The Theory of Multidimensional Differential Equations in Lie Group and Conditions for Their Exact Solvability

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Abstract. In this study, we investigate multidimensional differential equation with special coefficient. We built a new theorem for necessary and sufficient conditions for solvability of these differential equations. The proof of this theorem gives, n-dimensional vector spaces, $\mathbb{R}^n_x$ which transform Lie algebra. As a result of this transformation, we find new conditions for exact integrable of multidimensional differential equations.

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1. Introduction

Let $\mathbb{R}^n_x$ be an n-dimensional space of vectors and $E$ be an arbitrary complex Banach space, $L(\mathbb{R}^n_x; E)$ be space of operators space which acts from $\mathbb{R}^n_x$ to $E$ and $C(\mathbb{R}^n_x; L(E; E))$ be space of strong continuous operators acting from $\mathbb{R}^n_x$ to Endomorphism algebra $L(E; E)$ of Banach space E. The spectral theory of this kind of operator family has been investigated in [1]

2. Nonhomogeneous Multidimensional Differential Equation

Consider the multidimensional differential equation below

$$u'(x)h = A(Q(x)h)u(x) + f(x)h$$

(2.1)

The initial condition is;

$$u(x_0) = u_0, (x_0 \in S, u_0 \in E)$$

(2.2)
Here, the meaning of the differentiation $u'(x) \in L(R^n_x; E)$ is Freshe. In addition, $x \in S, h \in R^n_x; u(x) \in E, Q(x) \in GL(n; R)$ and $S$ is zero neighborhood in $R^n_x$. Let $Q$ be an element of $C^\infty(S)$ and $Q(0) = I$ (identity matrices), $A$ be an element of $C(R^n_x; L(E; E))$ at that $(Ah \neq 0)$ on $h \neq 0$ and the operator function $f$ which acts from $R^n_x$ to $L(R^n_x, E)$ be differentiable and continuous is the neighborhood of $S \subset R^n_x$.

**Definition 1.** If for an arbitrary point $(x_0, u_0)$(i.e. $x_0 \in S, u_0 \in E$), (2.1) is defined around $x_0$ and satisfies the initial conditions $u(x_0) = u_0$, then it has a continuous differentiable solution and is called solvable or exact integrable.

**Definition 2.** Let’s donate $L(E; E)$ as $F$. Following Shilov [2], the mapping $A : R^n_x \rightarrow F$ will be called morphism if the following relations hold true

1. $A(h_1 + h_2) = Ah_1 + Ah_2$, for any arbitrary $h_1, h_2 \in R^n_x$,
2. $A(\alpha h) = \alpha Ah$, for any arbitrary $h \in R^n_x, \alpha \in C$.

If the morphism $A$ maps the space $R^n_x$ onto the whole spaces $F$, it is called epimorphism. If the morphism $A$ maps the space $R^n_x$ at least not onto the whole spaces $F$ but one-to-one (i.e. from $h_1 \neq h_2$, it follows that $A(h_1) \neq A(h_2)$, it is called monomorphism.

**Definition 3.** Following N.Bourbaki [3], if the below multiplication

$$(Ah, Bk) \implies [Ah, Bk], \forall k, h \in R^n_x$$

holds the following three conditions in $F$ algebra, then $F$ is called Lie algebra.

1. $[Ah, Ah] = 0, h \in R^n_x, Ah \in F$;
2. $[Ah, Bk] = AhBk - BkAh = -[BkAh - AhBk] = -[Bk, Ah], \forall Ah, Bk \in F, h, k \in R^n_x$;
3. $[Ah, [Bk, C\xi]] + [Bk, [C\xi, Ah]] + [C\xi, [Ah, Bk]] = 0, \forall h, k, \xi \in R^n_x$.

Here $Ah, Bk, C\xi$ are in $F$ and the symbol $[Ah, Bk]$ is the commutator of operators $Ah, Bk$ (i.e. $[Ah, Bk] = (AhBk - BkAh), \forall h, k \in R^n_x$).

