A Managed Bayesian Risk Approach for Decision Making Alternatives

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Abstract

Bayesian risk analysis is widely used in industry management for many years. But some of uncertain factors unfortunately existed such as the bias of modeling and the obstacle of large volume data access from an online system. The purpose of a managed Bayesian risk approach proposed is to solve these problems. The approach we proposed in this article clarifies various situations for decision making alternatives. Furthermore, a case study of vehicle recognition failure event is demonstrated to verify the model we propose.

Keywords: Decision making alternatives, Security Level Requirement (SLR), managed Bayesian risk; Simulation

1. Introduction

Bayesian risk analysis is popularly used in decision making field for many years. But some of problems unfortunately existed and needed to solved, such as the bias of modeling and the obstacles of large volume data access from an online system. In this paper, a managed Bayesian risk approach is proposed to meet these needs [12,13]. The approach we proposed in this article clarifies various situations for decision making alternatives. Furthermore, a case study of vehicle recognition failure event is demonstrated to verify the model we propose.

This approach mainly includes several processes: (1) Building simulation model & related experiment data; (2) Verify the prior distribution; (3) Constructing the posterior distribution of Bayesian model & calculating the risk indexes [14, 1]; (4) Decision making.

2. Mathematical Modeling

The applied mathematical model definition and formulation of this study are explained as follows.

2.1 Formula A: An inverse function simulation modeling

Let x be a variable with cumulative distribution of $F_j(x)$, j denotes each data sampling event. For $F_j(x)$ is a non-decreasing function, the inverse simulation function of $F_j^{-1}(y)$ could be defined as

$$F_j^{-1}(y) = \inf\{x; F_j(x) \geq y\} \quad 0 \leq y \leq 1 \quad (1)$$

• Assume that sampling data can be grouped into n adjacent intervals $[b_0, b_1)$, $[b_1, b_2)$, ..., $[b_{n-1}, b_n]$, and define the piecewise-linear function:

$$F_j(x) = \begin{cases} 0 & , x < b_0 \\ F_j(b_{i-1}) + \frac{x - b_{i-1}}{b_i - b_{i-1}} [F_j(b_i) - F_j(b_{i-1})] & , b_{i-1} \leq x < b_i \text{ for some } i < n \\ 1 & , x > b_n \end{cases} \quad (2)$$

$$\text{where } F_j(b_0) = 0, \quad F_j(b_i) = \sum_{k=1}^i r_k / r, \quad i = 1, 2, \dots, n$$

r_i is the number of observations in the i th interval (b_{i-1}, b_i) and $r = \sum_{i=1}^n r_i$

• Find the minimum positive integer k ($0 \leq k \leq n-1$), such that $Y \leq F(X_k)$ and return

$$X = F_j^{-1}(Y) = b_k + [Y - F_j(b_k)] \times (b_{k+1} - b_k) / [F_j(b_{k+1}) - F_j(b_k)] \quad (3)$$

(This inverse function (3) associated random generating number is used to generate random variable simulation data as an input of Bayesian risk calculation in this study [7, 10])

2.2 Formula B: Bayesian risk definition and formula

For the purpose of mathematical illustration, an Bayesian method with the deduced formula is described as follows [3, 2, 4, 5]:

Given the collected data $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ with the associated parameters $\theta \in S$, where S (parameter space), α is defined as the possible action space. In Bayesian decision method, loss functions are defined as follows.

A. Square loss function $l(\alpha, \theta) = (q(\theta) - \alpha)^2$;

B. Absolute loss function $l(\alpha, \theta) = |q(\theta) - \alpha|$

In general, risk function is defined as the expected value of loss functions, owing to the different ways of calculation, there are two types of risk functions:

a. When x is a continuous variable :

$$w(\varphi, G) = \int R(\varphi, \theta) dG(\theta) = \int \sum_x \{ \varphi(x) - \theta \}^2 f(x|\theta) dG(\theta) \quad (4)$$

b. When x is a discrete variable :

$$R(\theta, \delta) = E\{l(\theta, \delta(X)) | \theta\} = \sum_x l(\theta, \delta(x)) p(x|\theta) \quad (5)$$

where $\delta(x)$ is defined as decision function of x .

Bayesian risk, which is the expected risk, will be minimized to make an optimal strategy:

$$w(\varphi, G) = \int R(\varphi, \theta) dG(\theta) = \int \sum_x \{ \varphi(x) - \theta \}^2 f(x|\theta) dG(\theta)$$

The key point is to find the minimum Bayesian risk function $\min\{w(\phi_i, G)\}, i = 1, 2, \dots, n$.; Consequently, φ will be called as Bayesian minimum risk estimate φ_G .

