Existence and stability of periodic solution
in impulsive neural networks
with both variable and distributed delays

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Abstract

In this paper, a generalized model of impulsive neural networks
with periodic coefficients and both time-varying and distributed delays
is investigated. By employing analytic methods, inequality technique
and M-matrix theory, some sufficient conditions ensuring the existence,
uniqueness and global exponential stability of the periodic oscillatory
solution for impulsive neural networks with both time-varying and dis-
tributed delays are obtained. Several examples are given to show the
effectiveness of the obtained results.

Keywords: neural networks; impulses; delays; global exponential stabil-
ity; periodic oscillatory solution

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1 Introduction

Recently, stability analysis and existence of periodic solutions have been widely researched for various neural networks with and without delays, and many important results on the global asymptotic stability and global exponential stability of both equilibrium point and periodic solution have been presented, see, for example, [1]-[18] and references cited therein. On the other hand, most neural networks can be classified as either continuous or discrete. However, there are many real-world systems and natural processes that behave in a piecewise continuous style interlaced with instantaneous and abrupt changes (impulses). Many interesting results on impulsive effect have been gained, e.g., Refs. [19]-[32]. As artificial electronic systems, neural networks such as Hopfield neural networks, cellular neural networks, bidirectional neural networks and recurrent neural networks often are subject to impulsive perturbations which can affect dynamical behaviors of the systems just as time delays. Therefore, it is necessary to consider both impulsive effect and delay effect on the stability of neural networks.

Motivated by the above discussions, in this paper, we consider a class of impulsive neural networks with both variable and distributed delays described by the following system of integro-differential equations:

\[
\begin{align*}
\frac{dx_i(t)}{dt} &= -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) \\
&\quad + \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} K_{ij}(t-s)f_j(x_j(s))ds + I_i(t), \quad t > 0, \quad t \neq t_k \\
\Delta x_i(t_k) &= x_i(t_k^+) - x_i(t_k^-) = -\delta_{ik}x_i(t_k), \\
&\quad t = t_k, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots 
\end{align*}
\]

where \( n \) is the number of neuron in the network, \( x_i(t) \) is the state of the \( i \)th neuron at time \( t \); \( f_i \) denote the activation function; \( I_i(t) \) denotes external input to the \( i \)th neuron; \( d_i(t) > 0 \) is the rate at which the \( i \)th neuron resets the state when isolated from the system; \( a_{ij}(t), b_{ij}(t), c_{ij}(t) \) denotes the connection strengths of the \( j \)th neuron on the \( i \)th neuron, respectively; \( \tau_{ij}(t) \) corresponds to the transmission delay and satisfies \( 0 \leq \tau_{ij}(t) \leq \tau \) (\( \tau \) is a constant); \( K_{ij} \) is the delay kernel; \( \delta_{ik} \) represents impulsive perturbation of the \( i \)th unit at time \( t_k \), \( 0 = t_0 < t_1 < t_2 < \cdots \), \( \lim_{k \to \infty} t_k = +\infty \).

To the best of our knowledge, few authors has considered dynamical behavior of impulsive neural networks with both variable and distributed delays.
This paper studies the existence, uniqueness and global exponential stability of the periodic oscillatory solution for impulsive neural networks with both variable and distributed delays. Several sufficient conditions ensuring the existence, uniqueness and global exponential stability of the periodic oscillatory solution for impulsive neural networks with distributed delays will be established for the system (1).

The remainder part of this paper is organized as follows. some notations and preliminaries are given in section 2. In section 3, several sufficient conditions will be established ensuring model (1) to the existence, uniqueness and global exponential stability of the periodic oscillatory solution. Remark and examples are given to illustrate our theory in section 4. Finally, in section 5 we give the conclusion.

2 Preliminaries

Throughout this paper we assume that:

\textbf{(H1)} $d_i(t), a_{ij}(t), b_{ij}(t), c_{ij}(t), I_i(t)$ are continuously periodic functions defined on $t \in [0, +\infty)$ with common period $\omega > 0$, \( i, j = 1, 2, \ldots, n \).

\textbf{(H2)} There exists a constant $F_i > 0$ such that

$$|f_i(x) - f_i(y)| \leq F_i|x - y|, \quad i = 1, 2, \ldots, n$$

for any $x, y \in R$.

\textbf{(H3)} The delay kernel $K_{ij} : [0, +\infty) \to [0, +\infty)$ are piecewise continuous function and satisfies:

(i) $\int_0^\infty K_{ij}(s)ds = 1, \quad i, j = 1, 2, \ldots, n.$

(ii) $\int_0^\infty sK_{ij}(s)ds < \infty, \quad i, j = 1, 2, \ldots, n.$

(iii) There exists a positive number $\mu$ such that

$$\int_0^\infty s e^{\mu s} K_{ij}(s)ds < \infty, \quad i, j = 1, 2, \ldots, n.$$

\textbf{(H4)} There exists a positive integer $q$ such that

$$t_{k+q} = t_k + \omega, \quad \delta_{i(k+q)} = \delta_{ik}, \quad k = 1, 2, \ldots, i = 1, 2, \ldots, n,$$

where $\delta_{ik}$ satisfies $0 < \delta_{ik} < 2$.

