Statistical Convergence of Double Sequences of Real-Valued Functions

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Abstract

In this study, we introduce the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. We also give the relations between these convergences and pointwise, uniform convergence. Furthermore, we introduce the concept of statistically Cauchy sequence and study statistical analogue of convergence and Cauchy criterion for double sequences of real-valued functions.

Keywords: Double sequence, Double natural density, Pointwise statistical convergence

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1. Introduction and background

By the convergence of a double sequence we mean the convergence in Pringsheim's sense[6]. A double sequence $x=(x_{jk})_{j,k \in \mathbf{N}}$ is said to be convergent in the Pringsheim's sense if there exists a number ℓ such that x_{jk} converges to ℓ as both j and k tend to ∞ independently of one another, in this case we shall write

$$\lim_{j,k \rightarrow \infty} x_{jk} = \ell.$$

It is trivial that $x=(x_{jk})$ converges in Pringsheim's sense if and only if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $|x_{jk} - \ell| < \varepsilon$, for all $j, k \geq n_0$. The limit ℓ is called double limit or the Pringsheim limit of x .

A double sequence $x=(x_{jk})$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $|x_{pq} - x_{jk}| < \varepsilon$ for all $p, q, j, k > n_0$. It is known

that a double sequence (x_{jk}) with complex terms is a Cauchy sequence if and only if it is convergent.

The concept of statistical convergence was first introduced by Fast [2] and also independently by Buck [1] and Shoenberg [7] for real and complex sequences. Further this concept for a real double sequence was studied by Mursaleen and Edely [5] and many others.

Let $A \subseteq \mathbf{N} \times \mathbf{N}$ be a two-dimensional set of positive integers. In case the double sequence $\left(\frac{|A(n,m)|}{nm}\right)$ has a limit in Pringsheim's sense then we say that A has a double natural density and is defined as

$$\lim_{n,m \rightarrow \infty} \frac{|A(n,m)|}{nm} = \delta_2(A),$$

where $A(n,m) = \{(j,k), j \leq n \text{ and } k \leq m : (j,k) \in A\}$ and $|A(n,m)|$ denotes the cardinality of set $A \subseteq \mathbf{N} \times \mathbf{N}$. Clearly we have $\delta_2(A^c) = \delta_2(\mathbf{N} \times \mathbf{N} - A) = 1 - \delta_2(A)$. It is obvious that all finite subsets of $\mathbf{N} \times \mathbf{N}$ have zero double natural density. Further some infinite subsets also have zero density. For example, the set $A = \{(j,k) : j \in [3^p, 3^p + p) \text{ and } k \in [3^q, 3^q + q) \mid p, q = 1, 2, \dots\}$ has double natural density zero. We shall particularly concern with the subset of two-dimensional positive integers sets having double natural density zero. If $x = (x_{jk})$ is a double sequence such that x_{jk} satisfies property P for all (j,k) except a set of double natural density zero, then we say that x_{jk} satisfies P for "almost all (j,k) ", and we abbreviate this by "a.a. (j,k) ".

A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$,

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{(j,k), j \leq n \text{ and } k \leq m : |x_{jk} - \ell| \geq \varepsilon\}| = 0,$$

i.e.,

$$|x_{jk} - \ell| < \varepsilon \quad \text{a.a. } (j,k).$$

In this case, we write $\text{st}_2 - \lim_{j,k \rightarrow \infty} x_{jk} = \ell$.

2. Pointwise and Uniform Convergence for Double Sequences of

Real-Valued Functions

In this paper we deal with double sequences $\{f_{jk}\}$ whose terms are real-valued functions having a common domain on the real line $\mathbf{R}$. For each x in

the domain we can form another double sequence $\{f_{jk}(x)\}$ whose terms are the corresponding function values.

Double sequences in which the elements are function are defined in the same way as are single sequences of real-valued functions.

