The Theory of Multidimensional Differential Equations in Lie Group and Conditions for Their Exact Solvability

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Abstract. In this study, we investigate multidimensional differential equation with special coefficient. We built a new theorem for necessary and sufficient conditions for solvability of these differential equations. The proof of this theorem gives, n -dimensional vector spaces, R_x^n which transform Lie algebra. As a result of this transformation, we find new conditions for exact integrable of multidimensional differential equations.

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1. INTRODUCTION

Let R_x^n be an n -dimensional space of vectors and E be an arbitrary complex Banach space, $L(R_x^n; E)$ be space of operators space which acts from R_x^n to E and $C(R_x^n; L(E; E))$ be space of strong continuous operators acting from R_x^n to Endomorphism algebra $L(E; E)$ of Banach space E . The spectral theory of this kind of operator family has been investigated in [1]

2. NONHOMOGENEOUS MULTIDIMENSIONAL DIFFERENTIAL EQUATION

Consider the multidimensional differential equation below

$$(2.1) \quad u'(x)h = A(Q(x)h)u(x) + f(x)h$$

The initial condition is;

$$(2.2) \quad u(x_0) = u_0, (x_0 \in S, u_0 \in E)$$

Here, the meaning of the differentiation $u'(x) \in L(R_x^n; E)$ is Freshe. In addition, $x \in S, h \in R_x^n; u(x) \in E, Q(x) \in GL(n; R)$ and S is zero neighborhood in R_x^n . Let Q be an element of $C^\infty(S)$ and $Q(0) = I$ (identity matrices), A be an element of $C(R_x^n; L(E; E))$ at that $(Ah \neq 0)$ on $h \neq 0$ and the operator function f which acts from R_x^n to $L(R_x^n; E)$ be differentiable and continuous is the neighborhood of $S \subset R_x^n$.

Definition 1. *If for an arbitrary point (x_0, u_0) (i.e. $x_0 \in S, u_0 \in E$), (2.1) is defined around x_0 and satisfies the initial conditions $u(x_0) = u_0$, then it has a continuous differentiable solution and is called solvable or exact integrable.*

Definition 2. *Let's denote $L(E; E)$ as F . Following Shilov [2], the mapping $A : R_x^n \rightarrow F$ will be called morphism if the following relations hold true*

1. $A(h_1 + h_2) = Ah_1 + Ah_2$, for any arbitrary $h_1, h_2 \in R_x^n$,
2. $A(\alpha h) = \alpha Ah$, for any arbitrary $h \in R_x^n, \alpha \in C$.

If the morphism A maps the space R_x^n onto the whole spaces F , it is called epimorphism. If the morphism A maps the space R_x^n at least not onto the whole spaces F but one-to-one (i.e. from $h_1 \neq h_2$, it follows that $A(h_1) \neq A(h_2)$), it is called monomorphism.

Definition 3. *Following N.Bourbaki [3], if the below multiplication*

$$(Ah, Bk) \implies [Ah, Bk], \forall k, h \in R_x^n$$

holds the following three conditions in F algebra, then F is called Lie algebra.

1. $[Ah, Ah] = 0, h \in R_x^n, Ah \in F$;
2. $[Ah, Bk] = AhBk - BkAh = -[BkAh - AhBk] = -[Bk, Ah], \forall Ah, Bk \in F, h, k \in R_x^n$;
3. $[Ah, [Bk, C\xi]] + [Bk, [C\xi, Ah]] + [C\xi, [Ah, Bk]] = 0, \forall h, k, \xi \in R_x^n$.

Here $Ah, Bk, C\xi$ are in F and the symbol $[Ah, Bk]$ is the commutator of operators Ah, Bk (i.e. $[Ah, Bk] = (AhBk - BkAh), \forall h, k \in R_x^n$).