Let $PC = C((-\infty, 0], R^n)$ be the linear space of bounded and continuous functions which map $(-\infty, 0]$ into $R^n$. The initial conditions associated with
model (1) are of the form

\[ x_i(t) = \varphi_i(t), \quad -\infty < t \leq 0 \]  \hspace{1cm} (2)

in which \( \varphi_i(\cdot) \) are bounded continuous \( (i = 1, 2, \cdots, n) \). For \( \varphi \in PC \), \( \| \varphi \| \) is defined as

\[ \| \varphi \| = \sup_{-\infty < s \leq 0} \left( \sum_{i=1}^{n} |\varphi_i(s)|^r \right)^{\frac{1}{r}}, \]

then \( PC \) is a Banach space of continuous functions which map \( (-\infty, 0] \) into \( \mathbb{R}^n \) with the topology of uniform convergence.

To begin with, we introduce some notation and recall some basic definitions.

For an \( n \times n \) matrix \( A \), \( |A| \) denotes the absolute value matrix given by

\[ |A| = (|a_{ij}|)_{n \times n}. \]

Let \( h(t) \) is a continuous periodic \( \omega \)-function, we denote

\[ |h| = \min_{t \in [0,\omega]} |h(t)|, \quad |\overline{h}| = \max_{t \in [0,\omega]} |h(t)|, \]

in particular, when \( h(t) > 0 \), we denote

\[ h = \min_{t \in [0,\omega]} h(t). \]

**Definition 1** A function \( x : (-\infty, +\infty) \to \mathbb{R}^n \) is said to be the special solution of system (1) with initial condition (2) if the following two conditions are satisfied

(i) \( x \) is piecewise continuous with first kind discontinuity at the points \( t_k, k = 1, 2, \cdots \). Moreover, \( x \) is left continuous at each discontinuity point.

(ii) \( x \) satisfies model (1) for \( t \geq 0 \), and \( x(s) = \varphi(s) \) for \( s \in (-\infty, 0] \).

Henceforth, we let \( x(t, \varphi) \) denote the special solution of (1) with initial condition \( \varphi \in PC \).

**Definition 2** The periodic solution \( x(t, \varphi) \) of system (1) is said to be globally exponentially stable, if there exist positive constants \( \varepsilon \) and \( \kappa \) such that every solution \( x(t, \phi) \) of (1) satisfies

\[ \| x(t, \phi) - x(t, \varphi) \| \leq \kappa \| \phi - \varphi \| e^{-\varepsilon t} \quad \text{for all } t \geq 0. \]

**Definition 3** \[ 33 \] A real matrix \( D = (d_{ij})_{n \times n} \) is said to be a non-singular \( M \)-matrix if \( d_{ij} \leq 0, i, j = 1, 2, \cdots, n, i \neq j, \) and all successive principal minors of \( D \) are positive.
To the non-singular $M$-matrix, we have

**Lemma 1** [33] $D$ is a nonsingular $M$-matrix if and only if the diagonal elements of $D$ are all positive and there exists a positive vector $d$ such that $Dd > 0$ or $D^Td > 0$.

**Lemma 2** [34] Let $a, b \geq 0, p > 1$, then

$$a^{p-1}b \leq \frac{p-1}{p} a^p + \frac{1}{p} b^p.$$  

### 3 Periodic oscillatory solution

In this section, we will discuss the existence, uniqueness and global exponential stability of the periodic oscillatory solution of model (1).

**Theorem 1** Under hypothesis (H1)-(H4), there exists exactly one $\omega$-periodic solution of model (1) and all other solutions of model (1) converge exponentially to it as $t \to +\infty$, if there exist real constants $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \sigma_{ij}$ ($i, j = 1, 2, \cdots, n$) and $r > 1$ such that $M = D - P - Q$ is a nonsingular $M$-matrix, where

$$D = \text{diag}(d_1, d_2, \cdots, d_n),$$

$$P = \text{diag}(p_1, p_2, \cdots, p_n) \quad \text{with} \quad p_i = \frac{r-1}{r} \sum_{j=1}^{n} F_j^{r-\sigma_{ij}} \left( |\overline{a}_{ij}|^{r-\alpha_{ij}} + |\overline{b}_{ij}|^{r-\beta_{ij}} + |\overline{c}_{ij}|^{r-\gamma_{ij}} \right),$$

$$Q = (q_{ij})_{n \times n} \quad \text{with} \quad q_{ij} = \frac{1}{r} F_j^{r-\sigma_{ij}} \left( |\overline{a}_{ij}|^{\alpha_{ij}} + |\overline{b}_{ij}|^{\beta_{ij}} + |\overline{c}_{ij}|^{\gamma_{ij}} \right).$$

**Proof.** Let $x(t, \phi) = (x_1(t, \phi), x_2(t, \phi), \cdots, x_n(t, \phi))^T$ and $x(t, \varphi) = (x_1(t, \varphi), x_2(t, \varphi), \cdots, x_n(t, \varphi))^T$ be an arbitrary pair of solutions of system (1). Since $M$ is a nonsingular $M$-matrix, from Lemma 1, we know that there exists a vector $l = (l_1, l_2, \cdots, l_n)^T > 0$ such that $Ml > 0$, that is

\begin{equation}
 l_i(d_i - p_i) - \frac{1}{r} \sum_{j=1}^{n} l_j F_j^{r-\sigma_{ij}} \left( |\overline{a}_{ij}|^{\alpha_{ij}} + |\overline{b}_{ij}|^{\beta_{ij}} + |\overline{c}_{ij}|^{\gamma_{ij}} \right) > 0, \quad i = 1, 2, \cdots, n. \tag{3}
\end{equation}

