A Study on F-Algebras and Almost F-Algebras

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Abstract. In this paper the properties of almost f-algebras and f-algebras given by [1], [2], [3], [4] have been studied. The most important one among these properties is that almost f-algebras and archimedean almost f-algebras are commutative. Furthermore nilpotent elements of f-algebras and Archimedean f-algebras have been studied.

Mathematics Subject Classification: 06F20

Keywords: f-Algebras, Archimedean f-Algebras, linear space

1. INTRODUCTION

Let A is a Riesz space and simultaneous algebras. For every $0 \leq a, b \in A$ if $0 \leq a \cdot b$, A is called Riesz algebra. An A Riesz algebra providing following characteristic, " $a \cdot b \in A$ and for every $c \geq a, a \wedge b = 0 \Rightarrow ac \wedge b = ca \wedge b = 0$ " is called f-algebra. Let X be an topological space, linear space $c(X)$ of real valued continuos functions, defined on X is a typical example of f-algebras. The definition of f-algebras has been firstly defined in the article called "lattice ordered rings. An. Acad. Brasil. C:28 (1956), (41-49) by G.Birkhoff and R.S.Pierce [5]".

In their article Birkhoff and Pierce showed that f-algebra A provides the property of $n|ab-ba| \leq a^2+b^2$ for every $a \cdot b \in A$ and $n \in N$. This indicates that Archimedean f-algebras are commutative. C.B.Huijmans and B.De Pagter also showed that Archimedean f-algebras are commutative using different method in their article called "Ideal-theory in f-algebras. Trans. Amer. Math. Soc. 269, 225-245". [6].

A Riesz algebra A is called almost f -algebras if it provides $a \wedge b = 0 \Rightarrow a \cdot b = 0$ for every $a, b \in A$. But [3] have stated that the reverse is not true.

In their article these authors studied the commutative property of Archimedean f -algebras.

Definition 1. *Let E be an ordered vector space. If the order of $\leq$ provides $y \leq 0$ when $nx \leq y$ for each $n \in \mathbb{N}$, E is called Archimedean.*

Definition 2. **i:** *As [3] suggested, L is a vector space and partially ordered, if for each $x, y \in L$*

1. *For each $z \in L, x + z \leq y + z$ while $x \leq y$*

2. *For $0 \leq \alpha \in R, \alpha x \geq 0$ while $x \geq 0$ are provided, L is called partially ordered reel vector spaces*

ii: *L is partially ordered reel vector space if for each $x, y \in L, x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$ exists on L , L is called Riesz space (vector lattice).*

iii: *Let L is an Riesz space and L is a subspace of K . If for each $x, y \in K, x \vee y$ and $x \wedge y$ are the elements of K , K is called Riesz subspace of L .*

iv: *Let L is an Riesz space and J is a subspace of L . If $x \in L, y \in J$ and $0 \leq |x| \leq |y|$, J is called ordered ideal in L .*

v: *Let B is an ordered ideal of L . If any subset of B has a supremum in L and it also bounds to, B is called a bound in L*

i: *The minimum ordered ideal including x element of L is called ordered ideal produced by x and it is showed indicated I_x*

$$I_x = \{y \in L : |y| \leq |\lambda x|, \lambda \in R\}$$

ii: *It is obviously that each ordered ideal is a Riesz subspace. The smallest band including x element is called band produced by x . For $x, y \in L$ if $|x| \wedge |y| = 0$, x and y elements are perpendicular and it is shown by $x \perp y$.*

iii: *For $D \subset L$, the perpendicular of D is*

$$D^d = \{x \in L : x \perp y \text{ for } y \in D\}$$

for each $D \subset L, D^d$ is a band in Archimedean Riesz space. D^{dd} is showed $(D^d)^d$. If upper bounded set which is not empty has supremum in L , L is called exact Dedekind.

The set of $\{x : x \in E \text{ and } x \geq 0\}$ showed with E_+ is called positive cone of E and its elements are positive. Let L is Riesz space for each $x \in L$, the

notations of

$$x^+ = \sup(x, 0), x^- = \inf(x, 0) \text{ and } x \vee y = \sup(x, y), x \wedge y = \inf(x, y)$$

can be used.

