EQUATION RELATED TO DERIVATIONS ON STANDARD OPERATOR ALGEBRAS¹

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Abstract

In this paper we prove the following result. Let X be a real or complex Banach space, let $A(X)$ be a standard operator algebra on X and let $m \geq 0$ and $n \geq 0$ be integers with $m + n \neq 0$. Suppose $D : A(X) \rightarrow L(X)$ is an additive mapping satisfying the relation $D(A^{m+n+1}) = (m + n + 1)A^m D(A)A^n$ for all $A \in A(X)$. In this case $D = 0$.

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PRELIMINARIES

Throughout this paper R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be n -torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if $aRb = (0)$

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implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \rightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [7] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation (see [1] for an alternative proof). Cusack [6] has generalized Herstein's theorem to 2-torsion free semiprime rings (see [2] for an alternative proof). A mapping F of a ring R into itself is called commuting (centralizing) on R in case $[F(x), x] = 0$ ($[F(x), x] \in Z(R)$) holds for all $x \in R$. The theory of commuting and centralizing mappings was initiated by a result of Posner [8] (Posner's second theorem), which states that the existence of a nonzero centralizing derivation $D : R \rightarrow R$, where R is a prime ring, forces the ring to be commutative. For results concerning commuting and centralizing mappings we refer to [3, 4, 9, 10] where further references can be found. Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem.

RESULTS

Brešar and Vukman [3] have proved the following result.

Theorem A. Let R be a 2 and 3-torsion free noncommutative prime ring and let $D : R \rightarrow R$ be an additive mapping satisfying the relation

$$D(x^2) = 2xD(x) \quad (1)$$

for all $x \in R$. In this case $D = 0$.

In general the result above cannot be proved in case we have a noncommutative semiprime ring (see [3] for an example). Vukman [11] has proved

that in case there exists an additive mapping $D : R \rightarrow R$, where R is a 2 and 3-torsion free semiprime, satisfying the relation (1), then D is a derivation which maps R into $Z(R)$. In [9] one can find the following result.

Theorem B. Let R be a 2 and 3-torsion free noncommutative prime ring with identity element and let $D : R \rightarrow R$ be an additive mapping satisfying the relation

$$D(x^3) = 3xD(x)x \quad (2)$$

for all $x \in R$. In this case $D = 0$.

Let us point out that any commuting derivation on an arbitrary ring satisfies the relations (1) and (2). The results we have just mentioned above lead to the following conjecture.

Conjecture 1. Let R be a semiprime ring with suitable torsion restrictions and let $m \geq 0$ and $n \geq 0$ be integers with $m + n \neq 0$. Suppose $D : R \rightarrow R$ is an additive mapping such that

$$D(x^{m+n+1}) = (m + n + 1)x^m D(x)x^n$$

is fulfilled for all $x \in A$. In this case D maps R into $Z(R)$. In case R is a noncommutative prime ring we have $D = 0$.

We adopt the convention $x^0 y = y x^0 = y$ for all pairs $x, y \in R$. It is our aim in this paper to prove the following result which is in the spirit of the conjecture above.

Theorem. Let X be a real or complex Banach space, let $A(X)$ be a standard operator algebra on X and let $m \geq 0$ and $n \geq 0$ be integers with $m + n \neq 0$. Suppose $D : A(X) \rightarrow L(X)$ is an additive mapping satisfying the relation

$$D(A^{m+n+1}) = (m+n+1)A^m D(A)A^n$$

for all $A \in A(X)$. In this case $D = 0$.

In the proof of Theorem we shall use Theorem A and the result below proved by Brešar [4].

Theorem C. Let R be a 2-torsion free prime ring. Suppose there exists an additive mapping $F : R \rightarrow R$ such that the mapping $x \mapsto [F(x), x]$ is commuting on R . In this case F is commuting on R .

We are ready to prove Theorem.

Proof of Theorem. We have the relation

$$D(A^{m+n+1}) = (m+n+1)A^m D(A)A^n \quad (3)$$

for all $A \in A(X)$. First we shall restrict D on $F(X)$. We intend to prove that D vanishes on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$, be a projection with $AP = PA = A$. From the above relation we obtain

$$D(P) = (m+n+1)PD(P)P.$$

Multiplying the relation above from both sides by P we obtain

$$PD(P)P = 0,$$

which gives

$$D(P) = 0. \quad (4)$$

Putting $A + P$ for A in the relation (3) and using (4), we obtain

$$\sum_{i=0}^{m+n+1} \binom{m+n+1}{i} D(A^{m+n+1-i}) =$$

$$(m+n+1) \left(\sum_{i=0}^m \binom{m}{i} A^{m-i} \right) D(A) \left(\sum_{i=0}^n \binom{n}{i} A^{n-i} \right). \quad (5)$$

Using (3) and collecting together terms of (5) involving the same number of factors of P , we obtain

$$\sum_{i=1}^{m+n} f_i(A, P) = 0, \quad (6)$$

where $f_i(A, P)$ stands for the expression of terms involving i factors of P .