Let us define function

\begin{equation}
 g_i(\theta) = l_i \left( \frac{\theta}{r} - d_i + p_i \right) + \frac{1}{r} \sum_{j=1}^{n} l_j F_j^{r-\sigma_{ij}} \left( |\overline{a}_{ij}|^{\alpha_{ij}} + |\overline{b}_{ij}|^{\beta_{ij}} e^{\tau \theta} + |\overline{c}_{ij}|^{\gamma_{ij}} \int_{0}^{+\infty} e^{\theta s} K_{ij}(s) ds \right) \tag{4}
\end{equation}
for $i = 1, 2, \cdots, n$. Obviously, $g_i(\theta)$ is continuous on $[0, +\infty)$, for $i = 1, 2, \cdots, n$. From (3) and assumption (H3), we know that $g_i(0) < 0$, $i = 1, 2, \cdots, n$, and $g_i(\theta) \to +\infty$ as $\theta \to +\infty$. Also, $\frac{dg_i(\theta)}{d\theta} > 0$, it follows that $g_i(\theta)$ is strictly monotone increasing functions on $[0, +\infty)$. Thus there exists a constant $\theta_i \in [0, +\infty)$ such that

$$g_i(\theta_i) = l_i\left(\frac{\theta_i}{r} - \frac{d}{r} + p_i\right)$$

$$+ \frac{1}{r} \sum_{j=1}^{n} l_j F_j^\sigma \left(\left|\tilde{a}_{ij}\right|^\alpha \left|\tilde{b}_{ij}\right|^\beta e^{\theta_i} + \left|\tilde{c}_{ij}\right|^\gamma \int_0^{+\infty} e^{\theta_i s} K_{ij}(s) ds\right) = 0 \quad (5)$$

for $i = 1, 2, \cdots, n$. Choose $\varepsilon$ such that $0 < \varepsilon < \min\{\theta_1, \theta_2, \cdots, \theta_n\}$, then

$$g_i(\varepsilon) = l_i\left(\frac{\varepsilon}{r} - \frac{d}{r} + p_i\right)$$

$$+ \frac{1}{r} \sum_{j=1}^{n} l_j F_j^\sigma \left(\left|\tilde{a}_{ij}\right|^\alpha \left|\tilde{b}_{ij}\right|^\beta e^{\varepsilon} + \left|\tilde{c}_{ij}\right|^\gamma \int_0^{+\infty} e^{\varepsilon s} K_{ij}(s) ds\right) < 0 \quad (6)$$

for $i = 1, 2, \cdots, n$.

**Case 1.** $t \geq 0$, $t \neq t_k$, $k = 1, 2, \cdots$. Let $y(t) = |x_i(t, \phi) - x_i(t, \varphi)|$, $i = 1, 2, \cdots, n$, we have

$$\frac{d^+ y_i(t)}{dt} = -d_i(t) y_i(t) + \sum_{j=1}^{n} a_{ij}(t) (f_j(x_j(t, \phi)) - f_j(x_j(t, \varphi)))$$

$$+ \sum_{j=1}^{n} b_{ij}(t) [f_j(x_j(t - \tau_{ij}(t), \phi)) - f_j(x_j(t - \tau_{ij}(t), \varphi))]$$

$$+ \sum_{j=1}^{n} c_{ij}(t) \int_{-\infty}^{t} K_{ij}(t-s) [f_j(x_j(s, \phi)) - f_j(x_j(s, \varphi))] ds,$$

$$\leq -d_i y_i(t) + \sum_{j=1}^{n} |a_{ij}(t)||f_j(x_j(t, \phi)) - f_j(x_j(t, \varphi))|$$

$$+ \sum_{j=1}^{n} |b_{ij}(t)||f_j(x_j(t - \tau_{ij}(t), \phi)) - f_j(x_j(t - \tau_{ij}(t), \varphi))|$$

$$+ \sum_{j=1}^{n} |c_{ij}(t)||\int_{-\infty}^{t} K_{ij}(t-s) [f_j(x_j(s, \phi)) - f_j(x_j(s, \varphi))] ds,$$
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\[
\begin{align*}
\leq & \ -\frac{d}{dt}y_i(t) + \sum_{j=1}^{n} |\bar{a}_{ij}| F_j |x_j(t, \phi) - x_j(t, \varphi)| \\
& + \sum_{j=1}^{n} |\bar{b}_{ij}| F_j |x_j(t - \tau_{ij}(t), \phi) - x_j(t - \tau_{ij}(t), \varphi)| \\
& + \sum_{j=1}^{n} |\bar{c}_{ij}| F_j \int_{-\infty}^{t} K_{ij}(t - s) |x_j(s, \phi) - x_j(s, \varphi)| ds, \\
& = \ -\frac{d}{dt}y_i(t) + \sum_{j=1}^{n} |\bar{a}_{ij}| F_j y_j(t) + \sum_{j=1}^{n} |\bar{b}_{ij}| F_j y_j(t - \tau_{ij}(t)) \\
& + \sum_{j=1}^{n} |\bar{c}_{ij}| F_j \int_{-\infty}^{t} K_{ij}(t - s)y_j(s)ds,
\end{align*}
\]