**Theorem 1.** The necessary and sufficient conditions for the exact solvability of (2.1) are;

1. In $R^n_x$ vector spaces there is a Lie product defined as;
   $$\gamma(h, k) = [h, k] = Q'(x)(h, k) - Q'(x)(k, h), \forall h, k \in R^n_x$$

2. According to the above product the following conditions hold
   - **a):** $A[h, k] = [Ah, Ak]$;
   - **b):** $Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k]$;
   - **c):** $L_{hh}\{Ah\varphi(x)k\} + L_{hk}\{\varphi'(x)hk\} = 0, \forall h, k \in R^n_x$

Here the symbol $[..]$ is the commutator of operators $Ah$ and $Ak$ which are in $F$ and $L_{hh}$ is the co-symmetric operator.

**Proof.** Assume that (2.1) is exact solvable. This shows that for an arbitrary point $(x_0, u_0), x_0 \in S, u_0 \in E$ this equation which is defined around $x_0$ and
satisfy the initial condition \( u(x_0) = u_0 \) has continuous differentiable solution. The function \( u(x) \) which holds the above properties is the solution for (2.1). If we put this solution into (2.1) then we obtain the following equation

\[
(2.3) \quad u'(x)h = A(Q(x)h)u(x) + f(x)h
\]

If we differentiate (2.3), then we obtain

\[
u''(x)hk = A(Q'(x)(h,k))u(x) + A(Q(x)h)u'(x)k + f'(x)hk
\]

\[
= A(Q'(x)(h,k))u(x) + A(Q(x)h)[A(Q(x)k)u(x) + f(x)k] + f'(x)hk
\]

(2.4)

\[
= A(Q'(x)(h,k))u(x) + A(Q(x)h)A(Q(x)k)u(x) + A(Q(x)h)f(x)k + f'(x)hk.
\]

Since the second order differential solution is symmetric with respect to \( h \) and \( k \), we get from (2.4),

\[
A(Q'(x)(h,k))u(x) + A(Q(x)h)A(Q(x)k)u(x) + A(Q(x)h)f(x)k + f'(x)hk
\]

(2.5)

\[
A(Q'(x)(k,h))u(x) + A(Q(x)k)A(Q(x)h)u(x) + A(Q(x)k)f(x)h + f'(x)kh.
\]

If we group (2.5) we get

(2.6)

\[
A[(Q'(x)(h,k) - Q'(x)(k,h))]u(x) = [A(Q(x)h)A(Q(x)k) - A(Q(x)k)A(Q(x)h)]u(x)
\]

(2.7)

\[
A(Q(x)h)f(x)k - A(Q(x)k)f(x)h + f'(x)hk - f'(x)kh = 0
\]

In (2.6) if we write \( x_0 \) instead \( x \) then the initial condition \( u(x_0) = u_0 \) holds. Here \( u_0 \) is an arbitrary vector of Banach space \( E \). Therefore from (2.6) for any arbitrary \( x_0 \in S \) we find

\[
A[(Q'(x_0)(h,k) - Q'(x_0)(k,h))] = [A(Q(x_0)h)A(Q(x_0)k) - A(Q(x_0)k)A(Q(x_0)h)]
\]

\[
= [A(Q(x_0)h), A(Q(x_0)k)].
\]

Then from this equation we find

(2.8)

\[
A[(Q'(x_0)(h,k) - Q'(x_0)(k,h))] = [A(Q(x_0)h), A(Q(x_0)k)].
\]

Right hand side of (2.8) is commutator of operators \( A(Q(x_0)h), A(Q(x_0)k) \) which is in \( F \). If we choose \( x=0 \) in (2.8) then

(2.9)

\[
A(Q'(0)(h,k) - Q'(0)(k,h)) = [A, Ak].
\]

Let’s define the symbol \( \gamma(h,k) \) as

(2.10)

\[
\gamma(h,k) = [h,k] = Q'(0)(h,k) - Q'(0)(k,h)
\]

In virtue of (2.8) and \( A \) is a monomorph operator in \( C(R^n_x, F) \) then \( \gamma(h,k) \) characterizes the Lie product in space of vectors \( R^n_x \).

