generalized limiting subdifferential of φ at $\bar{x}$ with respect to the pair of norms $(\|\cdot\|_1, \|\cdot\|_2)$. We set $\partial^{1,2}\varphi(\bar{x}) = \emptyset$ if $\bar{x} \notin \text{dom}\varphi$.

Let us remark that in general, $\partial^{1,2}\varphi(\bar{x}) \not\subset \partial^2\varphi(\bar{x})$ and $\partial^2\varphi(\bar{x}) \not\subset \partial^{1,2}\varphi(\bar{x})$. But if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, then $\partial^1\varphi(\bar{x}) = \partial^2\varphi(\bar{x}) = \partial^{1,2}\varphi(\bar{x})$. So, the generalized limiting subdifferential is a generalized version of the limiting subdifferential.

3 Sum rule

In this section we extend the result of sum rule for the limiting subdifferential given by Mordukhovich and Shao in [11] to generalized limiting subdifferential.

Theorem 3.1 Let $\varphi : X \rightarrow R$ be a $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable mapping at $\bar{x} \in X$ and let $\psi : X \rightarrow R$ be a l.s.c. function around $\bar{x}$ with respect to the norm $\|\cdot\|_1$. Assume that φ is l.s.c. around $\bar{x}$ with respect to the norm $\|\cdot\|_1$. Then

$$\partial^{1,2}(\varphi + \psi)(\bar{x}) = \nabla^{1,2}\varphi(\bar{x}) + \partial^{1,2}\psi(\bar{x}).$$

Proof. First we verify that:

$$\partial^{1,2}(\varphi + \psi)(\bar{x}) \subset \nabla^{1,2}\varphi(\bar{x}) + \partial^{1,2}\psi(\bar{x}).$$

Since φ is $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable at $\bar{x}$, then there are sequences $\gamma_v \searrow 0$, $\delta_v \searrow 0$, such that

$$|\varphi(z) - \varphi(x) - \langle \nabla^{1,2}\varphi(\bar{x}), z - x \rangle| \leq \gamma_v \|x - z\|_2 \quad \forall x, z \in B_{\delta_v}^1(\bar{x}), \quad v = 1, 2, \dots$$

Consequently,

$$\varphi(z) - \varphi(x) - \langle \nabla^{1,2}\varphi(\bar{x}), z - x \rangle \geq -\gamma_v \|x - z\|_2 \quad \forall x, z \in B_{\delta_v}^1(\bar{x}), \quad v = 1, 2, \dots \quad (7)$$

Let $x^* \in \partial^{1,2}(\varphi + \psi)(\bar{x})$. Then, there are sequences $x_k \xrightarrow{\|\cdot\|_1} \bar{x}$, $x_k^* \xrightarrow{\omega^*} x^*$, $\varepsilon_k \searrow 0$ as $k \rightarrow \infty$ such that:

$$(\varphi + \psi)(x_k) \rightarrow (\varphi + \psi)(\bar{x}), \text{ and } \forall k = 1, 2, \dots \quad x_k^* \in \hat{\partial}_{\varepsilon_k}^{1,2}(\varphi + \psi)(x_k). \quad (8)$$

Since $x_k \xrightarrow{\|\cdot\|_1} \bar{x}$, then we can choose a subsequence x_{k_v} ($k_1 < k_2 < \dots < k_v < \dots$ positive integers) converging to $\bar{x}$ with respect to the norm $\|\cdot\|_1$ and satisfying $\|x_{k_v} - \bar{x}\|_1 \leq \frac{\delta_v}{2} \quad \forall v = 1, 2, \dots$

Fix $\varepsilon > 0$ and pick (η_v) such that $0 \leq \eta_v \leq \frac{\delta_v}{2} \quad (\forall v)$ and

$$\forall x \in B_{\eta_v}^1(x_{k_v}) \quad \frac{\psi(x) - \psi(x_{k_v}) + \varphi(x) - \varphi(x_{k_v}) - \langle x_{k_v}^*, x - x_{k_v} \rangle}{\|x - x_{k_v}\|_2} \geq -\varepsilon_{k_v} - \varepsilon. \quad (9)$$

Observe that if $x \in B_{\eta_v}^1(x_{k_v})$, then $x \in B_{\delta_v}^1(\bar{x})$. Therefore, by virtue of (7) and (9) we deduce that $\forall v$ positive integer

$$\psi(x) - \psi(x_{k_v}) - \langle x_{k_v}^* - \nabla^{1,2}\varphi(\bar{x}), x - x_{k_v} \rangle \geq -(\varepsilon_{k_v} + \gamma_v + \varepsilon) \|x - x_{k_v}\|_2 \quad \forall x \in B_{\eta_v}^1(x_{k_v});$$

i.e.

$$\forall v \text{ positive integer : } x_{k_v}^* - \nabla^{1,2}\varphi(\bar{x}) \in \hat{\partial}_{\varepsilon_v}^{1,2}\psi(x_{k_v}),$$

where $\varepsilon_v = \gamma_v + \varepsilon_{k_v}$, $(\forall v)$.

