Defining sets in (proper) vertex colorings of $P_m \times K_n$

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Abstract

What is the defining number of the vertex colorings of a graph? This question has been verified for many graphs (see references). Let $\chi(G)$ be the chromatic number of vertex colorings of $G$. In this note we show that the exact value of the defining number $d(G = P_m \times K_n, c)$ where $c > \chi(G)$, $n \geq 2$ and $m \geq 3$.

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1 Introduction

A $c$–coloring (proper $c$–coloring) of a graph $G$ is an assignment of $c$ different colors to the vertices of $G$, such that no two adjacent vertices receive the same color. The vertex chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number $c$, for which there exists a $c$–coloring for $G$. The maximum degree of the vertices in $G$ is $\Delta(G)$ and a graph in which every vertex has degree $k$ is $k$–regular graph (see [11]). In a given graph $G = (V, E)$, a set of vertices $S$ with an assignment of colors to them is said to be a defining set of the vertex coloring of $G$, if there exists a unique extension of the colors of $S$ to a $c \geq \chi(G)$ coloring of the vertices of $G$. A defining set with the minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, c)$. We will use standard notations such as $K_n$ for the complete graph on $n$ vertices, $P_m$ for the path on $m$ vertices, $C_m$ for the cycle of size $m$ and $G \times H$ for the cartesian product of $G$ and $H$. There are some papers on the defining set of graphs, specially $d(K_n \times K_n, \chi)$ (the critical set of Latin squares of order $n$), $d(C_m \times K_n, c \geq \chi)$, $d(G, \chi = k)$ where $G$ is a $k$–regular graph and the defining set on block designs. The interested reader may see [1], [3], [4], [5], [6], [7], [8], [9], [10] and their references.
The following results can be found in [5, 7].

(1) \( d(P_m \times K_n, \chi) = m(n - 3) + 2 \), for \( n \geq 6 \).

(2) \( d(C_m \times K_n, n + i) = m(n + i - 3) \) for \( i = 0 \) and \( n \geq 6 \) or \( i = 1, 2, 3 \) and \( n, m \geq 4 \).

(3) \( d(C_r \times K_3, 4) = r + 1 \) for even \( r \).

(4) \( r + 1 \leq d(C_r \times K_3, 4) \leq r + 2 \) for odd \( r \).

(5) \( d(C_r \times K_2, 4) = 2\lceil \frac{r}{2} \rceil \).

The followings are useful.

**Definition A.** [2] A graph \( G \) on \( n \) vertices, is called a uniquely 2-list colorable graph, if there exists \( S_1, S_2, \ldots, S_n \), a list of colors on its vertices, each of size 2, such that there is a unique coloring for \( G \) from this list of colors.

**Theorem B.** [2] A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.

Let \( G \) be a \( k \)-regular \( k \)-coloring graph. Let \( C \) be a cycle in \( G \), then each vertex of \( C \) has at least two choices for coloring, in other words \( C \) is at least 2-list vertex colorable, if all vertices of \( V(G) \setminus V(C) \) have been already colored. So by Theorem B the cycle \( C \) is not uniquely 2-list colorable. Now we have.

**Lemma C.** [6] Let \( G \) be a \( k \)-regular \( k \)-coloring graph. Then every cycle in \( G \) has a vertex in the defining set of \( G \).

If \( G = H \times K \), then each subgraph \( H \) of \( G \) is said to be a column and each subgraph \( K \) of \( G \) is said to be a row and \( H \times K \cong K \times H \).

It is well known that \( \chi(P_m \times K_n) = \chi(K_n \times P_m) = n \) for \( n \geq 2 \).

\[ 2 \quad d(P_m \times K_n, n + i) \]

In this section we derive \( d(P_m \times K_n, n + i) \) for \( n, m \geq 4 \) and \( i \geq 0 \). We start with the following lemma.

**Lemma 2.1.** Let \( G = P_m \times K_n \) be colored with \( n + i \) colors for \( 0 \leq i \leq 3 \). Then the first and the last rows have at least \( n + i - 2 \) vertices in the defining set and the other rows have at least \( n + i - 3 \) vertices in the defining set.

**Proof.** Assume that, the defining set contains \( k < n + i - 2 \) vertices of the first row, the last row or contains \( k < n + i - 3 \) vertices of the other than row. If we color all the other rows completely. Then the induced subgraph of the non-coloring vertices of this row is a perfect and cannot be forced (uniquely colorable) by Theorem B.