1.1. Ordered Bounded Operators.

Definition 3. Let T be an linear transformation, for each $D \subset L$ ordered bounded set, if $T(D)$ is ordered bounded in M . T is called ordered bounded operator. All the sets (of ordered bounded operators) from L to M is showed $L_b(L, M)$ if for $T_1, T_2 \in L_b$, $T_1 \leq T_2 \Leftrightarrow$ for each $x \in L$, $T_1.x \leq T_2.x$, $L_b(L, M)$ is a ordered vector space. Let T be an operator from L to M . If for each $x \in L$ $Tx \geq 0$. T is called positive operator. Each positive operator is ordered bounded. Therefore the difference between two operators is ordered bounded. The positive part of $L_b(L, M)$ consists of positive operators. If M is an exact Dedekind, $L_b(L, M)$ consists of Dedekind operators. If M is exact Dedekind, $L_b(L, M)$ exact Dedekind is Riesz space. In this case every ordered bounded can be written the difference of two positive operators.

Definition 4. While $T \in L_b(L, M)$ is an. We show the set of ordered continuous operators $L_n(L, M)$. $L_n(L, M)$ is the subspace of $L_b(L, M)$. If M is an exact Dedekind $L_n(L, M)$ is a band. If $M = R, L_0(L, R) = L^-$ is showed $L_n(L, R) = L_n^-$. L^- is called ordered dual of L , L_n^- is called ordered continuous dual of L .

Definition 5. Let T is an operator from L to M . If for each $x, y \in L$, $x \wedge y = 0, T(x) \wedge T(y) = 0$, T is called Riesz homomorphism (lattice homomorphism). Each Riesz homomorphism is positive. The necessary and sufficient condition for T being Riesz homomorphism of transformation from L to M is for $x, y \in L$, $T(x \wedge y) = T(x) \wedge T(y)$ or $T(x \vee y) = T(x) \vee T(y)$ or each $x \in L$, $T(x^+) = (Tx)^+$. Each kernel of Riesz homomorphism is ideal. If it is ordered continuous zero space is a band. If any for $f \in L^-$,

$$N_f = \{x \in L : \text{If } (|x|) = 0\}$$

is an ideal. In $f \in L_n^-$, it is a band. If f is Riesz homomorphism $\text{Krn} f = N_f$

1.2. Orthomorphism and Its Characteristics.

Definition 6. Let π be operator from L to L . For each B band of L , if $\pi(B) \subset B$, π is called an operator which makes the bands constant.

Lemma 1. See [12] For $\pi : L \rightarrow L$ operator followings are equivalent

i: π makes the band constant

- ii: For $\forall D \subset L$, where $D \neq 0; \pi(D^d) \subset D^d$
- iii: If $x, y \in L, x \wedge y = 0 \Rightarrow \pi \cdot x \perp y$
- iv: For $x, y \in L$, if $x \perp y \Rightarrow \pi \cdot x \perp y = 0$
- v: For each $x \in L, \pi \cdot x \in \{x\}^{dd}$

Definition 7. An ordered bounded transformation which makes bands of L constant is called orthomorphism on L . The sets of orthomorphism is written by $\text{Orth}(L)$

Theorem 2. Let I be the unit of Dedekind $\text{Orth}(L)$, $x, y \in L, 0 \leq x \leq y$ there is $\pi \in \text{Orth}(L), \pi x = y$ and $0 \leq \pi \leq I$. (see [12]) The characteristics of orthomorphism can be stated followings.

- i: Each orthomorphism of L is ordered continuous
- ii: For every $U \in L_+, \pi_1, \pi_2 \in \text{Orth}(L), (\pi_1 \wedge \pi_2)(U) = \pi_1(U) \wedge \pi_2(U)$ and $(\pi_1 \vee \pi_2)(U) = \pi_1(U) \vee \pi_2(U)$
- iii: For $\pi \in \text{Orth}(L), N_\pi = \{x \in L : \{|x| \mid |x| = 0\}\}$ zero space is a band.
- iv: Let $\pi_1, \pi_2 \in \text{Orth}(L)$ and $D \subset L$. Every $x \in D$ if $\pi_1(x) = \pi_2(x)$ for every $y \in D^{dd}, \pi_1(y) = \pi_2(y)$. As a result, if L has a weak ordered unit ($e^{dd} = L$), two orthomorphism can be considered equal if they are equal on e .
- v: If $\pi \in \text{Orth}(L), R_\pi$ is the image set of $\pi, N_\pi = R_\pi^d$
- vi: Let $\pi \in \text{Orth}(L)$ and $\pi^2 = \pi, R_\pi$ is a band projection and π is a band projection on R_π . The proof of this theorem can be seen in [10] and [12].