On the other hand, we have $(\varphi + \psi)(x_k) \rightarrow (\varphi + \psi)(\bar{x})$, $x_k \xrightarrow{\|\cdot\|_1} \bar{x}$, φ and ψ are l.s.c. with respect to the norm $\|\cdot\|_1$. Hence, $\psi(x_{k_v}) \rightarrow \psi(\bar{x})$. Consequently,

$$x^* - \nabla^{1,2}\varphi(\bar{x}) \in \partial^{1,2}\psi(\bar{x}).$$

Thus, $\partial^{1,2}(\varphi + \psi)(\bar{x}) \subset \nabla^{1,2}\varphi(\bar{x}) + \partial^{1,2}\psi(\bar{x})$.

Applying the last result to functions $-\varphi$ and $\varphi + \psi$, we deduce the opposite inclusion $\nabla^{1,2}\varphi(\bar{x}) + \partial^{1,2}\psi(\bar{x}) \subset \partial^{1,2}(\varphi + \psi)(\bar{x})$. Thus, we achieve the proof.

Let us remark that if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$, then we recapture the result of sum rule for the limiting subdifferential given by Mordukhovich and Shao in [11].

The following result is an immediate consequence of theorem 3.1.

Corollary 3.2 *Let $\varphi : X \rightarrow R$ be a $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable mapping at $\bar{x} \in X$ and let $\psi : X \rightarrow R$ be a l.s.c. function around $\bar{x}$ with respect to the norm $\|\cdot\|_1$ satisfying $\partial^{1,2}(\varphi + \psi)(\bar{x}) = \partial^2(\varphi + \psi)(\bar{x})$. Assume that φ is l.s.c. around $\bar{x}$ with respect to the norm $\|\cdot\|_1$. Then $\partial^{1,2}\psi(\bar{x}) = \partial^2\psi(\bar{x})$.*

4 Chain rule and other calculus rules

Analogously to section 3, we give a generalization of other calculus rules established by Mordukhovich and Shao in [11].

Let us start with the theorem of chain rule.

Theorem 4.1 *Let $\Phi : (X, \| \cdot \|_1) \longrightarrow (Y, \| \cdot \|_3)$ be a continuous single-valued mapping such that $\Phi : X \longrightarrow (Y, \| \cdot \|_4)$ is $(\| \cdot \|_1, \| \cdot \|_2)$ -locally Lipschitzian around $\bar{x} \in X$. Let $\varphi : X \times Y \longrightarrow R$ be a $(\| \cdot \|_5, \| \cdot \|_6)$ -Fréchet strictly differentiable mapping at $(\bar{x}, \Phi(\bar{x}))$. Assume that φ is l.s.c. with respect to the norm $\| \cdot \|_5$. Then*

$$\partial^{1,2}(\varphi \circ \Phi)(\bar{x}) = \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}) + \partial^{1,2}\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x}) \text{ with } \bar{y} = \Phi(\bar{x}),$$

where $\varphi \circ \Phi$ is the function acting from X into R defined by:

$$(\varphi \circ \Phi)(x) = \varphi(x, \Phi(x)).$$

Proof. Under our hypothesis, we can easily see that $\varphi \circ \Phi$ is l.s.c. around $\bar{x}$ with respect to the norm $\| \cdot \|_1$ and $\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle$ is continuous around $\bar{x}$ with respect to the norm $\| \cdot \|_1$. Therefore, we can talk about $\partial^{1,2}(\varphi \circ \Phi)(\bar{x})$ and $\partial^{1,2}\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x})$.

Denote by l the Lipschitz constant of Φ at $\bar{x}$. Then there exists $\delta > 0$ such that

$$\forall x, u \in B_\delta^1(\bar{x}) \quad \| \Phi(u) - \Phi(x) \|_4 \leq l \| u - x \|_2.$$

Since φ is $(\| \cdot \|_5, \| \cdot \|_6)$ -Fréchet strictly differentiable at $(\bar{x}, \bar{y})$ with $\bar{y} = \Phi(\bar{x})$, then there are sequences $\gamma_v \searrow 0$, $\rho_v \searrow 0$ such that: $\rho_v < \delta \forall v$ and

$$\begin{aligned} & \varphi(u, \Phi(u)) - \varphi(x, \Phi(x)) - \langle \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}), u - x \rangle - \langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi(u) - \Phi(x) \rangle \\ & \geq -\gamma_v(\| u - x \|_2 + \| \Phi(u) - \Phi(x) \|_4) \quad \forall x, u \in B_{\rho_v}^1(\bar{x}), v = 1, 2, \dots \end{aligned}$$