Theorem 1. *The necessary and sufficient conditions for the exact solvability of (2.1) are;*

1. In R_x^n vector spaces there is a Lie product defined as;

$$\gamma(h, k) = [h, k] = Q'(x)(h, k) - Q'(x)(k, h), \forall h, k \in R_x^n$$

2. According to the above product the following conditions hold

- a): $A[h, k] = [Ah, Ak]$;
- b): $Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k]$;
- c): $\Lambda_{hk}\{Ah\varphi(x)k\} + \Lambda_{hk}\{\varphi'(x)hk\} = 0, \forall h, k \in R_x^n$

Here the symbol $[.,.]$ is the commutator of operators Ah and Ak which are in F and Λ_{hk} is the co-symmetric operator.

Proof. Assume that (2.1) is exact solvable. This shows that for an arbitrary point $(x_0, u_0), x_0 \in S, u_0 \in E$ this equation which is defined around x_0 and

satisfy the initial condition $u(x_0) = u_0$ has continuous differentiable solution. The function $u(x)$ which holds the above properties is the solution for (2.1). If we put this solution into (2.1) then we obtain the following equation

$$(2.3) \quad u'(x)h = A(Q(x)h)u(x) + f(x)h$$

If we differentiate (2.3), then we obtain

$$\begin{aligned} u''(x)hk &= A(Q'(x)(h, k))u(x) + A(Q(x)h)u'(x)k + f'(x)hk \\ &= A(Q'(x)(h, k))u(x) + A(Q(x)h)[A(Q(x)k)u(x) + f(x)k] + f'(x)hk \\ (2.4) \quad &= A(Q'(x)(h, k))u(x) + A(Q(x)h)A(Q(x)k)u(x) + A(Q(x)h)f(x)k + f'(x)hk. \end{aligned}$$

Since the second order differential solution is symmetric with respect to h and k, we get from (2.4),

$$\begin{aligned} &A(Q'(x)(h, k))u(x) + A(Q(x)h)A(Q(x)k)u(x) + A(Q(x)h)f(x)k + f'(x)hk \\ (2.5) \quad &A(Q'(x)(k, h))u(x) + A(Q(x)k)A(Q(x)h)u(x) + A(Q(x)k)f(x)h + f'(x)kh. \end{aligned}$$

If we group (2.5) we get

$$(2.6) \quad A[(Q'(x)(h, k) - Q'(x)(k, h))]u(x) = [A(Q(x)h)A(Q(x)k) - A(Q(x)k)A(Q(x)h)]u(x)$$

$$(2.7) \quad A(Q(x)h)f(x)k - A(Q(x)k)f(x)h + f'(x)hk - f'(x)kh = 0$$

In (2.6) if we write x_0 instead x then the initial condition $u(x_0) = u_0$ holds. Here u_0 is an arbitrary vector of Banach space E. Therefore from (2.6) for any arbitrary $x_0 \in S$ we find

$$\begin{aligned} A[(Q'(x_0)(h, k) - Q'(x_0)(k, h))] &= [A(Q(x_0)h)A(Q(x_0)k) - A(Q(x_0)k)A(Q(x_0)h)] \\ &= [A(Q(x_0)h), A(Q(x_0)k)]. \end{aligned}$$

Then from this equation we find

$$(2.8) \quad A[(Q'(x_0)(h, k) - Q'(x_0)(k, h))] = [A(Q(x_0)h), A(Q(x_0)k)].$$

Right hand side of (2.8) is commutator of operators $A(Q(x_0)h), A(Q(x_0)k)$ which is in F. If we choose $x=0$ in (2.8) then

$$(2.9) \quad A(Q'(0)(h, k) - Q'(0)(k, h)) = [Ah, Ak].$$

Let's define the symbol $\gamma(h, k)$ as

$$(2.10) \quad \gamma(h, k) = [h, k] = Q'(0)(h, k) - Q'(0)(k, h)$$

In virtue of (2.8) and A is a monomorph operator in $C(R_x^n; F)$ then $\gamma(h, k)$ characterizes the Lie product in space of vectors R_x^n .