Definition 8. The set of

$$Z(L) = \{\pi \in L_b(L) : -\lambda I \leq \pi \leq \lambda I, \lambda \in R_T\}$$

is called center of L . If $Z(L) \subset \text{Orth}(L)$ and L is a Dedekind, the ideal produced by the unit $L_b(L)$ is $Z(L)$. If L Dedekind is exact, $Z(L)$ and $\text{Orth}(L)$ are Dedekind Riesz spaces. If $Z(L)$ is a norm of space and Banach lattice, $\text{Orth}(L) = Z(L)$ (see [12]), when L is regular then $Z(L)$ is a Banach space.

1.3. F-Algebras and Properties.

Definition 9. Let X is a vector space. The function called multiplication function from $X \times X$ to X for every $x, y, z \in X$ and $\alpha \in R$, provides following properties;

- i: $x(yz) = (xy)z$
- ii: $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$

iii: $\alpha(xy) = (\alpha x)y = x(\alpha y)$

X is an associative algebra.

Definition 10. Let A be a Riesz space and associative algebra. If for every $a, b \in A_+$, $a \cdot b \in A_+$, A is called Riesz algebra. Let A be Riesz algebra, when $a, b \in A$, $a \wedge b = 0$ if for every $c \in A_+$; $a \cdot c \wedge b = c \cdot a \wedge b = 0$, A is called f -algebra.

Definition 11. Let A is an f -algebra and $a \in A$. If for at least one k natural number, while $a^k = 0$, $a=0$, A is called half -prime f -algebra. The necessary and sufficient condition to make A , f -algebra semi-prime when $a^2 = 0$, $a=0$.

Theorem 3. If A is an f -algebra

i: For $a, b \in A$ and $c \in A_+$

$$c \cdot (a \wedge b) = c \cdot a \wedge c \cdot b, (a \wedge b) \cdot c = a \cdot c \wedge b \cdot c \text{ and } c \cdot (a \vee b) = c \cdot a \vee c \cdot b, (a \vee b) \cdot c = a \cdot c \vee b \cdot c$$

ii: For $a, b \in A$, if $a \perp b \Rightarrow a \cdot b = 0$

iii: If A is called semi -prime and $a, b \in A$, $a \perp b \Leftrightarrow a \cdot b = 0$

iv: If A is with unit, it is semi -prime.

v: If A is Archimedean, it is commutative.

The proof of this preposition can be seen in [12].

2. ALMOST F-ALGEBRAS

2.1. Every Archimedean F-Algebras is Commutative. Almost f -algebras for $\forall a, b \in A$, the difference between ab and ba is shown as [3];

$$[a, b] = -[b, a] = ab - ba$$

for every $a, b \in A$, since

$$\begin{aligned} (a - a \wedge b) \wedge (b - a \wedge b) &= (b - a \wedge b) \wedge (a - a \wedge b) = 0 \\ (a - a \wedge b) \cdot (b - a \wedge b) &= (b - a \wedge b) \cdot (a - a \wedge b) = 0 \end{aligned}$$

is obtained. Furthermore,

$$\begin{aligned} ab &= a(a \wedge b) + (a \wedge b)b - (a \wedge b)^2 \\ &= (a - a \wedge b)(a \wedge b) + (a \wedge b)(b - a \wedge b) + (a \wedge b)^2 \\ ba &= b(a \wedge b) + (a \wedge b)a - (a \wedge b)^2 \\ &= (b - a \wedge b)(a \wedge b) + (a \wedge b)(a - a \wedge b) + (a \wedge b)^2 \end{aligned}$$

Then

$$(2.1) \quad \begin{aligned} [a, b] &= [a - a \wedge b, a \wedge b] + [a \wedge b, b - a \wedge b] \\ &= -[a \wedge b, a - a \wedge b] + [a \wedge b, b - a \wedge b] \end{aligned}$$

are obtained.

