Time-like Hypersurfaces of 4-Dimensional Lorentzian Space Forms with Zero Mean Curvature

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Abstract

Let M be a connected space-like surface in a Lorentzian space form $\widetilde{M}_1^4(c)$. We define a time-like hypersurface M^* which is the image of a subbundle of the normal bundle of a space-like surface M spanned by a time-like unit normal vector field ξ on M in a 4-dimensional Lorentzian space form $\widetilde{M}_1^4(c)$ under the normal exponential mapping of M in $\widetilde{M}_1^4(c)$. We find the equations for a surface M and a time-like unit normal vector field on M such that M^* is a time-like hypersurface in $\widetilde{M}_1^4(c)$ with zero mean curvature. We also build up some examples.

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1 Introduction

Let M_s^m be a submanifold of a pseudo-Riemannian manifold $\widetilde{M}_t^n$. If the pseudo-Riemannian metric tensor of $\widetilde{M}_t^n$ induces a pseudo-Riemannian metric tensor, a Riemannian metric tensor or a degenerate metric tensor on M_s^m , then M_s^m is called a pseudo-Riemannian (or time-like) submanifold, a Riemannian (or space-like) submanifold or a degenerate submanifold, respectively.

A non-degenerate submanifold M_s^m of $\widetilde{M}_t^n$ is said to be minimal if the mean curvature of M_s^m in $\widetilde{M}_t^n$ vanishes, [3]. A space-like hypersurface of a pseudo-Riemannian manifold with zero mean curvature is said to be maximal, which

has been extensively investigated in pseudo-Riemannian geometry. Maximal space-like hypersurfaces share a certain type of geometric properties with minimal hypersurfaces of Riemannian spaces, for instance, maximal surfaces represent a maximum for the area integral [2]. However, time-like surfaces with zero mean curvature neither maximize nor minimize the area integral. Minimal surfaces in Minkowski spaces are also called the extremal surfaces which have well-known important applications in general relativity and conformal geometry. They have been investigated by several authors (e.g., [1, 9, 6, 4, 8] and references therein).

In [5], Kimura gave a construction of 3-dimensional minimal hypersurfaces M^* of 4-dimensional space forms $\widetilde{M}^4(c)$ defined by $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$, where M is a minimal surface in $\widetilde{M}^4(c)$ and ξ is a local unit normal vector field on M . Then, it is natural to construct the similar hypersurfaces of 4-dimensional Lorentzian space form $\widetilde{M}_1^4(c)$. However, in a Lorentzian space vectors with different causal characters usually turn into a wider variety of cases to consider.

In this paper, motivated by Kimura's work [5], we determine time-like hypersurfaces $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ of 4-dimensional Lorentzian space forms $\widetilde{M}_1^4(c)$ with zero mean curvature, where M is a connected space-like surface in $\widetilde{M}_1^4(c)$ and ξ is a local time-like unit normal vector field on M . (Such a hypersurface is foliated by the geodesics of $\widetilde{M}_1^4(c)$ and each leaf is a space-like surface of $\widetilde{M}_1^4(c)$.) We obtain differential equations for a space-like surface and a unit time-like normal vector ξ such that M^* has zero mean curvature. We also construct some examples.

2 Preliminaries

Let $\widetilde{M}_q^m$ be an m -dimensional pseudo-Riemannian manifold with pseudo-Riemannian metric tensor $\tilde{g}$ of index q . Denoting by $\langle \cdot, \cdot \rangle$ the associated nondegenerate inner product on $\widetilde{M}_q^m$, a tangent vector X to $\widetilde{M}_q^m$ is said to be *space-like* if $\langle X, X \rangle > 0$ (or $X = 0$), *time-like* if $\langle X, X \rangle < 0$ or *light-like (null)* if $\langle X, X \rangle = 0$ and $X \neq 0$.

Let M^m be a submanifold of a pseudo-Riemannian manifold $\widetilde{M}_q^{m+n}$. If the pseudo-Riemannian metric tensor $\tilde{g}$ of $\widetilde{M}_q^{m+n}$ induces a pseudo-Riemannian metric g on M^m , then M^m is called a pseudo-Riemannian submanifold of $\widetilde{M}_q^m$. If the index of g is zero (resp. one) then M is called a space-like (resp. Lorentzian or time-like) submanifold.

Let $\widetilde{M}_1^m(c)$ be an m -dimensional Lorentzian space form of constant curvature c , that is, $\widetilde{M}_1^m(c)$ is the de Sitter space-time $\mathbb{S}_1^m(c)$, Minkowski space-time $\mathbb{R}_1^m$ or the anti-de Sitter space-time $\mathbb{H}_1^m(c)$ according to $c > 0$, $c = 0$ or $c < 0$.

For simplicity, we suppose that the constant curvature c of $\widetilde{M}_1^m(c)$ is equal to 1, 0, -1 according to whether $c > 0$, $c = 0$, $c < 0$.

Let $\mathbb{R}_q^m$ be an m -dimensional pseudo-Euclidean space with metric tensor given by

$$\tilde{g} = - \sum_{i=1}^q (dx_i)^2 + \sum_{i=q+1}^m (dx_i)^2$$

where $(x_1, \dots, x_m)$ is a rectangular coordinate system of $\mathbb{R}_q^m$. So $(\mathbb{R}_q^m, \tilde{g})$ is a flat pseudo-Riemannian manifold of index q . For the de Sitter space-time $\mathbb{S}_1^m(c)$ and the anti-de Sitter space-time we put

$$\mathbb{S}_1^m(1) = \{x \in \mathbb{R}_1^{m+1} \mid \langle x, x \rangle = 1\} \quad \text{and} \quad \mathbb{H}_1^m(-1) = \{x \in \mathbb{R}_2^{m+1} \mid \langle x, x \rangle = -1\}.$$

Hence the hyperbolic space $\mathbb{H}^m(-1)$ is defined by

$$\mathbb{H}^m(-1) = \{x \in \mathbb{R}_1^{m+1} \mid \langle x, x \rangle = -1 \text{ and } x_1 > 0\},$$

where x_1 is the first coordinate in $\mathbb{R}_1^{m+1}$.