In order to derive φ_G so that $w(\varphi_G, G) \leq w(\varphi, G)$, where φ is an estimator

$$\begin{aligned} w(\varphi, G) &= \int \sum_x \{ \varphi(x) - \theta \}^2 f(x|\theta) dG(\theta) \\ &= \int \sum_x \{ (\varphi(x) - \varphi_G) + (\varphi_G - \theta) \}^2 f(x|\theta) dG(\theta) \\ &= \int \sum_x \{ (\varphi(x) - \varphi_G)^2 + (\varphi_G - \theta)^2 + 2(\varphi(x) - \varphi_G)(\varphi_G - \theta) \} f(x|\theta) dG(\theta) \end{aligned} \quad (6)$$

$$\begin{aligned} w(\varphi, G) &= w(\varphi_G(x), G) + \sum_x \int \{ \varphi(x) - \varphi_G(x) \}^2 f(x|\theta) dG(\theta) \\ &= w(\varphi_G(x), G) + \sum_x \int \{ \varphi(x) - \varphi_G(x) \}^2 f(x|\theta) dG(\theta) \\ &\quad + 2 \sum_x \int \{ \varphi(x) - \varphi_G(x) \} \{ \varphi_G(x) - \theta \} f(x|\theta) dG(\theta) \end{aligned} \quad (7)$$

$$w(\varphi_G(x), G) = w(\varphi, G) - \sum_x \int \{ \varphi(x) - \varphi_G(x) \}^2 f(x|\theta) dG(\theta)$$

$$2 \sum_x \int \{ \varphi(x) - \varphi_G(x) \} \{ \varphi_G(x) - \theta \} f(x|\theta) dG(\theta) \quad (8)$$

To minimize $w(\varphi_G(x), G)$, the differential algorithm can be employed to make $w(\varphi_G(x), G)$ function first order equals to zero.

$$-2 \sum_x \int \{ \varphi(x) - \varphi_G(x) \} \{ \varphi_G(x) - \theta \} f(x|\theta) dG(\theta) = 0$$

while $\int \{ \varphi_G(x) - \theta \} f(x|\theta) dG(\theta) = 0$, it is derived that

$$\phi_G(x) = \frac{\int \theta f(x|\theta) dG(\theta)}{\int f(x|\theta) dG(\theta)}$$

Considering the observation x , the expected value of the parameter θ posterior estimate is $\varphi_G(x) = E_G[\theta|x]$. For Gamma prior distribution, the decision function is $\varphi_G(x) = \frac{\alpha\beta + \beta x}{1 + \beta}$, the theoretical Bayesian risk of **Formula B** can be derived as

$$w(\phi_G) = \frac{\beta^2 \alpha}{1 + \beta} \quad (9)$$

2.3 Formula C: Adding a simulated-adjusted factor

The Bayesian risk can be derived as formula (9) $w(\varphi_G) = \frac{\beta^2 \alpha}{1 + \beta}$. However, this formula is a theoretic formula. In practice, some of bias between theory and practical application always existed. Hence, the authors propose to use a simulated-adjusted factor to modify the Bayesian risk as below to meet the application requirement.

$$w(\varphi, G) = w(\varphi_G) + K(\varphi_n) \quad (10)$$

Where $w(\varphi, G)$ denotes the adjusted Bayesian risk function (**Formula C**); $w(\phi_G)$ denotes the theoretic Bayesian risk function; $K(\phi_n)$ denotes the adjusted

factor of variable function, which is the sum of $(\varphi_n(x) - \varphi_G(x))^2 f_G(x)$; $f_G(x)$ denotes the posterior probability of population; $\varphi_n(x)$ denotes the decision function of observations.

2.4 Formula D: A minimum variance formula

If the prior distribution is neglected, then $W(T)$ the minimum variance $E(X) = \alpha \beta$ will be used for risk index calculation.

3. Example Explanation

An experiment case of harbor vehicle recognition project of Taiwan is used for illustration as following:

3.1. Applied example

The study using a harbor vehicle recognition system as an example is shown as Fig. 1. This picture presents a real implemented infrastructure of Koashiung Harbor, which provides around five thousands of vehicles to pass the security gates for one day automatically. Some of uncertain factors unfortunately existed such as bad weather, false license plat and night time recognition etc that result in undesirable recognition failure event happened. By using this analysis method, the Harbor Security Center can make a proper decision for the automatic system operation [6, 11].