Lemma 4. If $a, b \in A^+$ and $\theta > 0$

$$0 \leq (a - a \wedge b)(a \wedge b) - (a - a \wedge (1 + \theta)b)(a \wedge b \wedge \theta^{-1}(a - a \wedge b)) \leq \theta b^2 \text{ and } 0 \leq (a \wedge b)(a - a \wedge b) - (a - b \wedge \theta^{-1}(a - a \wedge b))(a - a \wedge (1 + \theta)b) \leq \theta b^2$$

Proof. $a \wedge (1 + \theta)b \leq a \wedge b + a \wedge \theta \cdot b$ is clear. Then $0 \leq a \wedge (1 + \theta)b - a \wedge b \leq \theta \cdot b$ is obtained. If the right part is multiplied by $a \wedge b$

$$0 \leq (a \wedge (1 + \theta))(a \wedge b) - (a \wedge b)^2 \leq \theta \cdot b(a \wedge b) \leq \theta \cdot b^2$$

is obtained. Writing once again

$$(2.2) \quad 0 \leq (a - a \wedge b)(a \wedge b) - (a - a \wedge (1 + \theta)b)(a \wedge b) \leq \theta b^2$$

is obtained. Then

$$\begin{aligned} 0 &\leq a \wedge b - a \wedge b \wedge \theta^{-1}(a - a \wedge b) \\ &= \theta^{-1}(\theta(a \wedge b) - (\theta(a \wedge b) \wedge (a - a \wedge b))) \\ &= \theta^{-1}(\theta(a \wedge b) - ((\theta + 1)(a \wedge b) \wedge a - (a \wedge b))) \\ &= \theta^{-1}((\theta + 1)(a \wedge b) - (\theta + 1)a \wedge (\theta + 1)b \wedge a) \\ &= \theta^{-1}((\theta + 1)(a \wedge b) - a \wedge (\theta + 1)b) \leq \theta^{-1}((\theta + 1)b - a \wedge (\theta + 1)b) \end{aligned}$$

Then $(a - a \wedge (\theta + 1)b) \wedge ((\theta + 1)b - a \wedge (\theta + 1)b) = 0$ can be seen easily. Then followings are obtained;

$$a \wedge b - a \wedge b \wedge \theta^{-1}(a - a \wedge b) \perp (a - a \wedge (\theta + 1)b) \text{ and } (a - a \wedge (\theta + 1)b) \cdot (a \wedge b - a \wedge b \wedge \theta^{-1}(a - a \wedge b)) = 0 \text{ and}$$

$$(2.3) \quad (a - a \wedge (\theta + 1)b) \cdot (a \wedge b) = (a - a \wedge (\theta + 1)b) \cdot (a \wedge b \wedge \theta^{-1}(a - a \wedge b))$$

is obtained. First inequality is obtained through 2.3 and 2.2. the second inequality can be obtained in the same way [3]. ■

Lemma 5. For every $a, b \in A^+$ and $\theta > 0$,

$$|[a, b] + [f_0(a, b, \theta), f_1(a, b, \theta)] - [f_0(b, a, \theta), f_1(b, a, \theta)]| \leq 2\theta(a^2 + b^2)$$

Proof. In 2.1 Using triangle inequality $[a, b] = -[a \wedge b, a - a \wedge b] + [a \wedge b, b - a \wedge b]$ the following is obtained:

$$\begin{aligned} & |[a \wedge b, a - a \wedge b] - [f_0(a, b, \theta), f_1(a, b, \theta)]| \\ &= |(a \wedge b)(a - a \wedge b) - (a - a \wedge b)(a \wedge b) - (f_0(a, b, \theta), f_1(a, b, \theta) - f_1(a, b, \theta), f_0(a, b, \theta))| \\ &= |((a \wedge b)(a - a \wedge b)) - (f_0(a, b, \theta) f_1(a, b, \theta) + (a - a \wedge b)(a \wedge b) - f_1(a, b, \theta) f_0(a, b, \theta))| \\ &\leq |((a \wedge b)(a - a \wedge b)) - (f_0(a, b, \theta) f_1(a, b, \theta))| + |(a - a \wedge b)(a \wedge b) - f_1(a, b, \theta) f_0(a, b, \theta)| \\ &\leq 2 \cdot \theta \cdot b^2 \end{aligned}$$

Beside, if a and b interchangeable

$$|[a \wedge b]b - a \wedge b[f_0(b, a, \theta), f_1(b, a, \theta)]|$$

is also obtained. Applying 2.1 and triangle inequality once more, the wanted inequality is obtained. For every n natural numbers, D_n shows all sets of sequences which consist of zeros and ones and whose length is n , that is for

$$D_n = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n), \forall 1 \leq i \leq n, \varepsilon_i = 0 \text{ or } \varepsilon_i = 1\}.$$