Let M be a space-like surface of a 4-dimensional Lorentzian space form $\widetilde{M}_1^4$. Let e_1, e_2 be a local tangent basis on M , and let ξ, η be a local orthonormal normal basis of M in $\widetilde{M}_1^4$ with signature $\varepsilon_1 = \langle \xi, \xi \rangle$ and $\varepsilon_2 = \langle \eta, \eta \rangle$. Then the *Gauss* formula and the *Weingarten* formula are, respectively, written as

$$\begin{aligned} \widetilde{\nabla}_{e_i} e_j &= \nabla_{e_i} e_j + \varepsilon_1 h_{ij}^\xi \xi + \varepsilon_2 h_{ij}^\eta \eta \\ \widetilde{\nabla}_{e_i} \xi &= -A_\xi(e_i) + \nabla_{e_i}^\perp \xi, \quad \widetilde{\nabla}_{e_i} \eta = -A_\eta(e_i) + \nabla_{e_i}^\perp \eta, \end{aligned} \quad (1)$$

where $\widetilde{\nabla}$ is the Riemannian connection of $\widetilde{M}$, ∇ and $\nabla^\perp$ are the induced Riemannian connection of M^m and the normal connection of M in $\widetilde{M}_1^4$, h_{ij}^ξ and h_{ij}^η are the components of the second fundamental tensors of M^m in $\widetilde{M}_1^4$, and A_ξ (resp. A_η) is the shape operator of M with respect to the normal vector ξ (resp. η). Then we get $h_{ij}^\xi = \langle A_\xi e_i, e_j \rangle$, $h_{ij}^\eta = \langle A_\eta e_i, e_j \rangle$, $h_{ij}^\xi = h_{ji}^\xi$ and $h_{ij}^\eta = h_{ji}^\eta$. Let ω and s be the connection forms for ∇ and $\nabla^\perp$, respectively, defined by $\omega(e_i) = \langle \nabla_{e_i} e_1, e_2 \rangle$ and $s(e_i) = \langle \nabla_{e_i}^\perp \xi, \eta \rangle$. We denote the components of ω and s by ω_i and s_i , respectively.

Then we have $\nabla_{e_i} e_1 = \omega_i e_2$, $\nabla_{e_i} e_2 = -\omega_i e_1$, $\nabla_{e_i}^\perp \xi = \varepsilon_2 s_i \eta$ and $\nabla_{e_i}^\perp \eta = -\varepsilon_1 s_i \xi$. Here it is seen that if either ξ or η is parallel in the normal space then the normal connection form for $\nabla^\perp$ is zero. We therefore suppose that ξ and η are non-parallel.

We define the covariant derivatives of the shape operators A_ξ and A_η by

$$\begin{aligned} h_{ijk}^\xi &= \langle \nabla_{e_k} (A_\xi e_i) - A_{\nabla_{e_k}^\perp \xi} e_i - A_\xi (\nabla_{e_k} e_i), e_j \rangle \\ h_{ijk}^\eta &= \langle \nabla_{e_k} (A_\eta e_i) - A_{\nabla_{e_k}^\perp \eta} e_i - A_\eta (\nabla_{e_k} e_i), e_j \rangle \end{aligned} \quad (2)$$

Since the ambient space is the space form the Codazzi equation is described as

$$h_{ijk}^{\xi} = h_{kji}^{\xi} \quad \text{and} \quad h_{ijk}^{\eta} = h_{kji}^{\eta}. \quad (3)$$

So h_{ijk}^{ξ} and h_{ijk}^{η} are symmetric with respect to i, j and k , respectively.

The covariant derivative of the 1-form s is defined by

$$s_{ij} = (\nabla_{e_j} s)(e_i) = e_j(s_i) - s(\nabla_{e_j} e_i).$$

As the ambient space is the space form the Ricci equation can be written as

$$\langle R^{\perp}(X, Y)\xi, \eta \rangle = \langle [A_{\xi}, A_{\eta}]X, Y \rangle, \quad ([10, \text{p. 125}]).$$

where $R^{\perp}$ denotes the normal curvature tensor of the normal connection $\nabla^{\perp}$ and $[A_{\xi}, A_{\eta}] = A_{\xi}A_{\eta} - A_{\eta}A_{\xi}$. Also we can see that

$$s_{ij} = \langle \nabla_{e_j}^{\perp} \nabla_{e_i}^{\perp} \xi - \nabla_{\nabla_{e_j} e_i}^{\perp} \xi, \eta \rangle. \quad (4)$$

So we can express the Ricci equation as

$$s_{12} - s_{21} = \langle [A_{\xi}, A_{\eta}]e_2, e_1 \rangle = h_{12}^{\xi}(h_{22}^{\eta} - h_{11}^{\eta}) + (h_{11}^{\xi} - h_{22}^{\xi})h_{12}^{\eta}. \quad (5)$$

Let ξ be a time-like unit normal vector field on M in $\widetilde{M}_1^4$. The normal exponential mapping of M in $\widetilde{M}_1^4$ in direction ξ can be given by

$$\exp(x, t\xi) = f_1(t)x + f_2(t)\xi,$$

where $x \in M$ and $t \in \mathbb{R}$. The functions $f_1(t)$ and $f_2(t)$ are given by $f_1(t) = 1$, $f_2(t) = t$ if $c = 0$; $f_1(t) = \cosh t$, $f_2(t) = \sinh t$ if $c = 1$, and $f_1(t) = \cos t$, $f_2(t) = \sin t$ if $c = -1$.