Replacing A by $A + 2P$, $A + 3P$, ..., $A + (m+n)P$ in turn in the equation (3), and expressing the resulting system of $m+n$ homogeneous equations, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+n} \\ \vdots & \vdots & \vdots & \vdots \\ m+n & (m+n)^2 & \cdots & (m+n)^{m+n} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular,

$$f_{m+n-2}(A, P) = \binom{m+n+1}{m+n-2} D(A^3) -$$

$$(m+n+1) \left(\binom{m}{m-2} A^2 D(A) P + \binom{m}{m-1} \binom{n}{n-1} A D(A) A + \binom{n}{n-2} P D(A) A^2 \right) =$$

$$0,$$

$$f_{m+n-1}(A, P) = \binom{m+n+1}{m+n-1} D(A^2) -$$

$$(m+n+1) \left(\binom{m}{m-1} A D(A) P + \binom{n}{n-1} P D(A) A \right) = 0,$$

and

$$f_{m+n}(A, P) = D(A) - PD(A)P = 0.$$

The above relations reduce to

$$(m+n)(m+n-1)D(A^3) = 3m(m-1)A^2D(A)P + 6mnAD(A)A + 3n(n-1)PD(A)A^2, \quad (7)$$

$$(m+n)D(A^2) = 2mAD(A)P + 2nPD(A)A, \quad (8)$$

and

$$D(A) = PD(A)P. \quad (9)$$

From the relation (9) one can conclude that D maps $F(X)$ into itself. Applying the relation (9) and the fact that $AP = PA = A$, we obtain $AD(A)P = A(PD(A)P) = AD(A)$. Similarly one obtains that $PD(A)A = D(A)A$ is fulfilled. The relation (8) can now be written as

$$(m+n)D(A^2) = 2mAD(A) + 2nD(A)A. \quad (10)$$

Multiplying the relation (9) first from the right side and then from the left side by P and comparing the relations so obtained we obtain

$$D(A)P = PD(A).$$

Using the above relation the relation (7) reduces to

$$(m+n)(m+n-1)D(A^3) = 3m(m-1)A^2D(A) + 6mnAD(A)A + 3n(n-1)D(A)A^2. \quad (11)$$

Let us point out that relations (10) and (11) are fulfilled for all $A \in F(X)$. Let us first assume that $n = 0$. In this case the relation (10) reduces to

$$D(A^2) = 2AD(A)$$

for all $A \in F(X)$. Now we have an additive mapping D which maps $F(X)$ into itself satisfying the relation above. Since $F(X)$ is prime it follows from Theorem A that $D(A) = 0$ for all $A \in F(X)$. Similarly, we obtain because of left right symmetry that $D(A) = 0$ for all $A \in F(X)$ in case $m = 0$. We have therefore proved that D vanishes on $F(X)$ in case $mn = 0$. Let us now assume that $mn \neq 0$. We intend to prove that in this case the mapping $A \mapsto [D(A), A]$ is commuting on $F(X)$. The substitution $A + B$ for A in (10) gives

$$(m+n)D(AB+BA) = 2mAD(B) + 2mBD(A) + 2nD(A)B + 2nD(B)A. \quad (12)$$

Putting $B = (m+n)A^2$ in the relation above we obtain

$$(m+n)^2D(A^3) = m(m+n)AD(A^2) + m(m+n)A^2D(A) + n(m+n)D(A)A^2 + n(m+n)D(A^2)A. \quad (13)$$

According to (10) the above relation reduces to

$$(m+n)^2D(A^3) = (3m^2 + mn)A^2D(A) + 4mnAD(A)A + (3n^2 + mn)D(A)A^2. \quad (14)$$

Subtracting (11) from (14) we obtain

$$(m+n)D(A^3) = m(n+3)A^2D(A) - 2mnAD(A)A + \\ n(m+3)D(A)A^2. \quad (15)$$

From the above relation we obtain

$$(m+n)^2D(A^3) = (m+n)m(n+3)A^2D(A) - 2(m+n)mnAD(A)A + \\ (m+n)n(m+3)D(A)A^2. \quad (16)$$