**Theorem 2.1.** For \( n, m \geq 4 \), \( d(P_m \times K_n, n + 1) = m(n - 2) + 2 \).
Proof. Let \( G = P_m \times K_n \). By Lemma 2.1, \( d(G, n + 1) \geq m(n - 2) + 2 \). To show equality we give a defining set \( S \) of size \( m(n - 2) + 2 \). For this order it is sufficient to delete the index of one of the suitable vertices of the first and the last rows of the table of \((n + 1)\)-colorings of \( C_m \times K_n \), (see [7, Th. 2.1]) and set them to the defining set. Therefore the non-indexed vertices of the revised table are the defining set of \((n + 1)\)-colorings of \( G = P_m \times K_n \). □

**Theorem 2.2.** For \( n, m \geq 4 \), \( d(P_m \times K_n, n + 2) = m(n - 1) + 2 \).

Proof. Let \( G = P_m \times K_n \). By Lemma 2.1, \( d(G, n + 2) \geq m(n - 1) + 2 \). To show equality we give a defining set \( S \) of size \( m(n - 1) + 2 \). For this order it is sufficient to delete the index of the suitable vertex of the first and the last rows of the table of \((n + 2)\)-colorings of \( C_m \times K_n \), (see [7, Th. 2.2]) and set them to the defining set. Therefore the non-indexed vertices of the revised table are the defining set of \((n + 2)\)-colorings of \( G = P_m \times K_n \). □

**Lemma 2.2.** Let \( G = (V, E) \) be a graph with \( c \geq \Delta(G) + 2 \). Then \( d(G, c) = |V| \).

Proof. Let \( S \) be a defining set of \( G \) and \( v \) be a vertex for which \( v \notin S \). So if all of the neighbors of vertex \( v \) are colored then the vertex \( v \) has at least two choices for coloring. □

**Theorem 2.3.** For \( n, m \geq 4 \), \( d(P_m \times K_n, n + i) = mn \) where \( i \geq 3 \).

Proof. The degree of any vertex in \( P_m \times K_n \) is \( n \) or \( n + 1 \), \( |V(P_m \times K_n)| = mn \) and for \( i \geq 3 \), \( n + i \geq \Delta(P_m \times K_n) + 2 \). Now we use the Lemma 2.2. □

3 \( d(K_3 \times P_m, c > \chi) \)

Note that \( \chi(K_3 \times P_m) = 3 \).

**Lemma 3.1.** Let \( G = K_3 \times P_r \). Then \( d(G, 4) \geq r + 2 \).

Proof. There exist at least two vertices of each of the first and the last columns and at least one vertex of the other columns in the defining set. Thus \( d(G, 4) \geq r + 2 \). □

**Theorem 3.1.** Let \( G = K_3 \times P_r \). Then \( d(G, 4) = r + 2 \).

Proof. Let \( G = K_3 \times P_r \). From Lemma 3.1 we obtain \( d(G, 4) \geq r + 2 \). We give a defining set \( S \) of size \( r + 2 \).

Let \( v_1, v_2, \cdots, v_r \) be the vertices of the first row, \( u_1, u_2, \cdots, u_r \) be the vertices of the second row and \( w_1, w_2, \cdots, w_r \) the vertices of the third row.

We determine the defining set with their colors as follows.
Finally, let $c(w_1) = 2$ and $c(w_r) = 4$. Therefore $d(G, 4) = r + 2$. 

**Lemma 3.2.** Let $G = (V, E)$ be a graph. Let $S$ be a defining set of $G$ with $c = \Delta(G) + 1$. If $v$ is a vertex and $\deg(v) \leq \Delta(G) - 1$ then $v \in S$. If $\deg(v) = \Delta(G)$ then $v \in S$ or all neighbors of $v$ are in $S$.

**Proof.** Any vertex $v$ with $\deg(v) \leq \Delta(G) - 1$ and not in $S$ has two choices for coloring eventually all of the neighbors are colored. If $\deg(v) = \Delta(G)$, vertex $u$ is a neighbor of $v$, $(u, v \not\in S)$ and all the other neighbors of $v$ are in $S$ then each of both have two choices for coloring. 

**Theorem 3.2.** Let $G = K_3 \times P_r$. Then $d(G, 5) = 2r + 2$.

**Proof.** Let $G = K_3 \times P_r$. From Lemma 3.2 we obtain $d(G, 5) \geq 2r + 2$. To show equality we give a defining set, $S$ of size $2r + 2$.

Let $v_1, v_2, \ldots, v_r$ be the vertices of the first row, $u_1, u_2, \ldots, u_r$ be the vertices of the second row and $w_1, w_2, \ldots, w_r$ the vertices of the third row.

We determine the defining set with their colors as follows.

$$
c(v_m) = \begin{cases} 
1 & \text{if } m \equiv 1 \pmod{6} \\
2 & \text{if } m \equiv 3 \pmod{6} \\
3 & \text{if } m \equiv 5 \pmod{6} \\
4 & \text{if } m = r \text{ is even}
\end{cases}
$$

$$
c(u_m) = \begin{cases} 
3 & \text{if } m \equiv 2 \pmod{6} \\
1 & \text{if } m \equiv 4 \pmod{6} \\
2 & \text{if } m \equiv 0 \pmod{6} \\
5 & \text{if } m = 1 \text{ and } m = r \text{ is odd}
\end{cases}
$$

$$
c(w_m) = \begin{cases} 
4 & \text{if } m \text{ is odd} \\
5 & \text{if } m \text{ is even}
\end{cases}
$$

\[\square\]
Defining sets in (proper) vertex colorings

4 \quad d(K_2 \times P_m, c > \chi)

Note that \( \chi(K_2 \times P_m) = 2 \).