3 Time-like Hypersurfaces with Zero Mean Curvature

Here we determine time-like hypersurfaces of a 4-dimensional Lorentzian space form $\widetilde{M}_1^4(c)$ with zero mean curvature. Let M be a connected space-like surface in a 4-dimensional Lorentzian space form $\widetilde{M}_1^4(c)$. We define the time-like hypersurface M^* which is the image of a subbundle of the normal bundle of the surface M spanned by a (local) non-parallel time-like unit normal vector in a 4-dimensional Lorentzian space form $\widetilde{M}_1^4(c)$ under normal exponential mapping of M in $\widetilde{M}_1^4(c)$. Then the position vector of the hypersurface $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ is expressed as

$$F(u, v, t) = f_1(t)x(u, v) + f_2(t)\xi(u, v),$$

where $x = x(u, v)$ and $\xi = \xi(u, v)$ described the position vector of M and a non-parallel time-like unit normal vector of M in $\widetilde{M}_1^4$ at x , respectively, and (u, v) is a local coordinate of M . Let $\eta = \eta(u, v)$ be a local field of unit normal to M in $\widetilde{M}_1^4$ orthogonal to ξ . So, $\varepsilon_1 = \langle \xi, \xi \rangle = -1$ and $\varepsilon_2 = \langle \eta, \eta \rangle = 1$.

The tangent vectors at (u, v, t) on M^* are expressed as

$$F_u = f_1 x_u + f_2 \xi_u, \quad F_v = f_1 x_v + f_2 \xi_v, \quad F_t = f'_1 x + f'_2 \xi, \quad (6)$$

where $F_u, F_v, F_t, \dots$, etc. denote the derivatives of F, x and ξ with respect to u, v and t , respectively, which are considered as vectors in the ambient space, and f'_j denotes the differentiation of $f_j = f_j(t)$. Then it can be seen that the induced metric g^* on M^* is given by

$$g^* = \begin{pmatrix} \langle F_u, F_u \rangle & \langle F_u, F_v \rangle & 0 \\ \langle F_v, F_u \rangle & \langle F_v, F_v \rangle & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with respect to the local coordinate (u, v, t) of M^* . The metric g^* is non-degenerate if and only if $\langle F_u, F_u \rangle \langle F_v, F_v \rangle - \langle F_v, F_u \rangle^2 \neq 0$.

The definition of M^* means that M^* is foliated by geodesics of $\widetilde{M}_1^4(c)$. Since the integral curves of $\partial/\partial t$ are time-like geodesics of $\widetilde{M}_1^4(c)$, then each leaf of the foliation is a space-like surface of M^* . Thus the most of the following calculation to obtain the necessary minimality condition for M^* will be the same as in [5]. However, for the completeness of the paper we repeat the required calculation here.

Because of the integral curves of $\partial/\partial t$ are geodesics of $\widetilde{M}_1^4(c)$, the mean curvature vector H^* of M^* in $\widetilde{M}_1^4$ is given by

$$H^* = \frac{1}{3 \det g^*} (\|F_v\|^2 (F_{uu})^\perp + \|F_u\|^2 (F_{vv})^\perp - \langle F_u, F_v \rangle (F_{uv} + F_{vu})^\perp),$$

where $(\)^\perp$ denotes the projection of the normal space of M^* in $\widetilde{M}_1^4(c)$, and $\det g^* = \|F_v\|^2 \|F_u\|^2 - \langle F_u, F_v \rangle^2$. Suppose that $c \neq 0$. Then M^* has zero mean curvature in $\widetilde{M}_1^4(c)$ if and only if

$$x \wedge \xi \wedge F_u \wedge F_v \wedge (\|F_v\|^2 F_{uu} + \|F_u\|^2 F_{vv} - \langle F_u, F_v \rangle (F_{uv} + F_{vu})) = 0, \quad (7)$$

as $F \wedge F_t = x \wedge \xi$. If $c = 0$ then M^* has zero mean curvature in $\mathbb{R}_1^4$ if and only if

$$\xi \wedge F_u \wedge F_v \wedge (\|F_v\|^2 F_{uu} + \|F_u\|^2 F_{vv} - \langle F_u, F_v \rangle (F_{uv} + F_{vu})) = 0. \quad (8)$$