Let $x^* \in \partial^{1,2}(\varphi \circ \Phi)(\bar{x})$. Then, there are sequences $x_k \xrightarrow{\| \cdot \|_1} \bar{x}$, $x_k^* \xrightarrow{\omega^*} x^*$, and $\varepsilon_k \searrow 0$ such that: $(\varphi \circ \Phi)(x_k) \longrightarrow (\varphi \circ \Phi)(\bar{x})$ and $x_k^* \in \hat{\partial}_{\varepsilon_k}^{1,2}(\varphi \circ \Phi)(x_k) \forall k = 1, 2, \dots$

Due to assumption $x_k \xrightarrow{\| \cdot \|_1} \bar{x}$ as $k \longrightarrow \infty$, we can choose a subsequence x_{k_v} ($k_1 < k_2 < \dots < k_v < \dots$ positive integers) such that $\| x_{k_v} - \bar{x} \|_1 \leq \frac{\rho_v}{2}$ for all $v = 1, 2, \dots$

Fix $\varepsilon > 0$ and choose a sequence (η_v) such that $0 \leq \eta_v \leq \frac{\rho_v}{2}$ and

$$\varphi(x, \Phi(x)) - \varphi(x_{k_v}, \Phi(x_{k_v})) - \langle x_{k_v}^*, x - x_{k_v} \rangle \geq -(\varepsilon_{k_v} + \varepsilon) \| x - x_{k_v} \|_2 \quad (10)$$

for all $x \in B_{\eta_v}^1(x_{k_v})$, $v = 1, 2, \dots$

Noting that $x \in B_{\rho_v}^1(\bar{x})$ if $x \in B_{\eta_v}^1(x_{k_v})$, we deduce that for all v

$$\varphi(x_{k_v}, \Phi(x_{k_v})) - \varphi(x, \Phi(x)) - \langle \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}), x_{k_v} - x \rangle - \langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi(x_{k_v}) - \Phi(x) \rangle$$

$$\geq -\gamma_v(\|x_{k_v} - x\|_2 + l\|x_{k_v} - x\|_2) \quad \forall x \in B_{\eta_v}^1(x_{k_v}). \quad (11)$$

Therefore, by virtue of (10) and (11), we obtain: for all v

$$\begin{aligned} & \langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi(x) \rangle - \langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi(x_{k_v}) \rangle - \langle x_{k_v}^* - \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}), x - x_{k_v} \rangle \\ & \geq -[\varepsilon_{k_v} + \varepsilon + \gamma_v(l + 1)]\|x - x_{k_v}\|_2 \quad \forall x \in B_{\eta_v}^1(x_{k_v}). \end{aligned}$$

Consequently, for all v

$$x_{k_v}^* - \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}) \in \hat{\partial}_{\bar{\varepsilon}_v}^{1,2}\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle(x_{k_v}) \text{ with } \bar{\varepsilon}_v = [\varepsilon_{k_v} + \gamma_v(l + 1)].$$

Hence,

$$x^* - \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}) \in \partial^{1,2}\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x}).$$

This proves the inclusion $\partial^{1,2}(\varphi \circ \Phi)(\bar{x}) \subset \nabla_x^{1,2}\varphi(\bar{x}, \bar{y}) + \partial^{1,2}\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x})$. To establish the opposite inclusion we employ the similar arguments starting with a point $x^* \in \partial^{1,2}\langle \nabla_y^{3,4}\varphi(\bar{x}, \bar{y}), \Phi \rangle(\bar{x})$. Thus, we achieve the proof.

Remark 4.2 Let us remark that if $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ and $\|\cdot\|_3$ is equivalent to $\|\cdot\|_4$, then we recapture the result of chain rule for the limiting subdifferential given by Mordukhovich and Shao in [11].

At the end of this section we give some additional calculus formulas for the generalized limiting subdifferential.

The first one is the product rules involving $(\|\cdot\|_1, \|\cdot\|_2)$ -locally Lipschitzian functions.

Theorem 4.3 Let $\varphi_i : X \longrightarrow R$, $i = 1, 2$ be two $(\|\cdot\|_1, \|\cdot\|_2)$ -locally Lipschitzian functions around $\bar{x} \in X$. Assume that all φ_i are continuous with respect to the norm $\|\cdot\|_1$. Then we have

$$\partial^{1,2}(\varphi_1 \cdot \varphi_2)(\bar{x}) = \partial^{1,2}(\varphi_2(\bar{x})\varphi_1 + \varphi_1(\bar{x})\varphi_2)(\bar{x}). \quad (12)$$

if in addition, one of the functions (say φ_1) is $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable at $\bar{x}$, then we have:

$$\partial^{1,2}(\varphi_1 \cdot \varphi_2)(\bar{x}) = \varphi_2(\bar{x})\nabla^{1,2}\varphi_1(\bar{x}) + \partial^{1,2}(\varphi_1(\bar{x})\varphi_2)(\bar{x}). \quad (13)$$