**Lemma 4.1.** Let \( G = K_2 \times P_r \). Let \( G \) be colored with 3 colors. Then
1. Every two successive columns have at least one vertex in the defining set.
2. Each of the first and the last columns has at least one vertex in the defining set.

**Proof.**
1. Every two successive columns consist a cycle \( C_4 \). So if two columns have no vertex in the defining set, then every vertex of the cycle \( C_4 \) has two choices for coloring. Since it has no vertex in the defining set, thus by Theorem B this is impossible.
2. Let the first column or the last column has no vertex in the defining set. Then we have a complete graph \( K_2 \) which its vertices has two choices for coloring. This is impossible either.

**Theorem 4.1.** Let \( G = K_2 \times P_{2n} \). Then \( d(G, 3) = n + 1 \).

**Proof.** Let \( G = K_2 \times P_{2n} \). From Lemma 4.1 we obtain \( d(G, 3) \geq n + 1 \). To show equality we give a defining set, \( S \) of size \( n + 1 \).

Let \( v_1, v_2, \ldots, v_{2n} \) be the vertices of the first row and \( u_1, u_2, \ldots, u_{2n} \) be the vertices of the second row. we determine the defining set with their colors as follows.

\[
c(v_m) = \begin{cases} 
1 & \text{if } m \equiv 1 \pmod{4} \\
2 & \text{if } m = 2n \text{ and } 2n \equiv 0 \pmod{4}
\end{cases}
\]

also

\[
c(u_m) = \begin{cases} 
1 & \text{if } m \equiv 3 \pmod{4} \\
2 & \text{if } m = 2n \text{ and } 2n \equiv 2 \pmod{4}
\end{cases}
\]

**Lemma 4.2.**
1. \( d(G = K_2 \times P_3, 3) \geq 3 \).
2. Let \( G = K_2 \times P_{2n+1} \) and \( G \) be colored with 3 colors. Let every two successive columns have one vertex in the defining set. Then there exist three successive columns such that they have at least three vertices in the defining set.

**Proof.**
1. It is obvious that two colorings of the vertices of \( G = K_2 \times P_3 \) doesn’t force the coloring of the other vertices.

2. Contrarily assume that every three successive columns have at most two vertices in the defining set then by part 1, the non-coloring vertices cannot be forced. This is a contradiction.

**Theorem 4.2.** Let \( G = K_2 \times P_{2n+1} \). Then \( d(G, 3) = n + 2 \).
Proof. Let $G = K_2 \times P_{2n+1}$. From Lemma 4.3 we obtain $d(G, 3) \geq n + 2$. To show equality we give a defining set, $S$ of size $n + 2$.

Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices of the first row and $u_1, u_2, \ldots, u_{2n+1}$ be the vertices of the second row. We determine the defining set with their colors as follows.

\[
c(v_m) = \begin{cases} 
1 & \text{if } m \equiv 1 \pmod{4} \\
2 & \text{if } m = 2n + 1 \quad \text{and} \quad 2n + 1 \equiv 3 \pmod{4}
\end{cases}
\]

Also

\[
c(u_m) = \begin{cases} 
1 & \text{if } m \equiv 3 \pmod{4} \\
2 & \text{if } m = 2n + 1 \quad \text{and} \quad 2n + 1 \equiv 1 \pmod{4}
\end{cases}
\]

\[\square\]

**Theorem 4.3.** If $G = K_2 \times P_r$ then $d(G, 4) = r + 2$.

**Proof.** Let $G = K_2 \times P_r$. From Lemma 3.2 we obtain $d(G, 4) \geq r + 2$. To show equality we give a defining set, $S$ of size $r + 2$.

Let $v_1, v_2, \ldots, v_r$ be the vertices of the first row, $u_1, u_2, \ldots, u_r$ be the vertices of the second row.

We determine the defining set with their colors as follows.

\[
c(v_m) = \begin{cases} 
1 & \text{if } m \equiv 1 \pmod{6} \\
3 & \text{if } m \equiv 3 \pmod{6} \\
2 & \text{if } m \equiv 5 \pmod{6}
\end{cases}
\]

\[
c(u_m) = \begin{cases} 
2 & \text{if } m \equiv 2 \pmod{6} \\
1 & \text{if } m \equiv 4 \pmod{6} \\
3 & \text{if } m \equiv 0 \pmod{6}
\end{cases}
\]

If $r$ is even we set $c(v_r) = c(u_1) = 4$. If $r$ is odd we set $c(u_1) = c(u_r) = 4$. \[\square\]

**Corollary 4.4.** $d(K_2 \times P_r, 5) = 2r$.

**Proof.** By Lemma 2.2, each of column has at least 2 vertices in defining set. Therefore all the vertices are in the defining set. \[\square\]

**References**


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