If $(\varepsilon = \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in D_n$ for $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, 0) \in D_{n+1}$ can be written [3]. It also valued for $(\varepsilon, 1)$. Furthermore, we write $|\varepsilon| = \sum_{r=1}^n \varepsilon_r$ ■

Definition 12. Let $a, b, \alpha, \beta \in A^+$ and $\tau = (\tau_1, \tau_2, \dots)$ be the sequence of increasing positive reel numbers. For every given $k = 1, 2, \dots$ and $\varepsilon \in D_k$, with induction method. Let's define $\alpha(\varepsilon) = \alpha(\tau, a, b, \varepsilon)$ elements of A^+ , when $\alpha(0) = a, \alpha(1) = b$ for

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) \in D_k$$

and $i = 0, 1$ as

$$\alpha((\varepsilon, 1)) = f_i(\alpha(\varepsilon), \alpha(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-1}, 1 - \varepsilon_k), \tau_k)$$

In this definition [3]

$$\begin{aligned} \alpha((1, 0)) &= f_0(\alpha(1), \alpha(0), \tau_1) = a \wedge b \wedge \tau_1^{-1}(b - a \wedge b) \\ \alpha((0, 0, 0, 1)) &= f_1(\alpha(0, 0, 0), \alpha(0, 0, 1), \tau_3) \\ &= \alpha((0, 0, 0)) - \alpha((0, 0, 0)) \wedge (1 + \tau_3) \alpha(0, 0, 1) \end{aligned}$$

Lemma 6. **i:** For $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) \in D_k$ and $1 \leq r \leq k, \alpha((\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)) \leq \alpha((\varepsilon_1, \dots, \varepsilon_r))$,

ii: For $\varepsilon, \varepsilon' \in D_k, \varepsilon \neq \varepsilon'$ and $i, j \in \{0, 1\}, \alpha((\varepsilon, i)) \wedge \alpha((\varepsilon', j)) = 0$

iii: For $\varepsilon \in D_k, \alpha((\varepsilon, 0)) \leq (1/k + 1)(\alpha(0) + \alpha(1))$ and $\alpha((\varepsilon, 0)) \leq \alpha(0) + \alpha(1)$

Proof. **i:** Using $0 \leq f_0(a, b, \theta) + f_1(a, b, \theta) \leq \alpha$ we can see $f_0(a, b, \theta) \leq a_1 f_1(a, b, \theta) \leq a$

ii: Using (i) for every $\varepsilon \in D_k, k = (0, 1, 2, \dots)$, $\varepsilon((\varepsilon, 0, i)) \wedge \alpha((\varepsilon, 1, i)) = 0$ for $k=0$ if $\varepsilon \notin 0$.

iii:

$$\begin{aligned}\alpha((0, 0)) &= f_0(\alpha(0), \alpha(1), \tau_1) \leq \tau_1^{-1}(\alpha(0) - \alpha(0) \wedge \alpha(1)) \\ &= \tau_1^{-1}(\alpha(0) - \alpha(1))^+ \in \{(\alpha(0) - \alpha(1))^+\}^{dd}\end{aligned}$$

and

$$\begin{aligned}\alpha((0, 1)) &= f_1(\alpha(0), \alpha(1), \tau_1) = \alpha(0) - \alpha(0) \wedge (1 + \tau_1)\alpha(1) \\ &\leq (\alpha(0) - \alpha(1))^+ \in \{(\alpha(0) - \alpha(1))^+\}^{dd}\end{aligned}$$

Similarly

$$\alpha((i, j)) \in \{(\alpha(1) - \alpha(0))^+\}^{dd}, (j = 1, 2)$$

So for

$$\forall i, j \in \{a, 1\}, \alpha((0, i)) \wedge \alpha((1, j)) = 0.$$

Then (ii) is obtained.

iv: Induction will be used for the first inequality For $k = 0$,

$$\alpha(0), \alpha(1) \leq \alpha(0) + \alpha(1)$$

For $k = 1$,

$$\alpha((0, 0)), \alpha((1, 0)) \leq \alpha(0) \wedge \alpha(1) \leq \frac{1}{2}(\alpha(0) + \alpha(1))$$