Furthermore, let \( Y_i(t) = e^{\varepsilon t} |x_i(t, \phi) - x_i(t, \varphi)| \), and calculate the upper right Dini derivatives \( D^+ Y_i(t) \) of \( Y_i(t) \) along the solution of (1), from (6), (7), assumption (H3) and Lemma 2, we get

\[
\begin{align*}
D^+ Y_i(t) & = \varepsilon Y_i(t) + re^{\varepsilon t} y_i(t)^{r-1} \frac{d^+ y_i(t)}{dt} \\
& \leq \varepsilon Y_i(t) + re^{\varepsilon t} y_i(t)^{r-1} \left[ -\frac{d}{dt}y_i(t) + \sum_{j=1}^{n} |\bar{a}_{ij}| F_j y_j(t) \\
& + \sum_{j=1}^{n} |\bar{b}_{ij}| F_j y_j(t - \tau_{ij}(t)) + \sum_{j=1}^{n} |\bar{c}_{ij}| F_j \int_{-\infty}^{t} K_{ij}(t - s)y_j(s)ds \right] \\
& = (\varepsilon - rd^3) Y_i(t) + e^{\varepsilon t} \left[ \sum_{j=1}^{n} r |\bar{a}_{ij}| F_j y_j(t)^{r-1} y_j(t) \\
& + \sum_{j=1}^{n} r |\bar{b}_{ij}| F_j y_j(t)^{r-1} y_j(t - \tau_{ij}(t)) + \sum_{j=1}^{n} \int_{-\infty}^{t} K_{ij}(t - s)r |\bar{a}_{ij}| F_j y_j(t)^{r-1}y_j(s)ds \right] \\
& = (\varepsilon - rd^3) Y_i(t) + e^{\varepsilon t} \left[ \sum_{j=1}^{n} r \left( |\bar{a}_{ij}| \frac{r-\alpha_{ij}}{r-1} F_j \frac{\delta_{ij}}{F_j} y_j(t) \right)^{r-1} \left( |\bar{a}_{ij}| \frac{\delta_{ij}}{F_j} Y_j(t) \right) \\
& + \sum_{j=1}^{n} \int_{-\infty}^{t} K_{ij}(t - s)r \left( |\bar{a}_{ij}| \frac{r-\alpha_{ij}}{r-1} F_j \frac{\delta_{ij}}{F_j} y_j(t - \tau_{ij}(t)) \right) \\
& + \sum_{j=1}^{n} \int_{-\infty}^{t} K_{ij}(t - s)r \left( |\bar{a}_{ij}| \frac{r-\alpha_{ij}}{r-1} F_j \frac{\delta_{ij}}{F_j} y_j(t) \right) \right] \\
& \leq (\varepsilon - rd^3) Y_i(t) + e^{\varepsilon t} \left[ \sum_{j=1}^{n} (r - 1) |\bar{a}_{ij}| \frac{r-\alpha_{ij}}{r-1} F_j \frac{\delta_{ij}}{F_j} y_j(t) + \sum_{j=1}^{n} |\bar{a}_{ij}|^{\alpha_{ij}} F_j^{\sigma_{ij}} y_j(t)^r \\
& + \sum_{j=1}^{n} (r - 1) |\bar{b}_{ij}| \frac{r-\beta_{ij}}{r-1} F_j \frac{\delta_{ij}}{F_j} y_j(t) + \sum_{j=1}^{n} |\bar{a}_{ij}|^{\beta_{ij}} F_j^{\sigma_{ij}} y_j(t - \tau_{ij}(t))^r \\
& + \sum_{j=1}^{n} \int_{-\infty}^{t} K_{ij}(t - s)(r - 1) |\bar{a}_{ij}| \frac{r-\alpha_{ij}}{r-1} F_j \frac{\delta_{ij}}{F_j} y_j(t) + \sum_{j=1}^{n} \int_{-\infty}^{t} K_{ij}(t - s)|\bar{a}_{ij}|^{\gamma_{ij}} F_j^{\tau_{ij}} y_j(s)^r ds \right]
\end{align*}
\]
\[ Y_i(t) \left[ (\varepsilon - \rho_i) + (r-1) \sum_{j=1}^{n} F^\sigma_j \left( |\bar{\alpha}_{ij}| - \phi_{ij} + |\bar{\beta}_{ij}| \right) + \int_{-\infty}^{t} e^{s(t-s)} K_{ij}(t-s) \varepsilon \, ds \right) \]
\[ \leq Y_i(t) \left[ (\varepsilon - r_i) + (r-1) \sum_{j=1}^{n} F^\sigma_j \left( |\bar{\alpha}_{ij}| - \phi_{ij} + |\bar{\beta}_{ij}| \right) + \int_{-\infty}^{t} e^{s(t-s)} K_{ij}(t-s) \varepsilon \, ds \right) \]
\[ = r \left[ Y_i(t) \left( \frac{\varepsilon}{r} - d_i + p_i \right) + \sum_{j=1}^{n} F^\sigma_j \left( |\bar{\alpha}_{ij}| - \phi_{ij} + |\bar{\beta}_{ij}| \right) + \int_{-\infty}^{t} e^{s(t-s)} K_{ij}(t-s) \varepsilon \, ds \right] \]
\[ \leq Y_i(t) \left[ (\varepsilon - r_i) + (r-1) \sum_{j=1}^{n} F^\sigma_j \left( |\bar{\alpha}_{ij}| - \phi_{ij} + |\bar{\beta}_{ij}| \right) + \int_{-\infty}^{t} e^{s(t-s)} K_{ij}(t-s) \varepsilon \, ds \right) \]
\[ + [b_{ij}]^{\beta_{ij}} e^{\varepsilon t} Y_j(t^* - \tau_{ij}(t^*)) + |c_{ij}|^{\gamma_{ij}} \int_{-\infty}^{t^*} e^{\varepsilon(t^*-s)} K_{ij}(t^* - s) Y_i(s) ds \]