Definition 2.1. A double sequence of functions $\{f_{jk}\}$ is said to pointwise convergent to f on a set $S \subset \mathbf{R}$ if, for each point x in S and for each $\varepsilon > 0$, there exists a positive integer $N(x, \varepsilon)$ such that $|f_{jk}(x) - f(x)| < \varepsilon$, for all $j, k > N$. We write $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ or $f_{jk} \rightarrow f$ on S . The function f is called the double limit function (or Pringsheim limit function) of the sequence $\{f_{jk}\}$ and we say that $\{f_{jk}\}$ converges (pointwise) to f on the set S .

Lemma 2.2. Let $\{f_{jk}\}$ be a double sequence of functions defined on a set S . Then the Pringsheim limit

$$\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$$

on S exists iff for all strictly increasing functions $j, k : \mathbf{N} \rightarrow \mathbf{N}$ the ordinary limit

$$\lim_{r \rightarrow \infty} f_{j(r), k(r)}(x) = f(x)$$

on S exists.

Proof: The proof is similar to that of Lemma 2 in [4]. Therefore we omit it.

Definition 2.3. A sequence of functions $\{f_{jk}\}$ is said to converge uniformly to f on a set S if, for every $\varepsilon > 0$, there exists an $N(=N(\varepsilon))$ such that $j, k > N$ implies

$$|f_{jk}(x) - f(x)| < \varepsilon, \text{ for all } x \text{ in } S.$$

We denote this symbolically by writing $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ or $f_{jk} \rightarrow f$ uniformly on S .

Now, let us give the following theorem which characterizes the sequence $\{f_{jk}\}$ uniform convergence or not.

Theorem 2.4. Let f and f_{jk} , $j, k=1, 2, 3, \dots$, be continious functions on $I = [a, b] \subset \mathbf{R}$. Then $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on I if and only if $\lim_{j,k \rightarrow \infty} c_{jk} = 0$, where

$$c_{jk} = \max_{x \in I} |f_{jk}(x) - f(x)|.$$

Proof: Suppose that $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on I . Since $|f_{jk}(x) - f(x)|$ is continuous on I for each $j, k \in \mathbf{N}$, it has absolute maximum value at some

point $x_{jk} \in I$, i.e., there exists $x_{11}, x_{12}, \dots, x_{1k}, \dots, x_{21}, \dots, x_{jk}, \dots \in I$ such that $c_{11} = |f_{11}(x_{11}) - f(x_{11})|$, $c_{21} = |f_{21}(x_{21}) - f(x_{21})|$, ... etc. Thus, we may write $c_{jk} = |f_{jk}(x_{jk}) - f(x_{jk})|$, $j, k=1, 2, \dots$. From the definition of uniform convergence, we may write, for every $\varepsilon > 0$

$$|f_{jk}(x_{jk}) - f(x_{jk})| < \varepsilon$$

for all $j, k > N(\varepsilon)$. Hence $\lim_{j,k \rightarrow \infty} c_{jk} = 0$.

The necessity is trivial.

It is clear that uniform convergence implies pointwise convergence with the same limit f on the set S . But the converse is not true, as the following example shows.

Let $f_{jk}(x) = \frac{j^2 k^2 x}{1 + j^3 k^3 x^2}$ if $x \in (0, 1)$, $\{f_{jk}\}$ converges pointwise to $f=0$ but not uniformly on $(0, 1)$ since $\lim_{j,k \rightarrow \infty} c_{jk} = \lim_{j,k \rightarrow \infty} \frac{\sqrt{jk}}{2}$ does not convergent.

3. Statistical Convergence for Double Sequences

of Real-Valued Functions

The notion of pointwise statistical convergence for single sequences was introduced by Gökhan and Güngör in [3]. Furthermore they introduce the concept of a statistically Cauchy sequence for functional sequences and prove that it is equivalent to pointwise statistical convergence of sequence of real-valued functions.

We define the statistical analogue for double sequence of real-valued functions $\{f_{jk}\}$ as follows.

