On Dual Versions of Krull's Intersection Theorem

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Abstract

Let R be a commutative Noetherian ring (with identity) and let M be an R -module. In this paper we will state and prove some dual versions of Krull's intersection Theorem.

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1 Introduction

Throughout this paper R will denote a commutative ring with an identity.

Let R be a Noetherian ring and let I be an ideal of R . A well known version of Krull's intersection Theorem states that if M is a finitely generated R -module and if $L = \bigcap_{i=1}^{\infty} I^n M$, then $L = IL$. Another version of this Theorem says that if R is a domain and I is a proper ideal of R , $\bigcap_{i=1}^{\infty} I^n = 0$. In this paper we state and prove the dual versions of the above mentioned results (see 3.1, 3.2, 3.5, and 3.6).

2 Auxiliary results

Definition 2.1.

- (a) An R -module M is said to be *finitely cogenerated* (see [1]) if for every set $\{M_{\lambda}\}_{\lambda \in \Lambda}$ of submodules of M , $\bigcap_{\lambda \in \Lambda} M_{\lambda} = 0$ implies $\bigcap_{i=1}^n M_{\lambda_i} = 0$ for some positive integer n .

- (b) An non-zero R -module M is said to be *secondary* (see [4]) if for each $\alpha \in R$, the homothety $\alpha : M \rightarrow M$ is either surjective or nilpotent.
- (c) A *secondary representation* of an R -module M is an expression of M as a finite sum of secondary submodules of M (see [4]).
- (d) Let M be an R -module. The dual notion of $Z(M)$, the set of zero divisors of M , is denoted by $W(M)$ and defined by

$$W(M) = \{a \in R : \text{the homothety } M \xrightarrow{a} M \text{ is not surjective}\}.$$

M is said to be *codomain* (see [2]) if $W(M) = 0$.

Example 2.2. Let p be a prime number. The $\mathbb{Z}$ -module $\mathbb{Z}(p^\infty)$ (here $\mathbb{Z}$ is the ring of integers) is a codomain.

Remark 2.3. (A dual of Nakayama Lemma (see [2]).) Let M be a finitely cogenerated R -module and let I be an ideal of R with $I \subseteq \text{Jac}(R)$, where $\text{Jac}(R)$ denotes the Jacobson radical of R . Further let $(0 :_M I) = 0$. Then $M = 0$.

Lemma 2.4 Let R be a Noetherian ring and I be an ideal of R . If M is a secondary R -module, then $IM = M$ or there exists a positive integer n such that $I^n M = 0$.

Proof. Suppose that $IM \neq M$ and $a \in I$. Then we have $aM \neq M$. Since M is a secondary R -module, there exists a positive integer k such that $a^k M = 0$ so that $I \subseteq \sqrt{\text{Ann}_R(M)}$. Further since R is a Noetherian ring, there exists a positive integer n such that $I^n \subseteq \text{Ann}_R(M)$ as desired.

3 Main results

Throughout this section R will denote a Noetherian ring.

Theorem 3.1. (First dual version of Krull's intersection Theorem) Let M be an Artinian R -module and let I be an ideal of R . Set $L = \sum_{i=1}^{\infty} (0 :_M I^i)$. Then $L = (L :_M I)$.

Proof. If $(0 :_M I) = 0$, then the claim is clear. So we assume that $(0 :_M I) \neq 0$. This implies that $(L :_M I) \neq 0$. Now by [3], $(L :_M \underline{a})$ has a secondary

representation. Let

$$(L :_M I) = \sum_{i=1}^n S_i$$

be a minimal secondary representation of $(L :_M I)$, where S_i is a secondary submodule of M for each $i = 1, 2, \dots, n$. It is enough to show that $S_i \subseteq L$ for each $i = 1, 2, \dots, n$. Suppose that for some j with $1 \leq j \leq n$, we have $S_j \not\subseteq L$. Now we have

$$IS_j \subseteq \sum_{i=1}^n IS_i \subseteq L.$$

Hence $IS_j \subseteq L$. Since S_j is secondary, by Lemma 2.4, $IS_j = S_j$ or there exists a positive integer t such that $I^t S_j = 0$. If $IS_j = S_j$, then $S_j \subseteq L$, which is a contradiction. If $I^t S_j = 0$, then $S_j \subseteq (0 :_M I^t) \subseteq L$ which is again a contradiction. Hence $S_i \subseteq L$ for all $i = 1, 2, \dots, n$ so that $(L :_M I) \subseteq L$. This completes the proof.

