Independent edge dominating set of certain graphs

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Abstract

Let $G = (V, E)$ be a finite undirected graph. A set F of the edges of a graph G is an edge dominating set if each edge of $E \setminus F$ is adjacent to an edge of F . The edge domination number $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set of G . An independent set of edges is a set of edges, no two of which have a vertex in common. An independent edge dominating set is an independent set of edges which is edge dominating set. In this paper we are concerned with the problems of verifying a minimum independent edge dominating set in the Cartesian Product of paths $P_k \times P_n$, $P_k \times C_n$ and $C_k \times C_n$ for every positive integer k and n .

1 Introduction

An edge in a graph G *dominates* itself and its adjacent. A set of edges F in a graph G is an *edge dominating set*, if each edge of G is dominated by some edge of F . The *domination number* $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set of G . An independent set of edges is a set of edges, no two of which have a vertex in common. An *independent edge dominating set* is an independent set of edges which is edge dominating set. If G is a graph it is easy to show that G has a minimum edge dominating set F that is independent, (see [1] and [2]). So we consider $\gamma'(G)$ as an independent edge dominating set. There are some papers on the edge dominating sets (see

$[1, 2, 4, 5, 6, 7, 8]$). The Cartesian product of G and H , denoted $G \times H$, has vertex set $V(G) \times V(H)$. Two vertices $(u, v), (u', v')$ in $V(G) \times V(H)$ are adjacent if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

2 Results

Notations $\gamma'_{k,n}$ denotes the size of a minimum independent edge dominating set of $P_k \times P_n$.

In a graph $P_k \times P_n$ the set $C_j = \{(1, j), (2, j), \dots, (k, j)\}$ denotes the vertices in j^{th} column of $P_k \times P_n$. A sequence $(s_1, s_2, \dots, s_n)$ of non-negative integer is said to be a dominating sequence for $P_k \times P_n$ if there exists an independent edge dominating set F of $P_k \times P_n$ such that $s_j = |C_j \cap S|$ for $j = 1, 2, \dots, n$ where S is the set of all saturated vertices under F . In this case it is clear that $\gamma'_{k,n}$ is the minimum of $\lceil (s_1 + s_2 + \dots + s_n)/2 \rceil$ over all dominating sequence $(s_1, s_2, \dots, s_n)$.

Proposition 1 For every n , $\gamma'_{3,n} = n$.

Proof. The graph $P_3 \times P_n$ has n copy of the graph P_3 in n columns. Suppose that the set F is composed of the first edge of $(2k - 1)^{th}$ columns for $k = 1, 2, \dots, \frac{n+1}{2}$ and the second edge of the $(2k)^{th}$ columns for $k = 1, 2, \dots, \frac{n}{2}$. Thus F is an independent edge dominating set of $P_3 \times P_n$ such that $|F| = n$. Now we prove $\gamma'_{3,n} \geq n$. On the contrary suppose $\gamma'_{3,n} \leq n - 1$. Then $\sum_{j=1}^n s_j < 2n - 1$. Without loss of generality we can assume that D' is a minimum independent edge dominating set of cardinality $n - 1$ and $\sum_{j=1}^n s_j = 2n - 2$. There are some points:

- 1) $1 \leq s_j \leq 3$ for $j = 1, 2, \dots, n$.
- 2) If $s_j = 1$ then $(2, j)^{th}$ vertex of j^{th} column is saturated.
- 3) $s_1 + s_2 \geq 4$ and $s_{n-1} + s_n \geq 4$.

The points (1) and (2) are clear. The point (3), has one of the three subcases (a) or (b) or (c).

- a) If $s_1 = 1$ then each 3 vertex of second column should be covered. Hence $s_1 + s_2 \geq 4$.

- b) If $s_1 = 2$ then at least 2 vertex of the second column be covered. Thus the case (3) holds.
- c) If $s_1 = 3$, since $s_2 \geq 1$ then $s_1 + s_2 \geq 4$.

Notice to the point (1), at least 2 column of the graph contain exactly 1 saturated vertex . Because

$$2n - 2 = \sum_{j=1}^n s_j = s_1 + s_2 + \sum_{j=3}^n s_j \geq 4 + \sum_{j=3}^n s_j \implies 2n - 2 - 4 = 2n - 6 \geq \sum_{j=3}^n s_j.$$

From the third column to the n^{th} column the sequence $(s_3, s_4, \dots, s_n)$ consisted of $n - 2$ components such that have at most $2n - 6$ saturated vertices. But for then $2n - 6 - (n - 2) = 2n - 6 - n + 2 = n - 4$. Thus $n - 4$ vertices remain which must repose in at most $n - 2$ columns and then at least 2 columns consist exactly 1 saturated vertex.

- 4) If the number of columns with 1 saturated vertex is $m > 2$ then the number of column with 3 saturated vertices is $m - 2$.

- 5) For each i , $(s_{i-1}, s_i, s_{i+1}) \notin \{(1, 1, 2), (1, 2, 2), (1, 2, 1), (2, 2, 1), (2, 1, 2)\}$.

- 6) $(s_i, s_{i+1}) = (1, 1)$ is not possible.

- 7) If $s_i = 1$ then we have

- a) $s_{i-1} \geq 2, s_{i+1} = 3$.
- b) $s_{i-1} = 3, s_{i+1} \geq 2$.

- 8) If 2 columns has exactly 1 saturated vertex, for instance $C_i, C_j (i, j \geq 3)$ then $C_t (3 \leq t \neq i, j)$ has 2 saturated vertices that by using (5) and (6) the edges $P_3 \times P_n$ are not dominated completely.