Definition 3.1. A double sequence of functions $\{f_{jk}\}$ is said to pointwise statistically convergent to f on a set S if, for every $\varepsilon > 0$

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m: |f_{jk}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$,

$$|f_{jk}(x) - f(x)| < \varepsilon \quad \text{a.a. } (j, k).$$

In this case, we write $\text{st}_2 - \lim f_{jk}(x) = f(x)$ or $f_{jk} \xrightarrow{\text{st}} f$ on S . The function f is called the double statistical limit (or Pringsheim statistical limit) function of the sequence $\{f_{jk}\}$.

It is clear that $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ implies $st_2 - \lim f_{jk}(x) = f(x)$ on S . But the converse of this is not true. For example, define

$$f_{jk}(x) = \begin{cases} (-x)^{jk}, & j \in [3^p, 3^p+p) \text{ and } k \in [3^q, 3^q+q) \text{ , } p, q=1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

if, $x \in \mathbf{R}[-1, 1]$, $j, k = 1, 2, \dots$. Then

$$\begin{aligned} \lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m: |f_{jk}(x) - 0| \geq \varepsilon\}| \\ \leq \frac{pq(p+1)(q+1)}{4 \cdot 3^{p+q}} \rightarrow 0 \end{aligned}$$

for each (fixed) $x \in \mathbf{R}[-1, 1]$ so $st_2 - \lim f_{jk}(x) = 0$ on $\mathbf{R}[-1, 1]$. But $\lim_{j,k \rightarrow \infty} f_{jk}(x)$ does not exist.

Theorem 3.2. Let $\{f_{jk}\}$ and $\{g_{jk}\}$ be two double sequences of functions defined on a set S . If $st_2 - \lim f_{jk}(x) = f(x)$ and $st_2 - \lim g_{jk}(x) = g(x)$ on S , then $st_2 - \lim(\alpha f_{jk}(x) + \beta g_{jk}(x)) = \alpha f(x) + \beta g(x)$ on S , where $\alpha, \beta \in \mathbf{R}$.

Proof: The proof is similar to that of Theorem 2.1 in [3]. Therefore we omit it.

Theorem 3.3. A double sequence of functions $\{f_{jk}\}$ is pointwise statistically convergent to a function $f(x)$ on S if and only if there exists a subset $K_x = \{(j, k)\} \subseteq \mathbf{N} \times \mathbf{N}$, $j, k = 1, 2, \dots$, for each (fixed) $x \in S$ such that $\delta_2(K_x) = 1$ and $\lim_{\substack{j,k \rightarrow \infty \\ (j,k) \in K_x}} f_{jk}(x) = f(x)$ for each (fixed) $x \in S$.

Proof: Let $st_2 - \lim f_{jk}(x) = f(x)$ on S . Put

$$K_{r,x} = \{(j, k) \in \mathbf{N} \times \mathbf{N}: |f_{jk}(x) - f(x)| \geq \frac{1}{r}\}$$

and

$$M_{r,x} = \{(j, k) \in \mathbf{N} \times \mathbf{N}: |f_{jk}(x) - f(x)| < \frac{1}{r}\} \quad (r=1, 2, \dots)$$

for each (fixed) $x \in S$. Then $\delta_2(K_{r,x}) = 0$ and

$$(1) \quad M_{1,x} \supset M_{2,x} \supset \dots \supset M_{i,x} \supset M_{i+1,x} \supset \dots$$

and

$$(2) \quad \delta_2(M_{r,x}) = 1, \quad r=1, 2, \dots$$

for each (fixed) $x \in S$. Now we have to show that for $(j, k) \in M_{r,x}$, $\{f_{jk}\}$ is convergent to $f(x)$ on S . Suppose that $\{f_{jk}\}$ is not convergent to $f(x)$ on S . Therefore there is $\varepsilon > 0$ such that

$$|f_{jk}(x) - f(x)| \geq \varepsilon$$

for infinitely many terms and some $x \in S$.