Directly

1. $\gamma(h, h) = [h, h] = Q'(0)(h, h) - Q'(0)(h, h) = 0;$
2. $\gamma(h, k) = Q'(0)(h, k) - Q'(0)(k, h) = [h, k] = -[Q'(0)(k, h) - Q'(0)(h, k)] = -\gamma(k, h) = -[k, h];$
3. $z \in R_x^n$ holds the Jacobi equality which is below

$$[h, \gamma(k, z)] + [k, \gamma(z, h)] + [z, \gamma(h, k)] = 0$$

Thus the space of vectors R_x^n is transformed to the Lie Algebra. If we put (2.10) in (2.9) we obtain

$$(2.11) \quad A[h, k] = [Ah, Ak] = AhAk - AkAh$$

It is obvious that the operator $A \in C(R_x^n; L(E; E))$ is a morphism which acts from the Lie algebra R_x^n to endomorphisms F of Banach space E . We also know that the operator A is monomorph operator. Now let us use the monomorphism of A , from (2.8) we find a commutator differential equation which depends on function $Q(x)$:

$$(2.12) \quad Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k]$$

Then if we divide both sides of (2.7) by 2,

$$\frac{1}{2}[A(Q(x)h)f(x)k - A(Q(x)k)f(x)h] + \frac{1}{2}[f'(x)hk - f'(x)kh] = 0$$

or

$$(2.13) \quad \Lambda_{hk}\{A(Q(x)h)f(x)k\} + \Lambda_{hk}\{f'(x)hk\} = 0, \forall h, k \in R_x^n$$

Therefore the following conditions are satisfied in order to get an exact solution of (2.1)

$$A[h, k] = [Ah, Ak];$$

$$\Lambda_{hk}\{A(Q(x)h)f(x)k\} + \Lambda_{hk}\{f'(x)hk\} = 0;$$

$$(2.14) \quad Q'(x)(h, k) - Q'(x)(k, h) = [Q(x)h, Q(x)k].$$

Since the theorem of G. Frobenius [4] for exact solvable multidimensional differential equation, the conditions in (2.14) are also sufficient conditions for (2.1).

■

Corollary 2. *Let $Q(x) = I$, then $\forall x \in R_x^n$ (2.1) and (2.2) transform mutually to the following equations:*

$$(2.15) \quad u'(x)h = Ahu(x) + f(x)h, \forall h \in R_x^n.$$

$$(2.16) \quad u(x_0) = u_0, x_0 \in S, u_0 \in E.$$

The necessary and sufficient conditions for exact solvable of (2.15) and (2.16) are the following

$$(2.17) \quad \Lambda_{hk}\{A(h)A(k)\} = 0, \forall h, k \in R_x^n,$$

$$(2.18) \quad \Lambda_{hk}\{A(h)f(x)k\} + \Lambda_{hk}\{f'(x)hk\} = 0$$

Here, for bilinear operator $B : R_x^n \oplus R_x^n \rightarrow C(R_x^n; F)$, Λ_{hk} is;

$$\Lambda_{hk}\{Bhk\} = \frac{1}{2}(Bhk - Bkh).$$

If the conditions (2.17) and (2.18) hold, then the solution of (2.15) and (2.16) follows

$$(2.19) \quad u(x) = e^{A(x-x_0)}u_0 + \int_{x_0}^x e^{A(x-\sigma)}f(\sigma)d\sigma$$

3. APPLICATIONS

Let G be m^2 -dimensional Lie group and $\alpha = (\alpha_{11}, \dots, \alpha_{mm})$ is the collection of m^2 parameters ($\det \alpha \neq 0$), characterizing the elements of this group, E_y is the complex Banach space. Let's denote by $M(G; L(E_y; E_y))$ the space of continuous operator valued maps, defined on the group G with the values in the algebra $L(E_y; E_y)$ of endomorphisms of Banach space E_y [6] - [7] - [8] - [9], with topology, given by the norm.