Fig 1. Infrastructure of harbor vehicle recognition system

Recognition Failure Inter-arrival Time (Hours)	Frequency	Relative Frequency	Cumulative Frequency	Inverse reference point
$0.5 \leq x < 2.0$	35	0.35	0.35	(0.35,2.00)
$2.0 \leq x < 4.0$	30	0.30	0.65	(0.65,4.00)
$4.0 \leq x < 6.0$	20	0.20	0.85	(0.85,6.00)
$6.0 \leq x < 8.0$	10	0.10	0.90	(0.90,8.00)
$8.0 \leq x \leq 10.0$	5	0.05	1.00	(1.00,10.00)

(○,1.Vehicle starting point; ○,2.Camera facilities; ○,3.Automatic gate; ○,4.
Manual gate)

3.2. Validation of prior distribution for recognition failure events

This study takes a group of sampling data from vehicle recognition system presented in Table 3.

Table 3. An example of sampling frequency of recognition failure event

The purpose of this tested case is to analyze the observations of recognition failure event by frequencies statistics. This can be considered as prior distribution of failure arrival event. To perform the verification of prior distribution by using the statistical Chi-square test of equal probabilities good-of-fit test [7,10]. The following hypotheses are formed. H_0 : the random variable of failure arrival time is exponentially distributed; H_1 : not exponentially distributed. Referencing the Table 3, this study has a statistical sampling data of total failure event 100 observations of inter-arrival time. The authors take the sampling average $\hat{\mu} = 3.48$ of this inter-arrival time frequencies. Let the number of a class interval $k=8$, and calculating the statistical data of Table 4 by using the following procedures.

Table 4 Chi-square goodness-of-fit test case study

Interval from exponential distribution	Observed Frequency O_i	Expected frequency E_i
$(a_0, a_1) = (0.0001, 0.2014)$	15	12.5
$(a_1, a_2) = (1.2126, 0.4338)$	17	12.5
$(a_2, a_3) = (2.0839, 0.7087)$	16	12.5
$(a_3, a_4) = (3.2055, 1.0452)$	14	12.5
$(a_4, a_5) = (4.7816, 1.4789)$	11	12.5
$(a_5, a_6) = (7.4563, 2.0904)$	9	12.5
$(a_6, a_7) = (7.4563, 3.1355)$	9	12.5
$(a_7, a_8) = (7.4563, 10.000)$	9	12.5

The values of expected frequencies

$$\hat{\lambda} = 1/\hat{u} = 1/3.48 = 0.288, p = 1/k = 1/8 = 0.125, E_i = n \times p = 100 \times 0.125 = 12.5 \quad (11)$$

The interval endpoints values by the following cumulative distribution formula:

$$F(a_i) = 1 - e^{-\lambda a_i} \quad (12); \quad ip = 1 - e^{-\lambda a_i} \quad (13); \quad a_i = (-1/\lambda) \ln(1-ip) \quad (14)$$

Where $i=0, 1, 2, \dots, k$ with interval groups $(a_0, a_1), (a_1, a_2), \dots, (a_7, a_8)$. By

using the SAS statistics software package tool, it is derived at the 0.05 level of

significance that $\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i} = 6.4 < \chi^2(0.95, 6) = 12.6$. Therefore, the null

H_0 can be accepted and the random variable is exponentially distributed. If the hypothesis is rejected, the prior distribution will be neglected and the classical statistical minimum variance will be then, used to estimate the risk index.

3.3 Calculation of risk indexes

Firstly, we take the Poisson condition probability and the Gamma prior distribution of ($\alpha=2$, $\beta=1$) as a combination to calculate the failure event probability [8, 9]. A failure event probability example can be deduced as follows (Table 5):

Table 5 Failure events of risk parameters for 500 users $\alpha=2$ $\beta=1$

x	F1	F2	F3	F4	F5	K
0	28	0.056	0.250	2.143	1.00	0.327
1	60	0.120	0.250	3.000	1.50	0.563
2	90	0.180	0.188	3.667	2.00	0.521
3	110	0.220	0.125	3.127	2.50	0.049
4	86	0.172	0.078	2.907	3.00	0.001
5	50	0.100	0.047	4.680	3.50	0.065
6	39	0.078	0.027	2.692	4.00	0.047
7	15	0.030	0.016	5.333	4.50	0.011
8	10	0.020	0.009	3.600	5.00	0.017
9	4	0.008	0.005	5.000	5.50	0.001
10	2	0.004	0.003	11.000	6.00	0.067
11	2	0.004	0.001	12.000	6.50	0.044
12	2	0.004	0.001	13.000	7.00	0.029
13	2	0.004	0.000	0.000	7.50	0.024
Total	500	1	1.000	72.149	59.5	1.765