Let us take a local isothermal coordinate (u, v) of M such that $x_u = \varphi e_1, x_v = \varphi e_2$, where e_1, e_2 forms an orthonormal basis and φ is a positive

function on some open set in M . Suppose that $c \neq 0$. Then, by (1) and (6) we have

$$\begin{aligned} F_u &= \varphi(f_1 e_1 + f_2 D_{e_1} \xi) = \varphi(f_1 e_1 + f_2 (-A_\xi(e_1) + \nabla_{e_1}^\perp \xi)) \\ &= \varphi(f_1 e_1 + f_2 (-h_{11}^\xi e_1 - h_{12}^\xi e_2 + s_1 \eta)) \\ F_v &= \varphi(f_1 e_2 + f_2 D_{e_2} \xi) = \varphi(f_1 e_2 + f_2 (-A_\xi(e_2) + \nabla_{e_2}^\perp \xi)) \\ &= \varphi(f_1 e_2 + f_2 (-h_{12}^\xi e_1 - h_{22}^\xi e_2 + s_2 \eta)), \end{aligned} \quad (9)$$

and

$$\begin{aligned} \langle F_u, F_u \rangle &= \varphi^2((f_1 - f_2 h_{11}^\xi)^2 + f_2^2 (h_{12}^\xi)^2 + f_2^2 s_1^2) \\ \langle F_v, F_v \rangle &= \varphi^2((f_1 - f_2 h_{22}^\xi)^2 + f_2^2 (h_{12}^\xi)^2 + f_2^2 s_2^2) \\ \langle F_u, F_v \rangle &= \varphi^2(-2f_1 f_2 h_{12}^\xi + f_2^2 h_{12}^\xi (h_{11}^\xi + h_{22}^\xi) + f_2^2 s_1 s_2) \\ \langle F_t, F_t \rangle &= -1, \end{aligned} \quad (10)$$

where D denotes the covariant differentiation in $\mathbb{R}_1^5$ or $\mathbb{R}_2^5$. It is seen that the tangent vectors F_u and F_v are space-like and F_t is time-like. Therefore, M^* is a time-like hypersurface.

Using (1), (2), (3) and (4), for the second derivatives F_{uu}, F_{uv}, F_{vu} and F_{vv} we obtain

$$\begin{aligned} F_{uu} &= (e_1 \varphi) F_u + \omega_1 \varphi F_v + c f_1 \varphi^2 (h_{11}^\xi f_2 - f_1) x + \varphi^2 (-f_1 h_{11}^\xi + f_2 ((h_{11}^\xi)^2 + \\ &\quad (h_{21}^\xi)^2 + s_1^2)) \xi + \varphi^2 (f_1 h_{11}^\eta \eta - f_2 ((h_{11}^\xi h_{11}^\eta + h_{12}^\xi h_{12}^\eta - s_{11}) \eta + \\ &\quad (2s_1 h_{11}^\eta + h_{111}^\xi) e_1 + (2s_1 h_{21}^\eta + h_{112}^\xi) e_2)) \end{aligned} \quad (11)$$

$$\begin{aligned} F_{uv} &= (e_2 \varphi) F_u + \omega_2 \varphi F_v + c f_1 f_2 \varphi^2 h_{21}^\xi x + \varphi^2 (-f_1 h_{12}^\xi + f_2 (h_{11}^\xi h_{12}^\xi + \\ &\quad h_{21}^\xi h_{22}^\xi + s_1 s_2)) \xi + \varphi^2 (f_1 h_{12}^\eta \eta - f_2 ((h_{11}^\xi h_{12}^\eta + h_{21}^\xi h_{22}^\eta - s_{12}) \eta + \\ &\quad (s_1 h_{21}^\eta + s_2 h_{11}^\eta + h_{112}^\xi) e_1 + (s_1 h_{22}^\eta + s_2 h_{12}^\eta + h_{122}^\xi) e_2)) \end{aligned} \quad (12)$$

$$\begin{aligned} F_{vu} &= (e_1 \varphi) F_v - \omega_1 \varphi F_u + c f_1 f_2 \varphi^2 h_{21}^\xi x + \varphi^2 (-f_1 h_{12}^\xi + f_2 (h_{11}^\xi h_{21}^\xi + \\ &\quad h_{21}^\xi h_{22}^\xi + s_1 s_2)) \xi + \varphi^2 (f_1 h_{12}^\eta \eta - f_2 ((h_{11}^\xi h_{12}^\eta + h_{21}^\xi h_{22}^\eta - s_{21}) \eta + \\ &\quad (s_1 h_{21}^\eta + s_2 h_{11}^\eta + h_{112}^\xi) e_1 + (s_1 h_{22}^\eta + s_2 h_{21}^\eta + h_{122}^\xi) e_2)) \end{aligned} \quad (13)$$

$$\begin{aligned} F_{vv} &= (e_2 \varphi) F_v - \omega_2 \varphi F_u + c f_1 \varphi^2 (h_{22}^\xi f_2 - f_1) x + \varphi^2 (-f_1 h_{22}^\xi + f_2 ((h_{22}^\xi)^2 + \\ &\quad (h_{12}^\xi)^2 + s_2^2)) \xi + \varphi^2 (f_1 h_{22}^\eta \eta - f_2 ((h_{22}^\xi h_{22}^\eta + h_{12}^\xi h_{12}^\eta - s_{22}) \eta + \\ &\quad (2s_2 h_{12}^\eta + h_{122}^\xi) e_1 + (2s_2 h_{22}^\eta + h_{222}^\xi) e_2)) \end{aligned} \quad (14)$$