Proof. Consider the smooth function $\varphi : X \times R^2 \longrightarrow R$ and the $(\|\cdot\|_1, \|\cdot\|_2)$ -locally Lipschitzian mapping (around $\bar{x}$) $\Phi : X \longrightarrow R^2$ defined by:

$$\Phi(x) = (\varphi_1(x), \varphi_2(x)), \quad \varphi(x, y_1, y_2) = y_1 \cdot y_2.$$

Applying the above chain rule theorem to the composition $(\varphi \circ \Phi)(x) \equiv (\varphi_1 \cdot \varphi_2)(x)$, we deduce the result.

If now φ_1 is $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable at $\bar{x}$, then by virtue of Theorem 3.1, we obtain (13). Thus, we achieve the proof.

Similarly to Theorem 4.2 we obtain the following quotient rules.

Theorem 4.4 Let $\varphi_i : X \longrightarrow R$, $i = 1, 2$ be two $(\|\cdot\|_1, \|\cdot\|_2)$ -locally Lipschitzian functions around $\bar{x} \in X$. Assume that all φ_i are continuous with respect to the norm $\|\cdot\|_1$. Assume that $\varphi_2(\bar{x}) \neq 0$. Then

$$\partial^{1,2}(\varphi_1/\varphi_2)(\bar{x}) = \frac{\partial^{1,2}(\varphi_2(\bar{x})\varphi_1 - \varphi_1(\bar{x})\varphi_2)}{[\varphi_2(\bar{x})]^2}.$$

if in addition, one of the functions (say φ_1) is $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable at $\bar{x}$, then:

$$\partial^{1,2}(\varphi_1/\varphi_2)(\bar{x}) = \frac{\nabla^{1,2}\varphi_1(\bar{x})\varphi_2(\bar{x}) + \partial^{1,2}(-\varphi_1(\bar{x})\varphi_2)(\bar{x})}{[\varphi_2(\bar{x})]^2}.$$

Proof. Consider the smooth function $\varphi : X \times R^2 \longrightarrow R$ and the $(\|\cdot\|_1, \|\cdot\|_2)$ -locally Lipschitzian mapping (around $\bar{x}$) $\Phi : X \longrightarrow R^2$ defined by:

$$\Phi(x) = (\varphi_1(x), \varphi_2(x)), \quad \varphi(x, y_1, y_2) = \frac{y_1}{y_2}.$$

Applying the chain rule theorem (theorem 4.1) to the composition $(\varphi \circ \Phi)(x) \equiv \left(\frac{\varphi_1}{\varphi_2}\right)(x)$ and applying the sum rule theorem in the case where one of the functions (say φ_1) is $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable at $\bar{x}$, we deduce the result.

Finally we give an example of a non Fréchet strictly differentiable mapping with respect to the norm $\|\cdot\|_1$ for which we can apply our sum rule theorem (theorem 3.1), while the classical sum rule theorem of [11] given by Mordukhovich and Shao cannot be used.

Example. Let Ω be a bounded domain in R^2 and let $X = W_0^{1,2}(\Omega)$ be the Sobolev space under its usual norm $\|\cdot\|_2 = \|\cdot\|_{W_0^{1,2}(\Omega)}$. Let also p and ε such that $0 < \varepsilon < 1$, $\varepsilon + 2 < p < \infty$. Set $\|\cdot\|_1 = \|\cdot\|_{L^p(\Omega)}$. Remark that $(X, \|\cdot\|_2)$ is a Banach separable space. Since $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, then $(X, \|\cdot\|_1, \|\cdot\|_2)$ is a binormed space such that $\|\cdot\|_2$ is finer than $\|\cdot\|_1$. Set $g(u) = |u|^{\varepsilon+2}$ and consider the functional G defined on X by:

$$G(x) = \int_{\Omega} g(x(s)) ds$$

Then G is $\|\cdot\|_2$ -twice Fréchet differentiable at every $x \in X$. Moreover,

$$G^1(x)h = \int_{\Omega} g'(x(s))h(s) ds,$$

and

$$G^2(x)(h_1, h_2) = \int_{\Omega} g''(x(s))h_1(s)h_2(s) ds.$$

It is proved in [7] that G is $(\|\cdot\|_1, \|\cdot\|_2)$ -Fréchet strictly differentiable at every $x \in X$ and is not $\|\cdot\|_1$ -Fréchet strictly differentiable at any point x of X . Therefore, if for any reason we want to apply the sum rule theorem for the limiting subdifferential involving the norm $\|\cdot\|_1$, then our sum rule result given in theorem 3.1 is more adapted for this example than the one given by Mordukhovich and Shao in [11].

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