Subtracting (16) from (14), one obtains

$$mn(m+n+2)A^2D(A) - 2mn(m+n+2)AD(A)A + \\ mn(m+n+2)D(A)A^2 = 0.$$

Since we have assumed that $mn \neq 0$ the above relation reduces to $D(A)A^2 + A^2D(A) - 2AD(A)A = 0$, which can be written in the form

$$[[D(A), A], A] = 0 \quad (17)$$

for all $A \in F(X)$. Now Theorem C makes it possible to conclude that $[D(A), A] = 0$, for all $A \in F(X)$. In other words D is commuting on $F(X)$. The fact that D is commuting on $F(X)$ makes it possible to replace $D(A)A$ in (10) by $AD(A)$. The relation (10) reduces to $D(A^2) = 2AD(A)$. Applying again Theorem A we obtain that $D(A) = 0$ for all $A \in F(X)$. It remains to prove that D vanishes on $A(X)$ as well. Let $A \in A(X)$ and let $S = A + PAP - (AP + PA)$. Note that S can be written in the form $S = (I - P)A(I - P)$, where I denotes the identity operator on X . Since, obviously, $S - A \in F(X)$ and since D vanishes on $F(X)$ we have $D(S) = D(A)$. Besides, $SP = PS = 0$.

$$\begin{aligned}
(m+n+1)S^m D(S)S^n &= D(S^{m+n+1}) = D(S^{m+n+1} + P) = D((S+P)^{m+n+1}) = \\
(m+n+1)(S+P)^m D(S+P)(S+P)^n &= (m+n+1)(S^m + P)D(S)(S^n + P) = \\
(m+n+1)S^m D(S)S^n &+ (m+n+1)PD(S)S^n + \\
(m+n+1)S^m D(S)P &+ (m+n+1)PD(S)P.
\end{aligned}$$

We have therefore

$$PD(A)S^n + S^m D(A)P + PD(A)P = 0. \quad (18)$$

Multiplying both sides of the above relation by P we obtain

$$PD(A)P = 0 \quad (19)$$

which reduces the relation (18) to

$$PD(A)S^n + S^m D(A)P = 0.$$

Right multiplication of the above relation by P gives

$$S^m D(A)P = 0. \quad (20)$$

We intend to prove that

$$S^{m-1}D(A)P = 0. \quad (21)$$

Putting $A + kB$ for A , $k = 1, 2, \dots, n$, where $B \in F(X)$ in the relation (20), using van der Monde procedure (see [5] for the details) and the fact that D vanishes on $F(X)$, we obtain

$$(S_1 S^{m-1} + S^2 S_1 S^{m-2} + \dots + S^{m-1} S_1) D(A) P = 0,$$

where S_1 stands for $(I-P)B(I-P)$. Putting in the above relation $D(A)PB$ for B and applying the relation (19) we obtain

$$(D(A)P B S^{m-1} + S D(A)P B S^{m-2} + \dots + S^{m-2} D(A)P B) D(A) P = 0.$$

Multiplying the above relation from the left side by S^{m-1} and applying the relation (20) we obtain $(S^{m-1} D(A) P) B (S^{m-1} D(A) P) = 0$, for all $B \in F(X)$ which gives $S^{m-1} D(A) P = 0$ by primeness of $F(X)$ which proves the relation (21). Now, according to the fact that the relation (20) implies the relation (21) one can conclude by induction that $S D(A) P = 0$, which gives

$$A D(A) P - P A D(A) P = 0.$$

Putting in the above relation again $A + B$ for A , where $A \in A(X)$ and $B \in F(X)$, we obtain

$$0 = (A + B) D(A) P - P (A + B) D(A) P = B D(A) P - P B D(A) P.$$

We have therefore proved that

$$B D(A) P - P B D(A) P = 0$$

holds for any $A \in A(X)$ and all $B \in F(X)$. Putting in the above relation $D(A)PB$ for B and applying the relation (19), we obtain $(D(A)P)B(D(A)P) = 0$, for all $B \in F(X)$, whence it follows $D(A)P = 0$. Since P is an arbitrary one-dimensional projection, one can conclude that $D(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem.

Let R be a ring and let $D : R \rightarrow R$ be an additive mapping satisfying the relation

$$(m+n)D(x^2) = 2mD(x)x + 2nxD(x),$$

for all $x \in R$. The relation above is suggested by the proof of Theorem (see the relation (10)). In case $m = 0$, $n = 1$ the above relation reduces to $D(x^2) = 2xD(x)$, $x \in R$. If in addition R is a 2 and 3-torsion free semiprime ring D is a derivation which maps R into $Z(R)$, as we have already mentioned. These observations lead to the following conjecture.

Conjecture 2. Let R be a semiprime ring with suitable torsion restrictions. Suppose $D : R \rightarrow R$ is an additive mapping satisfying the relation

$$(m+n)D(x^2) = 2mD(x)x + 2nxD(x)$$

for all $x \in R$ and some integers $m \geq 0$, $n \geq 0$, $m+n \neq 0$, $m \neq n$. In this case D is a derivation which maps R into $Z(R)$.

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