\[ \leq r \left[ \left( \frac{\varepsilon}{r} - d_0 + p_0 \right) \lambda_0^2 + \frac{1}{r} \sum_{j=1}^{n} F_j^{\sigma_{ij}} \left( |\alpha_{ij}| l_j k_0 \right. \right. \]

\[ + [b_{ij}]^{\beta_{ij}} e^{\varepsilon t} l_j k_0 + |c_{ij}|^{\gamma_{ij}} \int_{-\infty}^{t^*} e^{\varepsilon(t^*-s)} K_{ij}(t^* - s) l_j k_0 ds \right] \]

\[ = r k_0 \left[ \frac{\varepsilon}{r} - d_0 + p_0 \right) \lambda_0^2 + \frac{1}{r} \sum_{j=1}^{n} F_j^{\sigma_{ij}} \left( |\alpha_{ij}| \right. \]

\[ + [b_{ij}]^{\beta_{ij}} e^{\varepsilon t} + |c_{ij}|^{\gamma_{ij}} \int_{-\infty}^{t^*} e^{\varepsilon(t^*-s)} K_{ij}(t^* - s) ds \right] < 0, \]

this is a contradiction, so (10) holds. Let \( \kappa = \left( \frac{(1+\delta) \sum_{i=1}^{n} l_i}{\min_{1 \leq i \leq n} \{ l_i \}} \right)^{\frac{1}{r}} \), from (10) we get

\[ \| x(t, \phi) - x(t, \varphi) \| = \left( \sum_{i=1}^{n} |x_i(t, \phi) - x_i(t, \varphi)|^r \right)^{\frac{1}{r}} \]

\[ \leq \left( \sum_{i=1}^{n} k_0 l_i e^{-\varepsilon t} \right)^{\frac{1}{r}} \]

\[ = \left( \frac{(1+\delta) \sum_{i=1}^{n} l_i}{\min_{1 \leq i \leq n} \{ l_i \}} \right)^{\frac{1}{r}} \| \phi - \varphi \| e^{-\frac{E}{r} t} \]

\[ = \kappa \| \phi - \varphi \| e^{-\frac{E}{r} t}, \]

that is

\[ \| x(t, \phi) - x(t, \varphi) \| \leq \kappa \| \phi - \varphi \| e^{-\frac{E}{r} t} \] \hspace{1cm} (11)

for \( t \geq 0, \ t \neq t_k, \ k = 1, 2, \ldots \).

**Case 2.** \( t = t_k \). From (H4), we have

\[ Y_i(t_k^+) = e^{\varepsilon t_k^+} |x_i(t_k^+, \phi) - x_i(t_k^+, \varphi)|^r \]

\[ = e^{\varepsilon t_k} |x_i(t_k^+, \phi) - \delta_{ik} x_i(t_k, \phi) - x_i(t_k^-, \varphi) + \delta_{ik} x_i(t_k, \varphi)|^r \]

\[ = e^{\varepsilon t_k} |x_i(t_k, \phi) - \delta_{ik} x_i(t_k, \phi) - x_i(t_k, \varphi) + \delta_{ik} x_i(t_k, \varphi)|^r \]
\[ e^{\varepsilon t_k}(1 - \delta_k)(x_i(t_k, \phi) - x_i(t_k, \varphi)) \leq e^{\varepsilon t_k}|x_i(t_k, \phi) - x_i(t_k, \varphi)| = Y_i(t_k), \]

i.e.,

\[ Y_i(t_k^+) \leq Y_i(t_k), \quad i = 1, 2, \ldots, n, k = 1, 2, \ldots. \]  

(12)

Since \( x_i(t, \phi) \) and \( x_i(t, \varphi) \) are left continuous at time \( t_k \), \( Y_i(t) \) is also left continuous at time \( t_k \), it follows that \( Y_i(t_k^+) \leq Y_i(t_k) \), for \( i = 1, 2, \ldots, n \). This implies that \( \|x(t_k^+, \phi) - x(t_k^+, \varphi)\| \leq \kappa\|\phi - \varphi\|e^{-\varepsilon t_k^+}. \)

So we have

\[ \|x(t, \phi) - x(t, \varphi)\| \leq \kappa\|\phi - \varphi\|e^{-\varepsilon t} \]  

(13)

for \( t \geq 0. \)

Below, we prove that the system (1) has exactly one \( \omega \)-periodic solution. for each solution \( x(t, \phi) \) of (1) and each \( t \geq 0 \), we define a function \( x_i(\phi) \) in this fashion:

\[ x_i(\phi)(s) = x(t + s, \phi) \quad \text{for } s \in (-\infty, 0] \]

From (13), we can choose a positive integer \( N \) such that \( \kappa e^{-\varepsilon N\omega} \leq \frac{1}{6}. \)

Now, define a Poincare mapping \( PC \rightarrow PC \) by \( P(\varphi) = x_{\omega}(\varphi) \), then \( P^N(\varphi) = x_{N\omega}(\varphi) \). Let \( t = N\omega \), then

\[ \|P^N(\phi) - P^N(\varphi)\| \leq \frac{1}{6}\|\phi - \varphi\|. \]

This implies that \( P^N \) is a contraction mapping, hence there exists one unique fixed point \( \varphi^* \in PC \) such that \( P^N(\varphi^*) = \varphi^*. \)

Since \( P^N(P(\varphi^*)) = P(P^N(\varphi^*)) = P(\varphi^*) \), \( P(\varphi^*) \in PC \) is also a fixed point of \( P^N \), it follows that \( P(\varphi^*) = \varphi^* \), that is \( x_{\omega}(\varphi^*) = \varphi^* \).

