On the Principle of Uniform Boundedness for LSC Convex Processes in Banach Spaces

S. Lahrech, J. Hlal, A. Ouahab

Dept. of Mathematics, Faculty of Science, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

A. Jaddar

National School of Managment, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

A. Mbarki

National School of Applied Sciences, Mohamed first University
Oujda, Morocco, (GAFO Laboratory)

Abstract

The main purpose of this paper is to generalize the Banach-Steinhaus theorem for linear bounded operators to LSC convex processes in Banach spaces.

Mathematics Subject Classification: 49J52, 49J50

Keywords: LSC convex processes, principle of uniform boundedness, Banach-Steinhaus theorem

1 Introduction

Convex processes were introduced by Rockafellar ([1],[2]). These are set-valued maps whose graphs are convex cones. As such, they provide a powerful unifying formulation for the study of linear bounded operators, convex cones, and linear programming.

The purpose of this paper is to generalize the Banach-Steinhaus theorem for

linear bounded operators to LSC convex processes.

Throughout this section, $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ are two Banach spaces. For clarity, we denote the closed unit balls in X and Y by B_X and B_Y , respectively. A multifunction (or set-valued map) $\Phi : X \rightarrow Y$ is a map from X to the set of subsets of Y . The domain of Φ is the set

$$D(\Phi) = \{x \in X : \Phi(x) \neq \emptyset\}.$$

We say Φ has nonempty images if its domain is X . For any subset C of X we write $\Phi(C)$ for the image $\bigcup_{x \in C} \Phi(x)$ and the range of Φ is the set $R(\Phi) = \Phi(X)$.

We say Φ is surjective if its range is Y . The graph of Φ is the set

$$G(\Phi) = \{(x, y) \in X \times Y : y \in \Phi(x)\},$$

and we define the inverse multifunction $\Phi^{-1} : Y \rightarrow X$ by the relationship

$$x \in \Phi^{-1}(y) \Leftrightarrow y \in \Phi(x) \text{ for } x \in X \text{ and } y \in Y$$

A multifunction is convex, or closed if its graph is likewise.

A process is a multifunction whose graph is a cone. For example, we can interpret linear bounded operators as closed convex processes in the obvious way. For a convex process $\Phi : X \rightarrow Y$, we define the lower and upper norms respectively by:

$$\|\Phi\|_l = \inf\{r > 0 : \Phi(B_X) \cap rB_Y \neq \emptyset\}; \|\Phi\|_u = \inf\{r > 0 : \Phi(B_X) \subset rB_Y\}.$$

Clearly, Φ is bounded (that is, bounded sets have bounded images) if and only if its upper norm is finite.

Both norms generalize the norm of a linear bounded operator $A : X \rightarrow Y$, defined by $\|A\| = \sup\{\|Ax\|_Y : \|x\|_X \leq 1\}$. Let $x_0 \in D(\Phi)$, the limit of $\Phi(x)$ as $x \rightarrow x_0$ is the set

$$\lim_{x \rightarrow x_0} \Phi(x) = \{y \in Y : \forall x_n \rightarrow x_0, \exists y_n \in \Phi(x_n), y_n \rightarrow y\}.$$

we say Φ is continuous at x_0 if $\lim_{x \rightarrow x_0} \Phi(x) = \Phi(x_0)$. If this property holds for all points $x_0 \in X$, we say Φ is continuous.

On the other hands, we say Φ is LSC at x_0 if $\Phi(x_0) \subset \lim_{x \rightarrow x_0} \Phi(x)$.

Let us remark that if Φ is closed at x_0 , then $\lim_{x \rightarrow x_0} \Phi(x) \subset \Phi(x_0)$. A convex process Φ such that $\forall x_0 \in X$ Φ is LSC at x_0 is called a LSC convex process.

2 Banach-Steinhaus theorem and uniform boundedness principle for LSC convex processes

In this section, we give the main result of our paper related to principle of uniform boundedness for LSC convex processes.

Let now (Φ_n) be a sequence of multifunctions acting from X into Y . We say that (Φ_n) converges to multifunction $\Phi : X \rightarrow Y$ if

$$\lim_{n \rightarrow \infty} \Phi_n(x) \equiv \{y \in Y, \exists y_n \in \Phi_n(x) \text{ such that } y_n \rightarrow y\} = \Phi(x)$$

for each $x \in X$.

We say that the sequence (Φ_n) is uniformly bounded if its upper norm is bounded.

Our next step is to generalize the Banach-Steinhaus theorem for linear bounded operators to LSC convex processes, but before proving our main results, we first present the principle of uniform boundedness for LSC convex processes.