Let suppose the inequality is right for k . Let

$$\begin{aligned}\varepsilon &= (\varepsilon_1, \dots, \varepsilon_k, \varepsilon_{k+1}) \in D_{k+1} \\ \varepsilon^* &= (\varepsilon_2, \dots, \varepsilon_k, \varepsilon_{k+1}), \varepsilon = (\varepsilon_1, \varepsilon^*)\end{aligned}$$

and $\tau^* = (\tau_2, \dots)$ from induction and Definition 12, from

$$\begin{aligned}0 &\leq f_0(a, b, \theta) + f_1(a, b, \theta) \leq \alpha \\ \alpha((\varepsilon, 0)) &= \alpha(\tau^*, \alpha((\varepsilon_1, \varepsilon_2)), \alpha(\varepsilon_1, 1 - \varepsilon_2))\end{aligned}$$

$$(2.4) \quad (\varepsilon^*, 0) \leq \left(\frac{1}{k+1}\right)(\alpha((\varepsilon_1, \varepsilon_2)) + \alpha((\varepsilon_1, 1 - \varepsilon_2))) \leq \frac{1}{k+1} \cdot \alpha(\varepsilon_1)$$

Besides from the definition of $\alpha(\varepsilon)$, $\alpha((\varepsilon_1, \dots, \varepsilon_k, \theta))$. Repeating this, the following inequality is obtained.

$$(2.5) \quad \alpha((\varepsilon, 0)) \leq \alpha((\varepsilon, 0)) \leq (1 - \varepsilon_1)$$

second side of the inequality is obtained from

$$0 \leq f_0(a, b, \theta) + f_1(a, b, \theta) \leq a$$

From 2.4 and 2.5

$$(k+2) \alpha((\varepsilon, 0)) \leq \alpha(\varepsilon_1) + \alpha(1 - \varepsilon_1) = \alpha(0) + \alpha(1)$$

is obtained and the inequality of (iii) is also proved. For second equality, (i) and $\alpha((\varepsilon, 1)) \leq \alpha(\varepsilon_1)$ and $\alpha((\varepsilon, 1)) \leq \alpha(\varepsilon_1) \leq \alpha(0) + \alpha(1)$ are observed and then proof is completed [3].

■

Definition 13. For $k = 1, 2, \dots$

$$c_k = \sum_{\varepsilon \in D_k} (-1)^{|\varepsilon|+k} [\alpha((\varepsilon, 0)), \alpha((\varepsilon, 1))]$$

For example

$$\begin{aligned} c_1 &= \sum_{\varepsilon \in D_1} (-1)^{|\varepsilon|+1} [\alpha((\varepsilon, 0)), \alpha((\varepsilon, 1))] = -[\alpha((0, 0)), \alpha((0, 1))] + [\alpha((0, 1)), \alpha((1, 1))] = \\ &= -[f_0(a, b, \tau_1), f_1(a, b, \tau_1)] + [f_0(b, a, \tau_1), f_1(b, a, \tau_1)] \end{aligned}$$

from Lemma 5

$$|[\alpha, b] - c_1| \leq 2 \cdot \tau_1(a^2 + b^2) \text{ then } c_0 = [\alpha, b] \text{ so } |c_0 - c_1| \leq 2 \cdot \tau_1(a^2 + b^2)$$

is obtained [3].

Lemma 7.

$$|c_k - c_{k+1}| \leq 2 \cdot \tau_{k+1}(a^2 + b^2), (k = 0, 1, 2, \dots)$$

Proof. Inequality is obtained for $k = 0$ in Definition 13 so let $k \geq 1$,

$$\begin{aligned} c_{k+1} &= \sum_{\varepsilon \in D_{k+1}} (-1)^{|\varepsilon|+k+1} [\alpha((\varepsilon, 0)), \alpha((\varepsilon, 1))] = \sum_{\varepsilon \in D_k} (-1)^{|\varepsilon|+k} [\alpha((\varepsilon, 0, 0)), \alpha((\varepsilon, 0, 1))] + \\ &\quad \sum_{\varepsilon \in D_k} (-1)^{|\varepsilon|+k} [\alpha((\varepsilon, 1, 0)), \alpha((\varepsilon, 1, 1))] \\ c_k - c_{k+1} &= \\ \sum_{\varepsilon \in D_k} (-1)^{|\varepsilon|+k} \{ &[\alpha((\varepsilon, 0)), \alpha((\varepsilon, 1))] + [\alpha((\varepsilon, 0, 0)), \alpha((\varepsilon, 0, 1))] + [\alpha((\varepsilon, 1, 0)), \alpha((\varepsilon, 1, 1))] \} \end{aligned}$$