Clearly, $F_{vu} = F_{uv}$ holds and this is equivalent to the Ricci equation (5), $e_1\varphi = \varphi\omega_2$ and $e_2\varphi = -\varphi\omega_1$. Using the equations (10), (11)-(14), by calculation we obtain

$$\begin{aligned} & \|F_v\|^2 F_{uu} + \|F_u\|^2 F_{vv} - \langle F_u, F_v \rangle (F_{uv} + F_{vu}) \equiv \varphi^4 [f_1^3 - a_0\eta \\ & - f_1^2 f_2 (a_{10}\eta + a_{11}e_1 + a_{12}e_2) + f_1 f_2^2 (a_{20}\eta + a_{21}e_1 + a_{22}e_2) \\ & - f_2^3 (a_{30}\eta + a_{31}e_1 + a_{32}e_2)], \quad \text{mod}(x, \xi, F_u, F_v), \end{aligned} \quad (15)$$

where

$$a_0 = h_{11}^\eta + h_{22}^\eta,$$

$$a_{10} = -(s_{11} + s_{22}) + (h_{11}^\xi + 2h_{22}^\xi)h_{11}^\eta - 2h_{12}^\xi h_{12}^\eta + (2h_{11}^\xi + h_{22}^\xi)h_{22}^\eta,$$

$$a_{11} = h_{111}^\xi + h_{122}^\xi + 2s_1 h_{11}^\eta + 2s_2 h_{22}^\eta,$$

$$a_{12} = h_{112}^\xi + h_{222}^\xi + 2s_1 h_{12}^\eta + 2s_2 h_{22}^\eta,$$

$$\begin{aligned} a_{20} = & -2h_{22}^\xi s_{11} + 2h_{12}^\xi (s_{12} + s_{21}) - 2h_{11}^\xi s_{22} + h_{11}^\eta (h_{22}^\xi (2h_{11}^\xi + h_{22}^\xi) - (h_{12}^\xi)^2) \\ & + s_2^2 h_{11}^\eta - 2(h_{12}^\xi (h_{11}^\xi + h_{22}^\xi) + s_1 s_2) h_{12}^\eta (h_{11}^\xi (h_{11}^\xi + 2h_{22}^\xi) - (h_{12}^\xi)^2 + s_1^2) h_{22}^\eta, \end{aligned}$$

$$a_{21} = 2h_{22}^\xi h_{111}^\xi - 4h_{12}^\xi h_{112}^\xi + 2h_{11}^\xi h_{122}^\xi + 4(s_1 h_{22}^\xi - s_2 h_{12}^\xi) h_{11}^\eta + 4(s_2 h_{11}^\xi - s_1 h_{12}^\xi) h_{12}^\eta,$$

$$a_{22} = 2h_{22}^\xi h_{112}^\xi - 4h_{12}^\xi h_{122}^\xi + 2h_{11}^\xi h_{222}^\xi + 4(s_1 h_{22}^\xi - s_2 h_{12}^\xi) h_{12}^\eta + 4(s_2 h_{11}^\xi - s_1 h_{12}^\xi) h_{22}^\eta,$$

$$\begin{aligned} a_{30} = & -((h_{12}^\xi)^2 + (h_{22}^\xi)^2 + s_2^2) s_{11} + (h_{12}^\xi (h_{11}^\xi + h_{22}^\xi) + s_1 s_2) (s_{12} + s_{21}) \\ & - ((h_{11}^\xi)^2 + (h_{12}^\xi)^2 + s_1^2) s_{22} + (h_{22}^\xi (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) + s_2 (s_2 h_{11}^\xi \\ & - s_1 h_{12}^\xi)) h_{11}^\eta + (2h_{12}^\xi ((h_{12}^\xi)^2 - h_{11}^\xi h_{22}^\xi) + (s_1^2 + s_2^2) h_{12}^\xi - s_1 s_2 (h_{11}^\xi \\ & + h_{22}^\xi)) h_{12}^\eta + (h_{11}^\xi (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) + s_1 (s_1 h_{22}^\xi - s_2 h_{12}^\xi)) h_{22}^\eta, \end{aligned}$$

$$\begin{aligned} a_{31} = & ((h_{12}^\xi)^2 + (h_{22}^\xi)^2 + s_2^2) h_{111}^\xi - 2(h_{12}^\xi (h_{11}^\xi + h_{22}^\xi) + s_1 s_2) h_{112}^\xi \\ & + ((h_{11}^\xi)^2 + (h_{12}^\xi)^2 + s_1^2) h_{122}^\xi + (2s_1 ((h_{12}^\xi)^2 + (h_{22}^\xi)^2) \\ & - 2s_2 h_{12}^\xi (h_{11}^\xi + h_{22}^\xi)) h_{11}^\eta + (2s_2 ((h_{11}^\xi)^2 + (h_{12}^\xi)^2) - 2s_1 h_{12}^\xi (h_{11}^\xi + h_{22}^\xi)) h_{12}^\eta, \end{aligned}$$

$$\begin{aligned}
a_{32} = & ((h_{12}^\xi)^2 + (h_{22}^\xi)^2 + s_2^2)h_{112}^\xi - 2(h_{12}^\xi(h_{11}^\xi + h_{22}^\xi) + s_1s_2)h_{122}^\xi \\
& + ((h_{11}^\xi)^2 + (h_{12}^\xi)^2 + s_1^2)h_{222}^\xi + (2s_1((h_{12}^\xi)^2 + (h_{22}^\xi)^2) - 2s_2h_{12}^\xi(h_{11}^\xi \\
& + h_{22}^\xi))h_{12}^\eta + (2s_2((h_{11}^\xi)^2 + (h_{12}^\xi)^2) - 2s_1h_{12}^\xi(h_{11}^\xi + h_{22}^\xi))h_{22}^\eta.
\end{aligned}$$

By (9), the exterior product of F_u and F_v is written as

$$\begin{aligned}
F_u \wedge F_v = & \varphi^2 \{ f_1^2 e_1 \wedge e_2 - f_1 f_2 [(h_{11}^\xi + h_{22}^\xi) e_1 \wedge e_2 \\
& - s_2 e_1 \wedge \eta + s_1 e_2 \wedge \eta] + f_2^2 ((h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) e_1 \wedge e_2 \\
& + (s_1 h_{12}^\xi - s_2 h_{11}^\xi) e_1 \wedge \eta + (s_1 h_{22}^\xi - s_2 h_{12}^\xi) e_2 \wedge \eta) \}.
\end{aligned} \tag{16}$$