Proposition 2.1 *Let (Φ_n) , $(\Phi_n : X \rightarrow Y)$ be a sequence of LSC convex processes. Assume that $\forall x \in X, \bigcup_{n \geq 1} \Phi_n(x)$ is bounded in Y . Then (Φ_n) is uniformly bounded.*

Proof. For any integer $s \geq 1$ put

$$X_s = \{x \in X : \forall n \geq 1 \Phi_n(x) \subset B_Y(0, s)\}.$$

we claim that $\forall s \geq 1, X_s$ is closed. Indeed, fix $s \geq 1$ and let (x_r) be a sequence of X_s converging to $x \in X$. Fix $n \geq 1$, since $x_r \in X_s$ ($\forall r$), then $\forall r \geq 1 \Phi_n(x_r) \subset B_Y(0, s)$. Let $y \in \Phi_n(x)$ and for each r pick any $y_r \in \Phi_n(x_r)$ such that $y_r \rightarrow y$. For each r we have $\Phi_n(x_r) \subset B_Y(0, s)$. Therefore, $\|y_r\| \leq s$. Tending $r \rightarrow +\infty$, we deduce that $\Phi_n(x) \subset B_Y(0, s)$. Since the last inclusion holds for every $n \geq 1$, we conclude that $x \in X_s$. Hence, X_s is closed.

On the other hand, $\bigcup_{n \geq 1} \Phi_n(x)$ is bounded. Consequently, $\bigcup_{s \geq 1} X_s = X$. Applying Baire theorem, we deduce that $\text{int } X_{s_0} \neq \emptyset$ for some $s_0 \geq 1$. Taking $x_0 \in X$, $r > 0$ such that $x_0 + rB_X \subset X_{s_0}$ and observing that for each n $G(\Phi_n)$ is a convex cone shows $\forall n \geq 1 \forall z \in B_X \Phi_n(z) \subset \frac{2s_0}{r} B_Y$. Hence, $\|\Phi_n\|_u \leq \frac{2s_0}{r} \forall n \geq 1$. Thus, we achieve the proof.

The next result is an immediate consequence of the uniform boundedness principle.

Corollary 2.2 *Let (Φ_n) , $(\Phi_n : X \rightarrow Y)$ be a sequence of LSC convex processes. Assume that $\forall x \in X, \bigcup_{n \geq 1} \Phi_n(x)$ is bounded in Y . Then the limit convex process $\Phi \equiv \lim \Phi_n$ is bounded and it satisfies: $\|\Phi\|_u \leq \liminf \|\Phi_n\|_u$.*

Proof. If $\Phi \equiv \emptyset$, then Φ is bounded assuming that the empty set is bounded. Assume now that $\Phi \neq \emptyset$, then $\Phi(B_X) \neq \emptyset$. Let $x_0 \in B_X, y \in \Phi(x_0)$. Then, there exists $y_n \in \Phi_n(x_0)$ such that $y_n \rightarrow y$. Fix $n \geq 1$ and let $r > 0$ such that $\Phi_n(B_X) \subset rB_Y$. Then, $\|y_n\|_Y \leq r$. This implies that $\|y_n\|_Y \leq \|\Phi_n\|_u$.

Consequently, $\|y\|_Y \leq \liminf_{n \geq 1} \|\Phi_n\|_u$. On the other hand, $\forall x \in X \bigcup_{n \geq 1} \Phi_n(x)$ is bounded in Y . Therefore, applying the uniform boundedness principle to LSC convex processes (Φ_n) , we deduce that $\Phi(B_X) \subset \liminf_{n \geq 1} \|\Phi_n\|_u B_Y$ with $\liminf_{n \geq 1} \|\Phi_n\|_u < +\infty$. Hence, $\|\Phi\|_u < +\infty$. Thus, we achieve the proof.

References

- [1] R.T. Rockafellar : Monotone processes of Convex and Concave type, Memoirs of the American Mathematical Society, 1967. No. 77.
- [2] R.T. Rockafellar : Convex Analysis. Princeton University Press, Princeton, N.J., 1970.
- [3] Jonathan M. Borwein, Adrian S. Lewis: Convex Analysis and Nonlinear Optimization. CMS Books in Mathematics. Gargnano, Italy, September 1999.
- [4] S.M. Robinson : Normed convex processes, Transactions of the American Mathematical Society, 174:127-140, 1972.
- [5] S.M. Robinson : Regularity and stability for convex multivalued functions, Mathematics of Operations Research, 1:130-143, 1976.
- [6] Haim Brezis : Analyse fonctionnelle, Théorie et applications, Masson, Paris, 1983.

Received: Aptil 7, 2007