Let \( x(t, \varphi^*) \) be the solution of model (1) through \((0, \varphi^*)\), then \( x(t + \omega, \varphi^*) \) is also a solution of model (1). Obviously

\[ x_{t+\omega}(\varphi^*) = x_t(x_{\omega}(\varphi^*)) = x_t(\varphi^*) \]

for all \( t \geq 0. \) Hence

\[ x(t + \omega, \varphi^*) = x(t, \varphi^*). \]

This shows that \( x(t, \varphi^*) \) is exactly one \( \omega \)-periodic solution of model (1), and all solutions of model (1) converge exponentially to it as \( t \rightarrow +\infty. \) The proof is completed.
Corollary 1 Under hypothesis (H1)-(H4), there exists exactly one ω-periodic solution of model (1) and all other solutions of model (1) converge exponentially to it as $t \to +\infty$, if $M_1 = D - (|\bar{A}| + |\bar{B}| + |\bar{C}|)F$ is a nonsingular $M$-matrix, where

$$D = \text{diag}(d_1, d_2, \cdots, d_n), \quad |\bar{A}| = (|\bar{a}_{ij}|)_{n \times n}, \quad |\bar{B}| = (|\bar{b}_{ij}|)_{n \times n}, \quad |\bar{C}| = (|\bar{c}_{ij}|)_{n \times n},$$

$F = \text{diag}(F_1, F_2, \cdots, F_n)$.

Proof. Take $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = \sigma_{ij} = 1$, and let $r \to 1^+$, then $M$ in Theorem 1 turns to $M_1$. The proof is completed.

As $d_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$, $c_{ij}(t)$ and $I_i(t)$ are constants, model (1) becomes the following model:

$$\begin{cases}
\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij}(t))) \\
\quad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t - s) f_j(x_j(s)) ds + I_i, \quad t > 0, \quad t \neq t_k \\
\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = -\delta_{ik} x_i(t_k), \quad t = t_k, \quad i = 1, 2, \cdots, n, \quad k = 1, 2, \cdots
\end{cases} \tag{14}$$

In model (14), for any constant $T \geq 0$, we have $d_i = d_i(t + T) = d_i(t)$, $a_{ij} = a_{ij}(t + T) = a_{ij}(t)$, $b_{ij}(t + T) = b_{ij}(t)$, $c_{ij} = c_{ij}(t + T) = c_{ij}(t)$ and $I_i = I_i(t + T) = I_i(t)$ for $t \geq 0$. Hence, by the above results, when the sufficient conditions in Theorem 1 and Corollary 1 are satisfied, the unique periodic solution becomes a periodic solution with any positive constant as its period. So the unique periodic solution reduced to a constant solution, that is, an equilibrium point. Furthermore, all other solution globally exponentially converge to this equilibrium point as $t \to +\infty$. The equilibrium point of model (14) is global exponential stable. Therefore, by applying Theorem 1 and Corollary 1, we can easily obtain the following results, respectively.

Theorem 2 Under hypothesis (H1)-(H4), model (14) has one unique equilibrium point, which is global exponential stable if there exist real constants $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \sigma_{ij}$ $(i, j = 1, 2, \cdots, n)$ and $r > 1$ such that $M_2 = D_2 - P_2 - Q_2$ is a nonsingular $M$-matrix, where

$$D_2 = \text{diag}(d_1, d_2, \cdots, d_n),$$

$$P_2 = \text{diag}(p_1, p_2, \cdots, p_n) \quad \text{with} \quad p_i = \frac{r-1}{r} \sum_{j=1}^{n} F_j \frac{r_{-a_{ij}}}{r} \left| a_{ij} \right|_{\frac{r}{r-1}} + \left| b_{ij} \right|_{\frac{r}{r-1}} + \left| c_{ij} \right|_{\frac{r}{r-1}} + \left| b_{ij} \right|_{\frac{r}{r-1}},$$

$$Q_2 = (q_{ij})_{n \times n} \quad \text{with} \quad q_{ij} = \frac{1}{r} F_j \frac{r_{-\alpha_{ij}}}{r} \left( \left| a_{ij} \right|_{\frac{r}{r-1}} + \left| b_{ij} \right|_{\frac{r}{r-1}} + \left| c_{ij} \right|_{\frac{r}{r-1}} \right).$$
Corollary 2 Under hypothesis \((H1)-(H4)\), model (14) has one unique equilibrium point, which is global exponential stable if \(M_3 = D - (|A| + |B| + |C|)F\) is a nonsingular \(M\)-matrix, where \(D = \text{diag}(d_1, d_2, \ldots, d_n)\), \(|A| = (|a_{ij}|)_{n \times n}\), \(|B| = (|b_{ij}|)_{n \times n}\), \(|C| = (|c_{ij}|)_{n \times n}\), \(F = \text{diag}(F_1, F_2, \ldots, F_n)\).