Substituting (15) and (16) into (7), we have the following equation,

$$(f_1^5 b_0 - f_1^4 f_2 b_1 + f_1^3 f_2^2 b_2 - f_1^2 f_2^3 b_3 + f_1 f_2^4 b_4 - f_2^5 b_5) x \wedge \xi \wedge \eta \wedge e_1 \wedge e_2 = 0,$$

where

$$\begin{aligned}
b_0 &= a_0, \\
b_1 &= (h_{11}^\xi + h_{22}^\xi) a_0 + a_{10}, \\
b_2 &= (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) a_0 + (h_{11}^\xi + h_{22}^\xi) a_{10} + s_1 a_{11} + s_2 a_{12} + a_{20}, \\
b_3 &= (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) a_{10} + (s_1 h_{22}^\xi - s_2 h_{12}^\xi) a_{11} + (s_2 h_{11}^\xi - s_1 h_{12}^\xi) a_{12} \\
&+ (h_{11}^\xi + h_{22}^\xi) a_{20} + s_1 a_{21} + s_2 a_{22} + a_{30}, \\
b_4 &= (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) a_{20} + (s_1 h_{22}^\xi - s_2 h_{12}^\xi) a_{21} + (s_2 h_{11}^\xi - s_1 h_{12}^\xi) a_{22} \\
&+ (h_{11}^\xi + h_{22}^\xi) a_{30} + s_1 a_{31} + s_2 a_{32}, \\
b_5 &= (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) a_{30} + (s_1 h_{22}^\xi - s_2 h_{12}^\xi) a_{31} + (s_2 h_{11}^\xi - s_1 h_{12}^\xi) a_{32}.
\end{aligned} \tag{17}$$

As $f_1^5, f_1^4 f_2, \dots, f_2^5$ are mutually independent functions, we have $b_j = 0, j = 0, \dots, 5$. Hence the statements in (17) implies

$$a_0 = 0, \tag{18}$$

$$a_{10} = 0, \tag{19}$$

$$s_1 a_{11} + s_2 a_{12} + a_{20} = 0, \tag{20}$$

$$\begin{aligned}
& (s_1 h_{22}^\xi - s_2 h_{12}^\xi) a_{11} + (s_2 h_{11}^\xi - s_1 h_{12}^\xi) a_{12} \\
& + (h_{11}^\xi + h_{22}^\xi) a_{20} + s_1 a_{21} + s_2 a_{22} + a_{30} = 0,
\end{aligned} \tag{21}$$

$$\begin{aligned}
& (h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2) a_{20} + (s_1 h_{22}^\xi - s_2 h_{12}^\xi) a_{21} + \\
& (s_2 h_{11}^\xi - s_1 h_{12}^\xi) a_{22} + (h_{11}^\xi + h_{22}^\xi) a_{30} + s_1 a_{31} + s_2 a_{32} = 0,
\end{aligned} \tag{22}$$

$$(h_{11}^\xi h_{22}^\xi - (h_{12}^\xi)^2)a_{30} + (s_1 h_{22}^\xi - s_2 h_{12}^\xi)a_{31} + (s_2 h_{11}^\xi - s_1 h_{12}^\xi)a_{32} = 0. \quad (23)$$

We can see that (18)- (23) hold too, when $c = 0$. Note that the above equations are independent of the choice of orthonormal basis e_1, e_2 at each point of M .

Now we have

Theorem 3.1 *Let M be a connected space-like surface in a Lorentzian space form $\widetilde{M}_1^4(c)$. Let M^* be the time-like hypersurface which is the image of the subbundle, spanned by a non-parallel time-like unit normal vector field ξ , of the normal bundle under normal exponential mapping of M in $\widetilde{M}_1^4(c)$. Then M^* has zero mean curvature in $\widetilde{M}_1^4(c)$ if and only if M and the orthonormal normal basis ξ, η satisfy (18)- (23).*

Remark 3.2 *In Theorem 3.1, the assumption about M is equivalent to the following two conditions: (a) M^* is foliated by geodesics of $\widetilde{M}_1^4(c)$, (b) a 2-dimensional distribution on M orthogonal to the geodesics in (a) is locally integrable. Then Theorem 3.1 claims that M^* has zero mean curvature if and only if a leaf of the foliation of (b) satisfies (18)- (23).*

Note that because of (18), if $H \neq 0$, then the normal vector ξ is proportional to the mean curvature vector H of M in $\widetilde{M}_1^4(c)$. Then (18) and (19) imply that $s_{11} + s_{22} = -\text{trace} A_\xi A_\eta$.

4 Construction of Examples

We here construct some examples of time-like hypersurfaces with zero mean curvature, defined as in the previous section.

Example 4.1 Suppose that M is a connected totally geodesic space-like surface in $\widetilde{M}_1^4(c)$. Then we have $h_{ij}^\xi = 0$, $h_{ij}^\eta = 0$, and $h_{ijk}^\xi = 0$ for $i, j, k = 1, 2$. Thus Theorem 2.1 implies that

$$s_{11} + s_{22} = 0, \quad (24)$$

$$s_1^2 s_{22} + s_2^2 s_{11} - s_1 s_2 (s_{12} + s_{21}) = 0, \quad (25)$$

and Ricci equation (5) produces

$$s_{12} = s_{21} \quad (26)$$

We now consider a 2-dimensional totally geodesic Riemannian space form $M = \widetilde{M}^2(c)$ in $\widetilde{M}_1^4(c)$. We determine the time-like unit normal vector ξ such that $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ has zero mean curvature in $\widetilde{M}_1^4(c)$.