Corollary 3.2. Let M be an Artinian R -module and let I be an ideal of R such that $I \subseteq \text{Jac}(R)$. Then $\sum_{i=1}^{\infty} (0 :_M I^i) = M$.

Proof. Set $L = \sum_{i=1}^{\infty} (0 :_M I^i)$. By Theorem 3.1, $L = (L :_M I)$. This implies that $(0 :_{M/L} I) = 0$. Now by [1, Prop. 10.10], M/L is a finitely cogenerated R -module. Hence by Remark 2.3, $M = L$ as desired.

Example 3.3. For $\mathbb{Z}_4$ as a $\mathbb{Z}_4$ -module, we have $\sum_{i=1}^{\infty} (0 :_{\mathbb{Z}_4} (2\mathbb{Z}_4)^i) = \mathbb{Z}_4$.

Corollary 3.4. Let M be an Artinian R -module and let R be a local ring with unique maximal ideal I . Then we have the following.

- (a) If N is a submodule of M , then $\sum_{i=1}^{\infty} (N \cap (0 :_M I^i)) = N$.
- (b) If N and K are submodules of M such that $N \cap (K :_M I) \subseteq K$, then $N \subseteq K$.

Proof. (a) This is an immediate consequence of 3.2 by using the fact that

$$(0 :_N I^i) = N \cap (0 :_M I^i).$$

(b) Since $N \cap (K :_M I) \subseteq K$, we have

$$(N :_M I) \cap (K :_M I^2) = (N \cap (K :_M I) :_M I) \subseteq (K :_M I).$$

This in turn implies that

$$N \cap (K :_M I^2) \subseteq N \cap (K :_M I) \subseteq K.$$

Hence for each i , we have $N \cap (K :_M I^i) \subseteq K$. Thus by part (a),

$$N = \sum_{i=1}^{\infty} (N \cap (0 :_M I^i)) \subseteq \sum_{i=1}^{\infty} (N \cap (K :_M I^i)) \subseteq K$$

as desired.

Theorem 3.5. (Second dual version of Krull's intersection Theorem.) Let M be a finitely generated Artinian R -module which is a codomain and let I be a proper ideal of R . Then $\sum_{i=1}^{\infty} (0 :_M I^i) = M$.

Proof. Set $L = \sum_{i=1}^{\infty} (0 :_M I^i)$. Then by Theorem 3.1, $L = (L :_M I)$. This implies that $(0 :_{M/L} I) = 0$. Now set $S = 1 + I$. It is clear that S is a multiplicatively closed subset of R and $S^{-1}I \subseteq S^{-1}R$. Since I is finitely generated, $S^{-1}(0 :_{M/L} I) = (0 :_{S^{-1}(M/L)} S^{-1}I) = 0$ by [5, Ex. 9.13]. But $S^{-1}(M/L)$ is finitely cogenerated $S^{-1}R$ -module so that $S^{-1}(M/L) = 0$ by Remark 2.3. Now since M is finitely generated, there exists an element $x \in R$ with $x = 1 + a$, where $a \in I$, such that $x(M/L) = 0$. Hence $xM \subseteq L$. Since M is codomain and I is a proper ideal of R , $xM = M$. Thus $M = L$ as desired.

Corollary 3.6. Let R be an Artinian codomain and I be a proper ideal of R . Then $\sum_{i=1}^{\infty} (0 :_R I^i) = R$.

Question 3.7. (A third dual version of Krull's intersection Theorem) Let M be a finitely cogenerated R -module and let I be an ideal of R . Set $L = \sum_{i=1}^{\infty} (0 :_M I^i)$. Is $L = (L :_M I)$?

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