- 9) If the number of columns with 1 saturated vertex is $m > 2$ then $m - 1$ columns have 3 saturated vertices. By using (5), (4) this case is impossible. Then $\sum_{j=1}^n s_j > 2n - 2$, that means $\gamma'_{3,n} = n$. ■

Proposition 2 For every $n \equiv 0, 2 \pmod{3}$, $\gamma'_{4,n} = \lceil 4n/3 \rceil$.

Proof. We first show $\gamma'_{4,n} \leq \lceil 4n/3 \rceil$. Each row of graph $P_4 \times P_n$ consists of 2 distinct path P_3 .

i) Suppose $n \equiv 0 \pmod{3}$. We consider the first edge of path P_4 from the first and the third rows and the second edge of path P_4 from the second and forth rows, then the set F consisted of the above edges is an independent edge dominating set of $P_4 \times P_n$ such that $|F| = 4n/3$.

ii) Suppose $n \equiv 2 \pmod{3}$.

Using (i), the graph $P_4 \times P_{n-2}$ dominate by $4(n-2)/3$ edges and 2 columns remain that can be dominated by 3 edges. So $\gamma'_{4,n} \leq \lceil 4n/3 \rceil$.

Now we find an independent edge dominating set of size $\lceil 4n/3 \rceil$.

We note that to 6 points.

- 1) $s_i \geq 2$ for $i = 1, 2, \dots, n$.
- 2) $s_1 + s_2 \geq 5$ and $s_{n-1} + s_n \geq 5$.
- 3) $(s_{i-1}, s_i, s_{i+1}) \notin \{(3, 2, 2), (2, 2, 3), (2, 3, 2), (2, 2, 2)\}$.
- 4) $s_{i-1} + s_i + s_{i+1} \geq 8$ for $i \geq 2$.
- 5) Suppose $n \equiv 0 \pmod{3}$ then $\sum_{i=1}^n s_i \geq 8n/3$. In this case at least the number of dominating edges with $8n/3$ saturated vertices is $4n/3$.
- 6) Suppose $n \equiv 2 \pmod{3}$, ($n = 3k + 2$) then

$$\sum_{i=1}^n s_i = \sum_{i=1}^{n-2} s_i + s_{n-1} + s_n \geq 8(n-2)/3 + 5 = 8k + 5.$$

At least of number of edges for saturating of $8k + 5$ vertices is $4k + 3$ edges too. Thus

$$\lceil 4n/3 \rceil = \lceil 4(3k+2)/3 \rceil = \lceil (12k+8)/3 \rceil = 4k + \lceil 8/3 \rceil = 4k + 3.$$

Therefore the proof is complete. ■

Corollary 3 $\gamma'(P_n \times C_m) \leq mn/3$ such that $m \equiv 0 \pmod{3}$ also $\gamma'(P_n \times P_m) \leq mn/3$.

Proof. Each horizontal cycle C_m partitions to $m/3$ distinct P_3 paths. Thus There exist $mn/3$ distinct P_3 paths in $(P_n \times C_m) \leq mn/3$. Let F be consisted of the first edges of P_3 paths from all of the odd rows and the second edges of P_3 paths from all of the even rows. Thus F is an independent edge dominating set, so $\gamma'(P_n \times C_m) \leq mn/3$. This reason is valid for $\gamma'(P_n \times P_m)$. ■

Also we have.

Corollary 4 Suppose $m \equiv 0 \pmod{3}$ and n is even then $\gamma'(C_n \times C_m) \leq mn/3$.

Proof. If we use the sketch of the proof of Corollary 3, all of edges of C_m s are dominated, while the edges of C_n s are dominated for even n . ■

Corollary 5 If $m \equiv 0 \pmod{3}$, n is odd, Then $\gamma'(C_n \times C_m) \leq m(n+1)/3$.

Proof. If we use the proof of the Corollary 3, except one edge of each of the $3k^{th}$ columns for $1 \leq k \leq m/3$ the other edges of $C_n \times C_m$ are dominated by $mn/3$ edges, but in this case the $m/3$ residue edges must be added to the dominating set. Thus $\gamma'(C_n \times C_m) \leq m(n+1)/3$. ■

It is easily seen that

Corollary 6 If $m = 5, 7$ then $\gamma'(P_m \times P_n) \leq \lceil mn/3 \rceil$.

Example 7 For $m = 7$ and $n = 8$ consider the set of edges

$$\begin{aligned} &\{(1, 2), (1, 3)\}, \{(1, 5), (1, 6)\}, \{(2, 1), (2, 2)\}, \{(2, 4), (2, 5)\}, \{(3, 2), (3, 3)\}, \\ &\{(3, 5), (3, 6)\}, \{(4, 1), (4, 2)\}, \{(4, 4), (4, 5)\}, \{(4, 7), (4, 8)\}, \{(5, 2), (5, 3)\}, \\ &\{(5, 5), (5, 6)\}, \{(6, 1), (6, 2)\}, \{(6, 4), (6, 5)\}, \{(7, 2), (7, 3)\}, \{(7, 5), (7, 6)\}, \\ &\{(1, 8), (2, 8)\}, \{(2, 7), (3, 7)\}, \{(5, 8), (6, 8)\}, \{(6, 7), (7, 7)\} \end{aligned}$$

of $P_7 \times P_8) = \lceil mn/3 \rceil$. This edge set dominates $P_7 \times P_8$. Thus $\gamma'_{7,8} = 19 = \lceil 56/3 \rceil$.

Finally we have.

Problems

- 1) If $m \equiv 0 \pmod{3}$, n is odd, Then $\gamma'(C_n \times C_m) = m(n+1)/3$.
- 2) $\gamma'(P_n \times P_m) = \lceil mn/3 \rceil$.

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Received: April 19, 2006