Let

$$M_{\varepsilon, x} = \{ (j, k) : |f_{jk}(x) - f(x)| < \varepsilon \}$$

and $\varepsilon > \frac{1}{r}$ ($r=1, 2, \dots$). Then

$$\delta_2(M_{\varepsilon, x}) = 0,$$

and by (1), $M_{r, x} \subset M_{\varepsilon, x}$. Hence $\delta_2(M_{r, x}) = 0$ which contradicts (2). Therefore $\{f_{jk}\}$ is convergent to $f(x)$ on S .

Conversely, suppose that there exists a subset $K_x = \{(j, k)\} \subseteq \mathbf{N} \times \mathbf{N}$ for each (fixed) $x \in S$ such that $\delta_2(K_x) = 1$ and $\lim_{j, k \rightarrow \infty} f_{jk}(x) = f(x)$ on S , i.e., there exists an $N(x, \varepsilon)$ such that for each (fixed) $x \in S$ and each $\varepsilon > 0$, $j, k \geq N$ implies $|f_{jk}(x) - f(x)| < \varepsilon$. Now $K_{\varepsilon, x} = \{(j, k) : |f_{jk}(x) - f(x)| \geq \varepsilon\} \subseteq \mathbf{N} \times \mathbf{N} - \{(j_{N+1}, k_{N+1}), (j_{N+2}, k_{N+2}), \dots\}$ for each (fixed) $x \in S$. Therefore $\delta_2(K_{\varepsilon, x}) \leq 1 - 1 = 0$ for each (fixed) $x \in S$.

Hence $\{f_{jk}\}$ is pointwise statistically convergent to $f(x)$ on S .

Remark 1. If $\text{st}_2 - \lim_{j, k \rightarrow \infty} f_{jk}(x) = f(x)$ on S , then there exists a sequence $\{g_{jk}\}$ such that $\lim_{j, k \rightarrow \infty} g_{jk}(x) = f(x)$ on S and $\delta_2\{(j, k) : f_{jk}(x) = g_{jk}(x)\} = 1$ for each (fixed) $x \in S$, i.e., $f_{jk}(x) = g_{jk}(x)$ a.a. (j, k) for each (fixed) $x \in S$.

Definition 3.4. A double sequence of functions $\{f_{jk}\}$ is said to uniformly statistically converge to f on a set S if, for every $\varepsilon > 0$,

$$\lim_{n, m \rightarrow \infty} \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m : |f_{jk}(x) - f(x)| \geq \varepsilon\}| = 0, \text{ for all } x \in S$$

i.e., for all $x \in S$,

$$(3) \quad |f_{jk}(x) - f(x)| < \varepsilon \quad \text{a.a. } (j, k).$$

In this case, we write $\text{st}_2 - \lim_{j, k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on S or $f_{jk} \xrightarrow{\text{st}_2} f$ uniformly on S .

Theorem 3.3 can be given for uniform statistical convergence as below.

“ $f_{jk} \xrightarrow{\text{st}_2} f$ uniformly on $S \Leftrightarrow \forall \varepsilon > 0$ and for all $x \in S$, $\exists K \subset \mathbf{N} \times \mathbf{N}$, $\delta_2(K) = 1$ and $\exists (n_0, m_0) \in K$, where $n_0 = n_0(\varepsilon)$, $m_0 = m_0(\varepsilon)$, $\exists \forall j > n_0, k > m_0$ and $(j, k) \in K$, $|f_{jk}(x) - f(x)| < \varepsilon$.”

Theorem 3.5. Let f and f_{jk} , $j, k = 1, 2, \dots$ be continuous functions on $I = [a, b] \subset \mathbf{R}$. Then $\text{st}_2 - \lim_{j, k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on I if and only if $\text{st}_2 - \lim_{j, k \rightarrow \infty} c_{jk} = 0$, where $c_{jk} = \max_{x \in I} |f_{jk}(x) - f(x)|$.

Proof: Proof follows from Theorem 2.4 and Theorem 3.3.