Directly
1. \( \gamma(h, h) = [h, h] = Q'(0)(h, h) - Q'(0)(h, h) = 0; \)
2. \( \gamma(h, k) = Q'(0)(h, k) - Q'(0)(k, h) = [h, k] = -[Q'(0)(k, h) - Q'(0)(h, k)] = -\gamma(k, h) = [-k, h]; \)
3. \( z \in R^n_x \) holds the Jacobi equality which is below
\[
[h, \gamma(k, z)] + [k, \gamma(z, h)] + [z, \gamma(h, k)] = 0
\]
Thus the space of vectors \( R^n_x \) is transformed to the Lie Algebra. If we put (2.10) in (2.9) we obtain
\[
A[h, k] = [Ah, Ak] = AkAh - AkAh
\]
It is obvious that the operator \( A \in C(R^n_x; L(E; E)) \) is a morphism which acts from the Lie algebra \( R^n_x \) to endomorphisms \( F \) of Banach space \( E \). We also know that the operator \( A \) is monomorph operator. Now let us use the monomorphism of \( A \), from (2.8) we find a commutator differential equation which depends on function \( Q(x) \):
\[
Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k] \tag{2.12}
\]
Then if we divide both sides of (2.7) by 2,
\[
\frac{1}{2}[A(Q(x)h)f(x)k - A(Q(x)k)f(x)h] + \frac{1}{2}[f'(x)hk - f'(x)kh] = 0
\]
or
\[
\Lambda_{hk}\{A(Q(x)h)f(x)k\} + \Lambda_{hk}\{f'(x)hk\} = 0, \forall h, k \in R^n_x \tag{2.13}
\]
Therefore the following conditions are satisfied in order to get an exact solution of (2.1)
\[
A[h, k] = [Ah, Ak];
\]
\[
\Lambda_{hk}\{A(Q(x)h)f(x)k\} + \Lambda_{hk}\{f'(x)hk\} = 0;
\]
\[
Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k] \tag{2.14}
\]
Since the theorem of G. Frobenius [4] for exact solvable multidimensional differential equation, the conditions in (2.14) are also sufficient conditions for (2.1).

**Corollary 2.** Let \( Q(x) = I \), then \( \forall x \in R^n_x \) (2.1) and (2.2) transform mutually to the following equations:
\[
u'(x)h = Aju(x) + f(x)h, \forall h \in R^n_x \tag{2.15}
\]
\[
u(x_0) = u_0, x_0 \in S, u_0 \in E. \tag{2.16}
\]
The necessary and sufficient conditions for exact solvable of (2.15) and (2.16) are the following
\[
\Lambda_{hk}\{A(h)A(k)\} = 0, \forall h, k \in R^n_x \tag{2.17}
\]
\[
\Lambda_{hh}\{A(h)f(x)k\} + \Lambda_{hk}\{f'(x)hk\} = 0
\]

Here, for bilinear operator \( B : R^n_x \oplus R^n_x \rightarrow C(R^n_x; F) \), \( \Lambda_{hh} \) is:
\[
\Lambda_{hh}\{Bhk\} = \frac{1}{2}(Bhk - Bkh). 
\]

If the conditions (2.17) and (2.18) hold, then the solution of (2.15) and (2.16) follows
\[
u(x) = e^{A(x-x_0)}u_0 + \int_{x_0}^x e^{A(x-\sigma)}f(\sigma)d\sigma
\]

3. Applications

Let \( G \) be \( m^2 \)-dimensional Lie group and \( \alpha = (\alpha_{11}, ..., \alpha_{mm}) \) is the collection of \( m^2 \) parameters (\( \det \alpha \neq 0 \)), characterizing the elements of this group, \( E_y \) is the complex Banach space. Let’s denote by \( M(G; L(E_y; E_y)) \) the space of continuous operator valued maps, defined on the group \( G \) with the values in the algebra \( L(E_y; E_y) \) of endomorphisms of Banach space \( E_y \) [6] - [7] - [8] - [9], with topology, given by the norm.