from Lemma 5

$$\begin{aligned}
 |c_k - c_{k+1}| &\leq 2 \cdot \tau_{k+1} \sum_{\varepsilon \in D_k} \{\alpha((\varepsilon, 0))^2 + \alpha((\varepsilon, 1))^2\} \\
 &\leq 2\tau_{k+1} \sum_{\varepsilon \in D_k} \{\alpha((\varepsilon, 0)) + \alpha((\varepsilon, 1))\}^2 \\
 &\leq 2\tau_{k+1} \sum_{\varepsilon \in D_k} \alpha(\varepsilon)^2
 \end{aligned}$$

the last inequality B obtained from $0 \leq f_0(a, b, \theta) + f_1(a, b, \theta) \leq a$ but Since

$$\begin{aligned}
 \sum_{\varepsilon \in D_k} \alpha(\varepsilon)^2 &= \sum_{\lambda \in D_{k-1}} \{\alpha((\lambda, 0))^2 + \alpha((\lambda, 1))^2\} \\
 |c_k - c_{k+1}| &\leq 2 \cdot \tau_{k+1} \sum_{\lambda \in D_{k-1}} \{\alpha((\lambda, 0)) + \alpha((\lambda, 1))\}^2
 \end{aligned}$$

As a result

$$|c_k - c_{k+1}| \leq 2 \cdot \tau_{k+1} (\alpha(0)^2 + \alpha(1)^2) = 2\tau_{k+1} (a^2 + b^2)$$

is obtained [3]. ■

Theorem 8. *Every Archimedean almost f -algebra is commutative.*

Proof. Let $a, b \in A^+$ and $\theta \geq 0$. And let $\sum_{k=1}^{\infty} \tau_k < (1/2)\theta$ be the sequence of definite positive reel numbers. Take $\{\tau_k\}_{k=1}^{\infty}$, for $\alpha(\varepsilon)$ and $\varepsilon \in D_k$ ($k = 1, 2, \dots$), c_k is defined as in Definiton 13 and $c_0 = [a, b]$. From Lemma 7

$$|c_k - c_{k+1}| \leq 2 \cdot \tau_{k+1} (a^2 + b^2)$$

$$\begin{aligned}
 (2.6) \quad |[a, b] - c_k| &= \left| \sum_{r=1}^k (c_{r-1} - c_r) \right| \leq \sum_{r=1}^k |(c_{r-1} - c_r)| \\
 &\leq \sum_{r=1}^k 2\tau_r (a^2 + b^2) \leq \theta (a^2 + b^2)
 \end{aligned}$$

Besides since

$$\begin{aligned}
 c_k &= \sum_{\varepsilon \in D_k} \mp [\alpha((\varepsilon, 0)), \alpha((\varepsilon, 1))] \\
 |c_k| &\leq \sum_{\varepsilon \in D_k} |[\alpha((\varepsilon, 0)), \alpha((\varepsilon, 1))]| \leq \sum_{\varepsilon \in D_k} \{|\alpha((\varepsilon, 0))\alpha((\varepsilon, 1)) + \alpha((\varepsilon, 1))\alpha((\varepsilon, 0))|\}
 \end{aligned}$$

is obtained. From Lemma 6 (iii) and (ii)

$$\sum_{\varepsilon \in D_k} \alpha((\varepsilon, 1)) = \bigcup_{\varepsilon \in D_k} \alpha((\varepsilon, 1)) \leq \alpha(0) + \alpha(1) = a + b$$

so

$$(2.7) \quad |c_k| \leq \left(\frac{1}{k+1} \right) \{ \alpha(0) + \alpha(1) \} \sum_{\varepsilon \in D_k} \alpha((\varepsilon, 1)) \\ + \sum_{\varepsilon \in D_k} \alpha((\varepsilon, 1)) + (\alpha(0) + \alpha(1)) \leq 2(a+b)^2 / (k+1)$$

for $k=1, 2, \dots$ combining 2.6 and 2.7

$$|[a, b]| \leq \theta(a^2 + b^2) + 2(a+b)^2 / (k+1)$$

is obtained. Since A is an Archimedean Riesz space, $|[a, b]| \leq \theta(a^2 + b^2)$ for $\forall \theta > 0$. Using Archimedean property of A $|[a, b]| = 0$ is obtained, in other words $ab = ba$. This proves that A is commutative. Every non Archimedean almost f-algebras can not be commutative [3]. ■

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Received: May 6, 2006