Put $u = x_1$ and $u = x_2$. So the position vector of the space-like surface $M = \widetilde{M}^2(c)$ is expressed as

$$x(x_1, x_2) = \begin{cases} (0, x_1, x_2, 0) & \text{if } c = 0, \\ \frac{1}{r^2}(0, c(r^2 - 2), 2x_1, 2x_2, 0) & \text{if } c = \mp 1, \end{cases}$$

where $x_1, x_2 \in \mathbb{R}$, $r^2 = 1 + c(x_1^2 + x_2^2)$ and, for $c = -1$, $x_1^2 + x_2^2 < 1$.

From now on we will do all calculations for $c = \mp 1$. By calculation it is seen that the components of the induced first fundamental form on M are obtained as $\langle f_i, f_j \rangle = \frac{4}{r^4} \delta_{ij}$, $\{i, j = 1, 2\}$. This shows that the chosen coordinate system on M^m is isothermal and $\varphi = \frac{2}{r^2}$. Thus, $e_i = \frac{r^2}{2} \frac{\partial}{\partial x_i}$, $i = 1, 2$, is a local orthonormal tangent basis on M . In terms of this metric the Christoffel symbols are obtained

$$\Gamma_{ij}^k = -\frac{2c}{r^2}(x_i \delta_{kj} + x_j \delta_{ik} - x_k \delta_{ij}). \quad (27)$$

For the normal space of M in $\widetilde{M}_1^4(c)$ an orthonormal local basis can, generally, be chosen as

$$\xi = (\cosh \theta, 0, 0, 0, \sinh \theta), \quad \eta = (\sinh \theta, 0, 0, 0, \cosh \theta), \quad (28)$$

where $\theta = \theta(x_1, x_2)$. We will find θ which determines the time-like nonparallel unit normal vector ξ on M such that the time-like hypersurface M^* has zero mean curvature. To do this we need to calculate the components s_i of the normal connection s and their covariant derivatives s_{ij} . Then, from the definition of s_i we have

$$s_i = \langle \nabla_{X_i}^\perp \xi, \eta \rangle = \langle D_{X_i} \xi, \eta \rangle = \frac{r^2}{2} \langle \frac{\partial \xi}{\partial x_i}, \eta \rangle = \frac{r^2}{2} \frac{\partial \theta}{\partial x_i}. \quad (29)$$

that is, $s_i = \frac{r^2}{2} \theta_i$, $i = 1, 2$, and using (27), by straight calculation we obtain

$$s_{ij} = \frac{r^4}{4} (\theta_{ij} - \sum_{k=1}^m \Gamma_{ij}^k \theta_k), \quad (30)$$

where $\theta_{ij} = \partial^2 \theta / \partial x_i \partial x_j$. Here it is clear that $s_{12} = s_{21}$ if and only if $\theta_{12} = \theta_{21}$. Therefore, the equations (24) (25) and (26) are equivalent to

$$\begin{aligned} \theta_{11} + \theta_{22} &= 0, \quad \theta_{12} = \theta_{21}, \\ 2c(\theta_1^2 + \theta_2^2)(x_1 \theta_1 + x_2 \theta_2) + r^2 \sum_{i,j=1}^2 \theta_i \theta_j \theta_{ji} &= 0. \end{aligned} \quad (31)$$

For $c = 0$, choosing the normals ξ and η as

$$\xi = (\cosh \theta, 0, 0, \sinh \theta), \quad \eta = (\sinh \theta, 0, 0, \cosh \theta), \quad (32)$$

by straightforward calculation we can see that the equations in (31) are valid even if $c = 0$.

In [5], Kimura obtained the general solutions of the partial differential equations in (31). Using the some solutions of the differential equations in (31), we can have some examples of time-like hypersurfaces M^* of $\widetilde{M}_1^4(c)$ with zero mean curvature. However, we give the following particular examples. For $c = 0$, the linear solution $\theta(x_1, x_2) = A_1x_1 + A_2x_2 + A_3$ determines M^* in $\mathbb{R}_1^4$ by

$$F(x_1, x_2, t) = (t \cosh(A_1x_1 + A_2x_2 + A_3), x_1, x_2, t \sinh(A_1x_1 + A_2x_2 + A_3)).$$

It is easy to show that F is a 2-ruled hypersurface of the Minkowski space $\mathbb{R}_1^4$.

For $c = -1, 0, 1$, the function $\theta(x_1, x_2) = \arctan \frac{x_2}{x_1}$ is a solution of the equations in (31). When we change the coordinate system to $x_1 = u \cos v$, $x_2 = u \sin v$, then, the parameterization of the time-like hypersurface M^* is given as follows:

for $c = 0$,

$$F(u, v, t) = (t \cosh v, u \cos v, u \sin v, t \sinh v), \quad u > 0, \quad |v| < \pi/2,$$

which is a 2-ruled time-like hypersurface in Minkowski space $\mathbb{R}_1^4$;

for $c = 1$,

$$F(u, v, t) = \left(\sinh t \cosh v, \frac{u^2 - 1}{u^2 + 1} \cosh t, \frac{2u}{u^2 + 1} \cosh t \cos v, \right. \\ \left. \frac{2u}{u^2 + 1} \cosh t \sin v, \sinh t \sinh v \right),$$

and for $c = -1$;

$$F(u, v, t) = \left(\sin t \cosh v, \frac{1 + u^2}{1 - u^2} \cos t, \frac{2u}{1 - u^2} \cos t \cos v, \right. \\ \left. \frac{2u}{1 - u^2} \cos t \sin v, \sin t \sinh v \right),$$

where $|u| < 1$.