In the case \(\delta_{ik} = 0\) for \(i = 1, 2, \ldots, n\), \(k = 1, 2, \ldots\), model (1) will reduce to neural networks with time-varying and distributed delays:

\[
\frac{d(x_i(t))}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + \sum_{j=1}^{n} b_{ij}(t) \int_{-\infty}^{t} K_{ij}(t - s)f_j(x_j(s))ds + I_i(t),
\]

for \(i = 1, 2, \ldots, n\). Here, as direct results of Theorem 1 and Corollary 1, we have

Corollary 3 Under hypothesis \((H1)-(H3)\), there exists exactly one \(\omega\)-periodic solution of model (15) and all other solutions of model (15) converge exponentially to it as \(t \to +\infty\), if there exist real constants \(\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \sigma_{ij}\) \((i, j = 1, 2, \ldots, n)\) and \(r > 1\) such that \(M = D - P - Q\) is a nonsingular \(M\)-matrix, where

\[
D = \text{diag}(d_1, d_2, \ldots, d_n),
\]

\[
P = \text{diag}(p_1, p_2, \ldots, p_n) \quad \text{with} \quad p_i = \frac{1}{r} \sum_{j=1}^{n} F_j^{r-\sigma_{ij}} \left( |\bar{a}_{ij}|^{r-\alpha_{ij}} + |\bar{b}_{ij}|^{r-\beta_{ij}} + |\bar{c}_{ij}|^{r-\gamma_{ij}} \right),
\]

\[
Q = (q_{ij})_{n \times n} \quad \text{with} \quad q_{ij} = \frac{1}{r} F_j^{r-\sigma_{ij}} \left( |\bar{a}_{ij}|^{\alpha_{ij}} + |\bar{b}_{ij}|^{\beta_{ij}} + |\bar{c}_{ij}|^{\gamma_{ij}} \right).
\]

Corollary 4 Under hypothesis \((H1)-(H3)\), there exists exactly one \(\omega\)-periodic solution of model (15) and all other solutions of model (15) converge exponentially to it as \(t \to +\infty\), if \(M_1 = D - (|\bar{A}| + |\bar{B}| + |\bar{C}|)F\) is a nonsingular \(M\)-matrix, where

\[
D = \text{diag}(d_1, d_2, \ldots, d_n), \quad |\bar{A}| = (|\bar{a}_{ij}|)_{n \times n}, \quad |\bar{B}| = (|\bar{b}_{ij}|)_{n \times n}, \quad |\bar{C}| = (|\bar{c}_{ij}|)_{n \times n}, \quad F = \text{diag}(F_1, F_2, \ldots, F_n).
\]

4 Remark and example

Remark 1. Some famous neural network models become a special case of model (1). For example, when the impulsive operator \(\delta_{ik} = 0\) \((i, j =
1, 2, · · · , n, k = 1, 2, · · · , model (1) becomes neural networks model (15), it contains those models studied by many authors, see, for example, Refs.[1]-[13], [15]-[18]. Thus the results of this paper can be applied to the recurrent neural networks with and/or without delays. Moreover, our results need only the activation functions $f_i$ satisfies the assumption (H2), not requiring the activation functions $f_j$ to be bounded and monotone nondecreasing. Therefore, we improve some previous results.

**Remark 2.** When $A = (a_{ij})_{2n \times 2n} = 0, B = (b_{ij})_{2n \times 2n} = 0,$

\[
C = (c_{ij})_{2n \times 2n} = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix},
\]

model (1) turns into a impulsive bidirectional associative memory neural networks with distributed delays, which has been studied in [27] and the references cited therein. Theorem 3.1. and Theorem 4.1. in [27] are special cases of Corollary 2.

**Remark 3.** When $B = (b_{ij})_{n \times n} = 0, C = (c_{ij})_{n \times n} = 0$, model (1) becomes the model studied in [28]. Assumption (H5) in [28] (corresponding to symbols in this paper):

\[
d_i - F_i \sum_{j=1}^{n} |\pi_{ji}| > 0, \quad i = 1, 2, \cdots, n.
\]

this condition implies $D - |A|F$ is a nonsingular $M$-matrix, it is a special case of Corollary 1. Also, Corollary 1 does not require the activation functions $f_j$ to be bounded ((H3) in [28]) and monotone nondecreasing. Therefore, we improve the results in [28].

**Example 1.** Consider the following model

\[
\begin{cases}
\frac{dx_1(t)}{dt} = -(1.9 + \sin t)x_1(t) - 0.5 \sin t f_1(x_1(t)) + 0.3 \cos t f_2(x_2(t)) + 3 \cos t, \\
\Delta x_1(t_k) = -(1 + \frac{2}{3} \sin (1 + k\pi)) x_1(t_k), \quad t_k = \frac{\pi}{2} + (k - 1)\pi, \quad k = 1, 2, \cdots, \\
\frac{dx_2(t)}{dt} = -(1.5 + \cos t) x_2(t) + 0.3 \cos t f_1(x_1(t)) + 0.25 \sin t f_2(x_2(t)) - 4 \sin t, \\
\Delta x_2(t_k) = -(1 + \frac{2}{3} \cos (2 + k\pi)) x_2(t_k), \quad t_k = \frac{\pi}{2} + (k - 1)\pi, \quad k = 1, 2, \cdots,
\end{cases}
\]

where $0 < t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{t \to +\infty} t_k = +\infty$, and $t_{k+2} = t_k + 2\pi$, $\delta_{ik} = \delta_{i(k+2)}$; $f_i(x) = \frac{1}{2}(|x + 1| + |x - 1|), i = 1, 2$. 