It is clear that if the inequality in (3) holds for all but finitely many (j, k) , then $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on S implies $st_2 - \lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on S . But the converse is not true, as the following example shows. Let

$$f_{jk}(x) = \begin{cases} 1, & \text{if } j=p^2 \text{ and } k=q^2, p, q=1, 2, \dots \\ (x - \frac{1}{jk})^2, & \text{otherwise} \end{cases}$$

if $x \in [-1, 1]$, $j, k = 1, 2, \dots$. $\{f_{jk}\}$ is uniformly statistically convergent to $f(x) = x^2$ on $[-1, 1]$ since $st_2 - \lim_{j,k \rightarrow \infty} c_{jk} = 0$, where

$$c_{jk} = \max_{x \in [-1, 1]} |f_{jk}(x) - x^2| = \begin{cases} 1, & \text{if } j=p^2 \text{ and } k=q^2, p, q=1, 2, \dots \\ \frac{2}{jk} + \frac{1}{j^2k^2}, & \text{otherwise.} \end{cases}$$

But $\{f_{jk}\}$ is not uniformly convergent on $[-1, 1]$ since $\lim_{j,k \rightarrow \infty} c_{jk}$ does not exist.

Furthermore, it is obviously that uniform statistical convergence implies pointwise statistical convergence with the same limit f on the set S . Note that the converse of this theorem is not true. For this, let us consider the following example. Let

$$f_{jk}(x) = \begin{cases} 1, & j \in [3^p, 3^p + p), p=1, 2, \dots, k \in \mathbb{N} \\ \frac{k^2 x}{1+k^3 x^2}, & \text{otherwise} \end{cases}$$

if $x \in (0, 1)$, $k=1, 2, \dots$. Then the sequence $\{f_{jk}\}$ is pointwise statistically convergent to $f(x) = 0$ on $(0, 1)$. But $\{f_{jk}\}$ is not uniformly statistically convergent by Theorem 3.5. Because

$$c_{jk} = \max_{x \in (0, 1)} |f_{jk}(x) - 0| = \begin{cases} 1, & j \in [3^p, 3^p + p), p=1, 2, \dots, k \in \mathbb{N} \\ \frac{\sqrt{k}}{2}, & \text{otherwise.} \end{cases}$$

and $st_2 - \lim_{j,k \rightarrow \infty} c_{jk}$ does not exist.

Now, we can give the relations between well-known convergence models and our studied models as the following result.

Corollary 3.6.

- (i) $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on $S \Rightarrow \lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ on $S \Rightarrow st_2 - \lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ on S ,
- (ii) $\lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on $S \Rightarrow st_2 - \lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ uniformly on $S \Rightarrow st_2 - \lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ on S .

4. Statistically Cauchy Sequences

In this section we define statistically Cauchy double sequence of functions and prove some analogues.

Definition 4.1. Let $\{f_{jk}\}$ be a double sequence of functions defined on a set S . A double sequence $\{f_{jk}\}$ is said to be statistically Cauchy if for every ε

>0 , there exist $N(=N(\varepsilon))$ and $M(=M(\varepsilon))$ such that

$$|f_{jk}(x) - f_{NM}(x)| < \varepsilon \quad \text{a.a. } (j, k) \text{ and for each (fixed) } x \in S$$

i.e.,

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m: |f_{jk}(x) - f_{NM}(x)| \geq \varepsilon\}| = 0$$

for each (fixed) $x \in S$.

Theorem 4.2. Let $\{f_{jk}\}$ be a double sequence of functions defined on a set S . Then $\{f_{jk}\}$ is (pointwise) statistically convergent on S if and only if $\{f_{jk}\}$ is statistically Cauchy on S .