By considering these examples we have

Proposition 4.1 *Let M be a totally geodesic Riemannian space form $\widetilde{M}^2(c)$ in a 4-dimensional Lorentzian space form $\widetilde{M}_1^4(c)$, and let ξ be a time-like unit normal local vector field of M in $\widetilde{M}_1^4(c)$, given by (28) and (32), according to $c \neq 0$ and $c = 0$. Then the time-like hypersurface $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ has zero mean curvature in $\widetilde{M}_1^4(c)$ if and only if the function θ which determines ξ satisfies the equations in (31).*

Example 4.2 Let us consider the space-like surface $M = \mathbb{H}^1(-1) \times \mathbb{H}^1(-1) \subset \mathbb{H}_1^3(-1) \subset \mathbb{H}_1^4(-1)$. We determine the time-like unit vector field ξ on $\mathbb{H}^1(-1) \times \mathbb{H}^1(-1)$ such that the time-like hypersurface $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ has zero mean curvature in $\mathbb{H}_1^4(-1)$. The position vector of M in $\mathbb{R}_2^5$ is given by

$$x(u, v) = \frac{1}{\sqrt{2}}(\cosh u, \cosh v, \sinh u, \sinh v, 0)$$

for an isothermal coordinate (u, v) of M . Let $e_1 = \sqrt{2}\frac{\partial}{\partial u}$, $e_2 = \sqrt{2}\frac{\partial}{\partial v}$. These vectors are space-like and they form an orthonormal frame on M . Also, by a direct calculation we have $\nabla e_1 = 0$ and $\nabla e_2 = 0$.

An orthonormal normal basis on M in $\mathbb{H}_1^4(-1)$ is given by

$$\xi_1 = \frac{1}{\sqrt{2}}(\cosh u, -\cosh v, \sinh u, -\sinh v, 0), \quad \xi_2 = (0, 0, 0, 0, 1), \quad (33)$$

where ξ_1 is time-like and ξ_2 is space-like. Then the shape operators and normal connection of M in $\mathbb{H}_1^4(-1)$ are obtained as

$$A_{\xi_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{\xi_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nabla^\perp \xi_1 = 0 \text{ and } \nabla^\perp \xi_2 = 0.$$

Moreover, the covariant derivatives of the shape operators A_{ξ_1} and A_{ξ_2} are identically zero, that is, $h_{ijk}^{\xi_1} \equiv 0$ and $h_{ijk}^{\xi_2} \equiv 0$.

Any orthonormal normal basis ξ, η for M in $\mathbb{H}_1^4(-1)$ can be written as

$$\xi = \cosh \theta \xi_1 + \sinh \theta \xi_2, \quad \eta = \sinh \theta \xi_1 + \cosh \theta \xi_2,$$

where $\theta = \theta(u, v)$. Then we have $A_\xi = \cosh \theta A_{\xi_1}$ and $A_\eta = \sinh \theta A_{\xi_1}$, that is, $h_{11}^\xi = -\cosh \theta$, $h_{22}^\xi = \cosh \theta$, $h_{12}^\xi = 0$, $h_{11}^\eta = -\sinh \theta$, $h_{22}^\eta = \sinh \theta$, $h_{12}^\eta = 0$. Also, by straight forward calculation we see that $h_{ijk}^\xi = 0$, $i, j, k = 1, 2$, and the normal connection satisfies $s_1 = \sqrt{2}\theta_u$, $s_2 = \sqrt{2}\theta_v$, $s_{11} = 2\theta_{uu}$, $s_{12} = 2\theta_{uv}$, $s_{21} = 2\theta_{vu}$ and $s_{22} = 2\theta_{vv}$.

Suppose that $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ is a time-like hypersurface in $\mathbb{H}_1^4(-1)$ with zero mean curvature. Then the equations (18)-(23) hold. So, because of (19) we have

$$\theta_{uu} + \theta_{vv} = -\cosh \theta \sinh \theta \quad (34)$$

and, from (21) and (23) we obtain

$$(\theta_u^2 + \theta_v^2) \cosh \theta \sinh \theta = 0. \quad (35)$$

Then, by (34) and (35) we get $\theta(u, v) = 0$. In this case $\xi = \xi_1$ and M^* is totally geodesic $\mathbb{H}_1^3(-1)$ in $\mathbb{H}_1^4(-1)$.

Thus we obtain

Proposition 4.2 *Let M be the space-like surface $\mathbb{H}^1(-1) \times \mathbb{H}^1(-1) \subset \mathbb{H}_1^3(-1) \subset \mathbb{H}_1^4(-1)$, and ξ_1 and ξ_2 is be a natural orthonormal normal basis of M in $\mathbb{H}_1^4(-1)$ defined by (33). Define a time-like unit normal vector field ξ of M in $\mathbb{H}_1^4(-1)$ by $\xi = \cosh \theta \xi_1 + \sinh \theta \xi_2$, where θ is a function on M . If the time-like hypersurface $M^* = \{\exp_x(t\xi) : x \in M, t \in \mathbb{R}\}$ in $\mathbb{H}_1^4(-1)$ has zero mean curvature, then $\theta \equiv 0$. As a result, M^* is totally geodesic $\mathbb{H}_1^3(-1)$ in $\mathbb{H}_1^4(-1)$.*

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