Suppose that the unity e of the group G corresponds to the zero values of the parameters $\alpha_{11}, \dots, \alpha_{mm}$. Let $T \in M(G; L(E_y; E_y))$ since the giving of the parameters simply determines the element of the group G then the operator $T(g) \in M(G; L(E_y; E_y))$, $g \in G$ can be considered as the function of the parameters $\alpha_{11}, \dots, \alpha_{mm}$, i.e. $T(g) = T(\alpha_{11}, \dots, \alpha_{mm}) = T(\alpha)$. Consider the operator equation of the form:

$$(3.1) \quad T(fg) = T(f)T(g), (f, g \in G)$$

with initial condition

$$(3.2) \quad T(e) = I, (e \in G; I \in L(E_y; E_y))$$

Where $T \in M(G; L(E_y; E_y))$, and $I \in L(E_y; E_y)$ are identical operators. Denote by $\alpha_{ij}(f)$, $i, j = 1, 2, \dots, m$ the parameters characterizing element f of the group G . If determining of the parameters $\alpha_{ij}(f)$, $\alpha_{ij}(g)$, $i, j = 1, 2, \dots, m$ simply determines the element and $f, g \in G$ also determines their product $fg \in G$, and consequently, the parameters $\alpha_{ij}(fg)$, $i, j = 1, 2, \dots, m$. Let φ_{ts} , $t, s = 1, 2, \dots, m$ are the functions of $2m^2$ parameters

$$\alpha_{ij}(f), \alpha_{ij}(g) : \alpha_{ts}(fg) = \varphi_{ts}\{\alpha_{11}(f), \dots, \alpha_{mm}(f); \alpha_{11}(g), \dots, \alpha_{mm}(g)\} \quad t, s = 1, 2, \dots, m$$

such $\varphi_{ts}(t, s = 1, 2, \dots, m)$ that are continuously differentiable functions of co-multipliers. It is not difficult to see that the operator equation (3.1) - (3.2) is equivalent to the following vector equation

$$(3.3) \quad y(f) = T(fg)y(g^{-1}), (f, g, g^{-1} \in G)$$

$$(3.4) \quad y(e) = \eta(e \in G; \eta \in E_y; T(f)\eta = y(f))$$

The following problem is formulated by us: it is required to find the solutions $y(t) \in E_y, f \in G$ of the functional problems (3.3) - (3.4), analytical in the neighborhood of the unique element $f = e \in G$.

Theorem 3. *Let the operator $T : G \rightarrow L(E_y; E_y)$ has continuous private derivatives for each of the parameters $\alpha_{ij}(f), i, j = 1, 2, \dots, m$ at the point $g = f^{-1} \in G$ and let the function $y(f)(f \in G)$ with continuous private derivatives $\frac{\partial y(f)}{\partial \alpha_{ts}(f)}(t, s = 1, 2, \dots, m)$ is the solution of the vector equation (3.3). Then there exists the unique finite collection of linear operators $\{A_{11}, A_{12}, \dots, A_{mm}\} \subset L(E_y; E_y)$ such that at the same time and the solution of overdetermined system of differential equations:*

$$(3.5) \quad (L_{ij}y)(f) \equiv \frac{\partial y(f)}{\partial \alpha_{ij}(f)} - \sum_{t=1}^m \sum_{s=1}^m c_{ts}^{ij}(f) A_{ts} y(f) = 0,$$

The operators $L_{11}, L_{12}, \dots, L_{mm} [L_{ij} \equiv \frac{\partial}{\partial \alpha_{ij}(f)} - \sum_{t=1}^m \sum_{s=1}^m c_{ts}^{ij}(f) A_{ts}]$ are called characteristic differential operators (CDO) of Lie, $c(e) = \{c_{ts}^{ij}(e)\}$ is $m \times m$ matrix such that $c_{ts}^{ij}(e) = \delta_{ts}^{ij}$ is Kronecker symbol. System of equations (3.5), according to Lie [10], we'll call if characteristics differential equations [GCDE] of Lie of the presentation the group G . This notion plays the important role in quantum mechanics [11]. It is true the following inverse.