x denotes the occurred number of failure event; F1 denotes the vehicle number of failure events $f'_n(x)$; F2 denotes the probability of failure $f_n(x) = f'_n(x)/500$; F3 denotes the posterior probability of population $f_G(x)$

$$\begin{aligned} f_G(x) &= \int f(x/\theta) dG(\theta) = \int e^{-\theta} \frac{\beta^\alpha}{x! \Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta \\ &= \frac{\beta^\alpha \int e^{-(\alpha+1)\theta} \theta^{(\beta+x-1)} d\theta}{x! \Gamma(\alpha)} = \frac{\beta^\alpha}{x! \Gamma(\alpha)} \times \frac{\Gamma(\alpha+x)}{(\beta+1)^{\alpha+x}} \\ &= \left(\frac{\beta}{\beta+1}\right)^\alpha \times \frac{\Gamma(\alpha+x)}{\Gamma(\alpha)x!} \times \left(\frac{1}{\beta+1}\right)^x \end{aligned} \quad (15)$$

F4 denotes the decision function of observations $\phi_n(x) = (x+1) \frac{f_n(x+1)}{f_n(x)}$

F5 denotes decision function of theoretic distribution $\phi_G(x) = \frac{\beta(\alpha+x)}{1+\beta}$

K denotes an adjusted factor $(\phi_n(x) - \phi_G(x))^2 f_G(x)$; $W(T)$ denotes the minimum

variance $E(X) = \alpha\beta$; $K(\phi_n) = \sum_{i=0}^{13} (\phi_n(x) - \phi_G(x))^2 f_G(x) = 1.765$;

$$w(\phi_G) = \frac{\beta^2 \alpha}{1+\beta} = 1.000; \quad w(\phi_n) = w(\phi_G) + K(\phi_n) = 1.000 + 1.765 = 2.765$$

Secondly, this project defines the different managed alternatives. Table 6 shows the definition of managed rules of risk calculation results of an empirical experiment.

Table 6 Managed alternatives definition

Alternatives of Managed Strategies	Weather Status	Day/Night Time	Attribute Intensity	Distribution $G(\alpha\beta)$ Prior Distribution Approximated Parameters
1	Sunny	Day/Night	Temperature 10°C~30°C	$\alpha=1 \beta=1$
2	Windy	Day	Over 5 scale	$\alpha=1 \beta=3$
3	Windy	Day	Under 5 scale	$\alpha=2 \beta=1$
4	Windy	Night	Over 5 scale	$\alpha=2 \beta=3$
5	Windy	Night	Under 5 scale	$\alpha=2 \beta=2$
6	Rainy	Day	Over 100 ml/hour	$\alpha=3 \beta=3$
7	Rainy	Day	Under 100 ml/hour	$\alpha=3 \beta=1$
8	Rainy	Night	Over 100 ml/hour	$\alpha=3 \beta=4$
9	Rainy	Night	Under 100 ml/hour	$\alpha=3 \beta=2$

Finally, by repeating the processes of 3.1 & 3.2 in this section, the statistical & computing calculations for simulated data $n=500, 1000, 2000$ are listed as Table 7 & 8.

Table 7 Results of Bayesian adjusted factors

Decision Alternatives of Prior Distribution $G(\alpha, \beta)$	$K(\varphi_n)$ (**)		
	n		
	500	1000	2000
1: $\alpha=1$ $\beta=1$	3.11	2.341	2.268
2: $\alpha=1$ $\beta=3$	2.105	1.789	1.727
3: $\alpha=2$ $\beta=1$	<u>1.765</u>	<u>1.107</u>	<u>0.677</u>
4: $\alpha=2$ $\beta=3$	1.105	1.089	0.97
5: $\alpha=2$ $\beta=2$	0.809	0.586	0.523
6: $\alpha=3$ $\beta=3$	0.744	0.641	0.6
7: $\alpha=3$ $\beta=1$	0.863	0.598	0.536
8: $\alpha=3$ $\beta=4$	0.299	0.245	0.205
9: $\alpha=3$ $\beta=2$	0.571	0.547	0.52