Suppose that the unity \( e \) of the group \( G \) corresponds to the zero values of the parameters \( \alpha_{11}, ..., \alpha_{mm} \). Let \( T \in M(G; L(E_y; E_y)) \) since the giving of the parameters simply determines the element of the group \( G \) then the operator \( T(g) \in M(G; L(E_y; E_y)) \), \( g \in G \) can be considered as the function of the parameters \( \alpha_{11}, ..., \alpha_{mm} \), i.e.\( T(g) = T(\alpha_{11}, ..., \alpha_{mm}) = T(\alpha) \). Consider the operator equation of the form:
\[
T(fg) = T(f)T(g), \ (f, g \in G)
\]
with initial condition
\[
T(e) = I, \ (e \in G; I \in L(E_y; E_y))
\]
Where \( T \in M(G; L(E_y; E_y)) \), and \( I \in L(E_y; E_y) \) are identical operators. Denote by \( \alpha_{ij}(f), i, j = 1, 2, ..., m \) the parameters characterizing element \( f \) of the group \( G \). If determining of the parameters \( \alpha_{ij}(f), \alpha_{ij}(g) \), \( i, j = 1, 2, ..., m \) simply determines the element and \( f, g \in G \) also determines their product \( fg \in G \), and consequently, the parameters \( \alpha_{ij}(fg) \), \( i, j = 1, 2, ..., m \) are the functions of \( 2m^2 \) parameters
\[
\alpha_{ij}(f), \alpha_{ij}(g) : \alpha_{ts}(fg) = \varphi_{ts}\{\alpha_{11}(f), ..., \alpha_{mm}(f); \alpha_{11}(g), ..., \alpha_{mm}(g)\} \ (t, s = 1, 2, ..., m)
\]
such \( \varphi_{ts}(t, s = 1, 2, ..., m) \) that are continuously differentiable functions of co-multipliers. It is not difficult to see that the operator equation (3.1) - (3.2) is equivalent to the following vector equation
\[
y(f) = T(fg)y(g^{-1}), \ (f, g, g^{-1} \in G)
\]
\[
y(e) = \eta(e \in G; \eta \in E_y; T(f)\eta = y(f))
\]
The following problem is formulated by us: it is required to find the solutions
\[ y(t) \in E_y, f \in G \]
of the functional problems (3.3) - (3.4), analytical in the
neighborhood of the unique element \( f = e \in G \).

**Theorem 3.** Let the operator \( T : G \to L(E_y; E_y) \) has continuous private
derivatives for each of the parameters \( \alpha_{ij}(f), i, j = 1, 2, ..., m \) at the point
\( g = f^{-1} \in G \) and let the function \( y(f)(f \in G) \) with continuous private deriva-
tives \( \frac{\partial y(f)}{\partial \alpha_{ij}(f)} \) (\( t, s = 1, 2, ..., m \)) is the solution of the vector equation (3.3). Then
there exists the unique finite collection of linear operators \( \{A_{11}, A_{12}, ..., A_{mm}\} \subset L(E_y; E_y) \) such that at the same time and the solution of overdetermined system
of differential equations:

\[
(L_{ij}y)(f) \equiv \frac{\partial y(f)}{\partial \alpha_{ij}(f)} - \sum_{t=1}^{m} \sum_{s=1}^{m} c_{ts}^{ij}(f)A_{ts}y(f) = 0,
\]

The operators \( L_{11}, L_{12}, ..., L_{mm} \) are called characteristic differential operators (CDO) of Lie, \( c(e) = \{c_{ts}^{ij}(e)\} \) is \( m \times m \) matrix such that \( c_{ts}^{ij}(e) = \delta_{ts}^{ij} \) is Kronecker symbol. System of equations (3.5), according
to Lie [10], we’ll call if characteristics differential equations (GCDE) of
Lie of the presentation the group \( G \). This notion plays the important role in
quantum mechanics [11]. It is true the following inverse.