Theorem 4. *If SCDE of Lie (3.5) has unique analytical solution in the neighborhood of unity element $e \in G$ with initial condition (3.4), then it is also solution of the vector equation (3.3).*

Definition 4. *SCDE of Lie (3.5) called absolutely integrable (absolutely solvable) on the set $G_a \times D_y \subset G \times E_y$ for any point $(e, \eta) \in G_0 \times D_y$ there exists unique solution $y(f)$ of this system, defined in some neighborhood of unity element e in $G_a \subset G$ which has the values in $D_y \subset E_y$ and satisfies the initial condition $y(e) = \eta$.*

Theorem 5. *Let $A_{ts} \in L(E_y; E_y), t, s = 1, 2, \dots, m$ are bounded, infinitesimal operators of the presentation Lie group. Suppose that the functions $\{G_{ts}^{ij}(f), i, j, t, s = 1, 2, \dots, m\}$ are continuously differentiable. The neighborhood of the unity element $e \in G_0$. SCDE of Lie (3.5) has unique analytical solution in the neighborhood of unity element $e \in G$ with initial condition (3.6) if and only if the conditions*

$$(3.6) \quad A_{\alpha\beta} A_{ts} - A_{ts} A_{\alpha\beta} = \sum_{t=1}^m \sum_{s=1}^m c_{\beta\beta ts}^{ij} A_{ij}, (\alpha, \beta, t, s = 1, 2, \dots, m)$$

are fulfilled. If the condition of absolutely integrability is satisfied, then the solution of the problem (3.4) - (3.5) can be presented in the form

$$(3.7) \quad y(f) = \eta + \sum_{k=1}^{\infty} \frac{1}{k!} \left\{ \sum_{i_1 j_1 \dots i_k j_k} \left(\frac{\partial y(f)}{\partial \alpha_{i_1 j_1}(f) \dots \partial \alpha_{i_k j_k}(f)} \right) \prod_{f=c}^k (\alpha_{i_t j_t}(f) - \alpha_{i_t j_t}(e)) \right\}$$

Where $\sum_{i_1 j_1 \dots i_k j_k} = \sum_{i_1}^m \sum_{j_1}^m \dots \sum_{i_k}^m \sum_{j_k}^m$; $c_{\alpha \beta ts}^{ij}$ are the structure real constants of Lie group, satisfying the conditions of antisymmetry and Jakobi identities, and the derivatives of k order of the function $y(f)$ at the point $f = e \in G$ are determined from the following recurrent operator correlations:

$$A_{i_1 j_1 \dots i_k j_k i_{k+1} j_{k+1}} = \sum_{t_1 s_1 \dots t_k s_k} \{ C_{t_1 s_1 i_{k+1} j_{k+1}}^{i_1 j_1} \delta_{t_2 s_2}^{i_2 j_2} \dots \delta_{t_k s_k}^{i_k j_k} + C_{t_2 s_2 i_{k+1} j_{k+1}}^{i_2 s_2} \delta_{t_1 s_1}^{i_1 j_1} \delta_{t_3 s_3}^{i_3 j_3} \dots \delta_{t_k s_k}^{i_k j_k} + C_{t_k s_k i_{k+1} j_{k+1}}^{i_k j_k} \delta_{t_1 s_1}^{i_1 j_1} \dots \delta_{t_{k-1} s_{k-1}}^{i_{k-1} j_{k-1}} \} \times A_{t_1 s_1 \dots t_k s_k} + A_{i_1 j_1 \dots i_k j_k} A_{i_{k+1} j_{k+1}};$$

the row (3.7) converges for

$$\sum_{t=1}^m \sum_{s=1}^m |\alpha_{ts}(f) - \alpha_{ts}(e)| < \frac{1}{2d} (d \geq 1)$$

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