Table 8 Risk calculation results of an empirical experiment

Alternatives & Decided Actions	$W(T)$ (Formula D)	$w(\varphi_G)$ (*) (Formula B)	$w(\varphi, G)$ (C= * + **) (Formula C)			
			n			
			500	1000	2000	Averaged Risk Indexes (ARI)
1& A	1	0.50	3.610	2.841	2.768	<u>3.073</u>
2& A	3	2.25	4.355	4.039	3.977	<u>4.124</u>
3& A	<u>2</u>	<u>1.00</u>	<u>2.765</u>	<u>2.107</u>	<u>1.677</u>	<u>2.183</u>
4& B	6	4.50	5.605	5.589	5.470	<u>5.555</u>
5& A	4	2.66	3.469	3.246	3.183	<u>3.299</u>
6& C	9	6.75	7.494	7.391	7.350	<u>7.412</u>
7& A	3	1.50	2.363	2.098	2.036	<u>2.166</u>
8& D	12	9.60	9.899	9.845	9.805	<u>9.850</u>
9& A	6	4.00	4.571	4.547	4.520	<u>4.546</u>

From the previous study, a proper measure can be takes if undesirable case happened. In Table 8, the authors also show the possible decisions/actions of the Harbor Administration Center. Usually, HAC defines the SLR strategies as:

- A**. Normal operation (ARI less than 5)
- B**. Warning (ARI 5-7)
- C**. Warning/Partially changed to manual operation (ARI 7-9)
- D**. Serious warning/Mostly changed to manual operation (ARI 9-12)
- E**. Stop operation/Completely changed to manual operation (ARI greater than 12)

Fig 2. Risk comparison of different alternatives

4. CONCLUSION

. In this paper, the authors presented a simulated & managed risk analysis pattern to support decision making alternatives. In specific, a series of simulated procedures are provided for the readership to follow up by this article. At the same time, this evaluation model can also be used to meet the requirement of the management level's decision making activities in many other industries.

5. REFERENCE

- [1]Clemen, R. T. Winkler, R. L, 1999. Combining probability distributions from experts in risk analysis, *Risk Anal*, 19, pp: 187-203.
- [2]Denson, W., 1998, "The history of reliability prediction", *Reliability, IEEE Transactions on*, Volume, 47, Issue 3, Part 2, Sept. pp. SP321 - SP328.
- [3]D. H. Stamatis, 1995. *Failure Mode and Effect Analysis*, ISBN-0-87389-300-X, ASQC, USA, pp: 1-25.

- [4]Groen, F., Droguett, E.L, 2005, “Competing failure mode modeling in a Bayesian reliability assessment tool”, Reliability and Maintainability Symposium, Proceedings, Annual, Jan. 24-27, 2005 pp: 393 – 397.
- [5]Hsieh, P.I., Ling, J., 1999, “A framework of integrated reliability demonstration in system development”, Reliability and Maintainability Symposium, Proceedings, Annual, 18-21 Jan. pp: 258 – 264.
- [6]J. A. Hegt, R. J. de la Haye and N. A. Khan, 1998, “A high performance license plate recognition system”, IEEE International Conference on Systems, Man, and Cybernetics, Vol. 5, pp. 4357-4362, October.
- [7]Jerry Banks etc., 1996, “Discrete-event system simulation”, Prentice Hall International Edition, pp. 355-390.
- [8]Jones, J.; Hayes, J.; 2001 , “Estimation of system reliability using a "non-constant failure rate", model , Reliability, IEEE Transactions on, Volume 50, Issue 3, Sept. 2001 pp. 286 – 288.
- [9]Lin, J.C, 2002, “The use of indirect evidence for Bayesian reliability analysis”, Reliability and Maintainability Symposium, Proceedings, Annual, 28-31 Jan. 2002 Page(s):429 – 433.
- [10]Reuven Y. Rubinstein “Modern Simulation and Modeling”, A Wiley Inter-science Publication, 1998.
- [11]R. Parisi, E.D.Di Claudio, G. Lucarelli, G. Orlandi, 1998, “Car plate recognition by neural networks and image processing”, Proceedings of the IEEE International Symposium on Circuits and Systems, Vol. 3, pp. 195-198, May 31-June 3.
- [12]Samuel Kotz, 2002, “Modern Bayesian Statistics and Applications”, USA, George Washington University, pp. 1-80.
- [13]Terje Aven, 1993, Reliability and Risk Analysis, Elsevier Science Publishers LTD, pp. 1-36.
- [14]Wallace R Blischke, 2000, Reliability: Modeling, Prediction, Optimization, A Wiley-Interscience Publication, JOHN WILEY & SONS, INC, pp: 67-199

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