Proof: Suppose $\text{st}_2 - \lim_{j,k \rightarrow \infty} f_{jk}(x) = f(x)$ on S and let $\varepsilon > 0$. Then $|f_{jk}(x) - f(x)| < \frac{\varepsilon}{2}$ a.a. (j, k) and if N, M are chosen so that $|f_{NM}(x) - f(x)| < \frac{\varepsilon}{2}$ for each (fixed) $x \in S$, then we have

$$|f_{jk}(x) - f_{NM}(x)| \leq |f_{jk}(x) - f(x)| + |f_{NM}(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{a.a. } (j, k)$$

for each (fixed) $x \in S$. Hence $\{f_{jk}\}$ is a statistically Cauchy sequence.

Conversely let $\{f_{jk}\}$ be statistically Cauchy, then choose N, M so that the band $I = [f_{NM}(x) - 1, f_{NM}(x) + 1]$ contains $f_{jk}(x)$ a.a. (j, k) for each (fixed) $x \in S$. Now choose K, L such that $I' = [f_{KL}(x) - \frac{1}{2}, f_{KL}(x) + \frac{1}{2}]$ contains $f_{jk}(x)$ a.a. (j, k) for each (fixed) $x \in S$. We assert that $I_1 = I \cap I'$ contains $f_{jk}(x)$ a.a. (j, k) for each (fixed) $x \in S$;

$$\{(j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I \cap I'\}$$

$$= \{(j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I\}$$

$$\cup \{(j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I'\}$$

for each (fixed) $x \in S$, so

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I \cap I'\}| \\ & \leq \lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I\}| \end{aligned}$$

$$+ \lim_{n,m \rightarrow \infty} \frac{1}{nm} |\{ (j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I' \}| = 0$$

for each (fixed) $x \in S$.

Therefore I_1 is a closed band of height less than or equal to 1 that contains $f_{jk}(x)$ a.a. (j, k) for every $x \in S$. Now we proceed by choosing $N(2)$ and $M(2)$ so that $I'' = [f_{N(2), M(2)}(x) - \frac{1}{4}, f_{N(2), M(2)}(x) + \frac{1}{4}]$ contains $f_{jk}(x)$ a.a. (j, k) , and by the preceding argument $I_2 = I_1 \cap I''$ contains $f_{jk}(x)$ a.a. (j, k) for each (fixed) $x \in S$, and I_2 has height less than or equal to $\frac{1}{2}$. Continuing inductively we construct a sequence $\{I_p\}_{p=1}^{\infty}$ of closed band such that for each p , $I_p \supseteq I_{p+1}$, the height of I_p is not greater than 2^{1-p} , and $f_{jk}(x) \in I_p$ a.a. (j, k) for each (fixed) $x \in S$. Thus, there exists a function $f(x)$, defined on S , that $\{f(x)\}$ is equal to $\bigcap_{p=1}^{\infty} I_p$. Now we show that $\{f_{jk}\}$ is statistically convergent to $f(x)$ on S . Let $\varepsilon > 0$ be given, then there exist q such that $\varepsilon > 2^{1-q}$. Then from the above construction it follows that $f_{jk}(x) \in I_q$ a.a. (j, k) for each (fixed) $x \in S$. We have

$$\begin{aligned} & \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m : |f_{jk}(x) - f(x)| \geq \varepsilon \}| \\ & \leq \frac{1}{nm} |\{(j, k), j \leq n \text{ and } k \leq m : f_{jk}(x) \notin I_q \}| \rightarrow 0 \end{aligned}$$

for each (fixed) $x \in S$.

This completes the proof of the theorem.

From Theorem 3.3 and 4.2 we can state the following theorem for double sequences .

Theorem 4.3. Let $\{f_{jk}\}$ be a double sequence of functions defined on a set S . The following statements are equivalent:

- (i) $\{f_{jk}\}$ is (pointwise) statistically convergent to $f(x)$ on S ;
- (ii) $\{f_{jk}\}$ is statistically Cauchy on S ;
- (iii) There exists a subsequence $\{g_{jk}\}$ of $\{f_{jk}\}$ such that $\lim_{j,k \rightarrow \infty} g_{jk}(x) = f(x)$

on S .

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