**Theorem 4.** If SCDE of Lie (3.5) has unique analytical solution in the neigh-
borhood of unity element \( e \in G \) with initial condition (3.4), then it is also
solution of the vector equation (3.3).

**Definition 4.** SCDE of Lie (3.5) called absolutely integrable (absolutely solv-
able) on the set \( G_a \times D_y \subset G \times E_y \) for any point \((e, \eta) \in G_0 \times D_y\) there exists
unique solution \( y(f) \) of this system, defined in some neighborhood of unity el-
ment \( e \) in \( G_a \subset G \) which has the values in \( D_y \subset E_y \) and satisfies the initial
condition \( y(e) = \eta \).

**Theorem 5.** Let \( A_{ts} \in L(E_y; E_y) \), \( t, s = 1, 2, ..., m \) are bounded, infinitesimal
operators of the presentation Lie group. Suppose that the functions \( \{G_{ts}^{ij}(f), i, j, \)
\( t, s = 1, 2, ..., m\} \) are continuously differentiable. The neighborhood of the unity
element \( e \in G_0 \). SCDE of Lie (3.5) has unique analytical solution in the neigh-
brhood of unity element \( e \in G \) with initial condition (3.6) if and only if the conditions

\[
A_{\alpha\beta}A_{ts} - A_{ts}A_{\alpha\beta} = \sum_{t=1}^{m} \sum_{s=1}^{m} c_{\beta\gamma ts}^{ij}A_{ij}, (\alpha, \beta, t, s = 1, 2, ..., m)
\]

(3.6)
are fulfilled. If the condition of absolutely integrability is satisfied, then the solution of the problem (3.4) - (3.5) can be presented in the form

\[ y(f) = \eta + \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ \sum_{i_1 j_1 \ldots i_k j_k} \left( \frac{\partial y(f)}{\partial \alpha_{i_1 j_1}(f) \ldots \partial \alpha_{i_k j_k}(f)} \right) \right\} \prod_{t=1}^{k} (\alpha_{i_t j_t}(f) - \alpha_{i_t j_t}(e)) \]

Where \( \sum_{i_1 j_1 \ldots i_k j_k} = \sum_{i_1}^{m} \ldots \sum_{i_k}^{m} \ldots \sum_{j_1}^{m} \ldots \sum_{j_k}^{m} ; c_{ij}^{ij} \) are the structure real constants of Lie group, satisfying the conditions of antisymmetry and Jakobi identities, and the derivatives of \( k \) order of the function \( y(f) \) at the point \( f = e \in G \) are determined from the following recurrent operator correlations:

\[ A_{i_1 j_1 \ldots i_k j_k i_{k+1} j_{k+1}} = \sum_{t_1 s_1 \ldots t_k s_k} \left\{ C_{i_1 j_1 t_1 s_1 i_{k+1} j_{k+1}} \delta_{t_2 s_2} \ldots \delta_{t_k s_k} + C_{i_1 s_1 i_{k+1} j_{k+1}} \delta_{t_2 s_2} \ldots \delta_{t_k s_k} \right\} \times A_{t_1 s_1 \ldots t_k s_k} + A_{i_1 j_1 \ldots i_k j_k} A_{i_{k+1} j_{k+1}} ; \]

the row (3.7) converges for

\[ \sum_{i=1}^{m} \sum_{s=1}^{m} |\alpha_{ts}(f) - \alpha_{ts}(e)| < \frac{1}{2d} (d \geq 1) \]

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