We can easily check that (H2) holds, and for any \(x_1, x_2 \in R\), we have
\[
|f_1(x_1) - f_2(x_2)| \leq |x_1 - x_2|, \quad i = 1, 2,
\]
hence \(F_1 = F_2 = 1\). It follows that
\[
D = \begin{pmatrix}
0.9 & 0 \\
0 & 0.5
\end{pmatrix}, \quad |A| = \begin{pmatrix}
0.5 & 0.3 \\
0.3 & 0.25
\end{pmatrix}, \quad F = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
\]
Obviously, \(D - |A|F = \begin{pmatrix}
0.4 & -0.3 \\
-0.3 & 0.25
\end{pmatrix}\) is a nonsingular \(M\)-matrix. Also, \(\alpha_{1k} = 1 + \frac{1}{2} \sin(1 + k\pi), \quad \alpha_{2+k\pi} = 1 + \frac{2}{3} \cos(2k)\) such that \(0 < \alpha_{ik} < 2, \quad i = 1, 2, \quad k = 1, 2, \cdots\). From Corollary 1, we know that there exists exactly one \(2\pi\)-periodic solution of model (16), and all other solutions of model (16) converge exponentially to it as \(t \to +\infty\). But, we notice that
\[
d_2 - F_2 \sum_{j=1}^{2} |A_j| = 0.5 - 1 \times (0.3 + 0.25) = -0.05 < 0,
\]
this implies that assumption (H5) in [28] does not hold, which means that Theorems in [28] is not applicable to ascertain the existence and global exponential stability of periodic solution for model (16).

**Example 2.** Consider the following model:

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= -d_1(t)x_1(t) + b_{11}(t)f_1(x_1(t - \tau_{11}(t))) + c_{11}(t) \int_{-\infty}^{t} e^{-(t-s)}f_1(x_1(s))ds \\
&\quad + c_{12}(t) \int_{-\infty}^{t} e^{-2(t-s)}f_2(x_2(s))ds - 2 \cos t, \\
&\quad t \geq 0, \quad t \neq t_k \\
\Delta x_1(t_k) &= -(1 + \frac{4}{3} \sin (1 + k\pi))(x_1(t_k)), \quad t_k = 0.3 + (k - 1)\frac{2}{3}, \quad k = 1, 2, \cdots,
\end{align*}
\]

\[
\begin{align*}
\frac{dx_2(t)}{dt} &= -d_2(t)x_2(t) + b_{21}(t)f_1(x_1(t - \tau_{21}(t))) + b_{22}(t)f_2(x_2(t - \tau_{22}(t))) \\
&\quad + c_{21}(t) \int_{-\infty}^{t} e^{-2(t-s)}f_1(x_1(s))ds + c_{22}(t) \int_{-\infty}^{t} e^{-(t-s)}f_2(x_2(s))ds + 3 \sin t, \\
&\quad t \geq 0, \quad t \neq t_k \\
\Delta x_2(t_k) &= -(1 + \frac{4}{3} \cos (2 + k\pi))(x_2(t_k)), \quad t_k = 0.3 + (k - 1)\frac{2}{3}, \quad k = 1, 2, \cdots,
\end{align*}
\]
where \(d_1(t) = 5 + \sin t, \quad d_2(t) = 5 - 0.5 \cos t, \quad b_{11}(t) = 1 - 0.5 \sin t, \quad b_{12}(t) = 0, \quad b_{21}(t) = \sin t, \quad b_{22}(t) = \cos t, \quad \tau_{11}(t) = 0.2 + 3 |\cos\frac{t}{2}|, \quad \tau_{21}(t) = 0.3 + |\sin\frac{t}{2}|, \quad \tau_{22}(t) = 1 - \sin t, \quad c_{11}(t) = \cos t, \quad c_{12}(t) = 0.5 + 0.5 \sin t, \quad c_{21} = 0.5 + 0.5 \sin t, \quad c_{22} = 1 - 0.5 \sin t, \quad f_i(x) = \frac{1}{2}(|x + 1| + |x - 1|), \quad i = 1, 2.
It is easy to check that assumptions (H1)-(H4) hold, and $F_1 = F_2 = 1$, $d_1 = 4$, $d_2 = 4.5$, $\bar{b}_{11} = 2$, $\bar{b}_{12} = 0$, $\bar{b}_{21} = 1$, $\bar{b}_{22} = 1$, $\bar{c}_{11} = 1$, $\bar{c}_{12} = 1$, $c_{21} = 1$, $\bar{c}_{22} = 1.5$, $0.2 \leq \tau_{11}(t) \leq 3.2$, $0.3 \leq \tau_{21}(t) \leq 2.3$, $0 \leq \tau_{22}(t) \leq 2$. Thus

$$M_2 = \begin{pmatrix} 3.5 & 0 \\ 0 & 4.5 \end{pmatrix} - \left( \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1.5 \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1.5 & -1 \\ -2 & 2 \end{pmatrix}$$

in Corollary 1. Since $M_1$ is an $M$-matrix, from Corollary 1, model (17) has exactly one $2\pi$-periodic solution and all other solutions of model (17) converge exponentially to it as $t \to +\infty$.

## 5 Conclusions

Stability and periodic oscillatory behavior are important in the applications and theories of neural networks. By using analytic methods, inequality technique and $M$-matrix theory, We have obtained some sufficient conditions ensuring the existence, uniqueness and global exponential stability of the periodic solution for a class of impulsive neural networks with both time-varying and distributed delays. The method given in this paper is simple and valid for the periodicity analysis of impulsive neural networks with variable and/or distributed delays. Some existing results are improved and extended. The results what we obtain are less restrictive than previously known criteria, and the hypothesis for boundedness on the activation functions and differentiability on time-varying delays are removed. It is believed that these results are significant and useful for the design and applications of neural networks.

## References


Existence and stability of periodic solution


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