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The structure of module amenable Banach algebras

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Abstract. In this paper, we study the module amenability of Banach algebras and characterize it in terms the concepts spliting and admissiblity of short exact sequences.

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1. MODULE AMENABILITY

All of the definitions in the following are in [A].
Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}-$bimodule with compatible actions,that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, a(\alpha \cdot b) = (a \cdot \alpha)b \qquad (a, b \in A, \alpha \in \mathfrak{A})$$

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions,that is

$$\alpha\cdot(a\cdot x) = (\alpha\cdot a)\cdot x, (a\cdot\alpha)\cdot x = a\cdot(\alpha\cdot x), (\alpha\cdot x)\cdot a = \alpha\cdot(x\cdot a) \quad (a \in A, \alpha \in \mathfrak{A}, x \in X)$$

and the same for the right or two-sided actions. Then we say that X is a Banach A$-\mathfrak{A}-$module. If morever

$$\alpha \cdot x = x \cdot \alpha \qquad (\alpha \in \mathfrak{A}, x \in X),$$

Then X is called a commutative A$-\mathfrak{A}-$module .
A bounded map $D : A \longrightarrow X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b), \quad D(ab) = (Da) \cdot b + a \cdot (Db) \qquad (a, b \in A)$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), D(a \cdot \alpha) = D(a) \cdot \alpha \qquad (\alpha \in \mathfrak{A}, a \in A).$$

Note that $D : A \longrightarrow X$ is bounded if there exist $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for each $a \in A$. Although D is not necessarily linear ,but still its boundedness implies its norm continuity. When X is commutative , each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \qquad (a \in A).$$

These are called inner module derivations.

Definition 2.1. A is called module amenable (as an $\mathfrak{A}-$module) if for any commutative Banach A$-\mathfrak{A}$-module X , each module derivation $D : A \longrightarrow X^*$ is inner.

Next let $A\widehat{\otimes}_{\mathfrak{A}}A$ be the projective module tensor product of A and A [R] . This is the quotient of the usual projective tensor product $A\widehat{\otimes}A$ by the closed ideal I generated by elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathfrak{A},\ a, b \in A$. We have

$$(A\widehat{\otimes}_{\mathfrak{A}}A)^* \cong L_{\mathfrak{A}}(A, A^*) \ ,$$

where the right hand side is the space of all $\mathfrak{A}-$module morphisms from A to A^*[R] . In particular $A\widehat{\otimes}_{\mathfrak{A}}A$ is a Banach A$-\mathfrak{A}-$module consider $w \in L(A\widehat{\otimes}A, A)$ define by $w(a \otimes b) = ab$ $(a, b \in A)$ and extended by linearity .

Then both w and its second conjugate $w^{**} \in L((A\widehat{\otimes}A)^{**}, A^{**})$ are A$-\mathfrak{A}-$module homomorphisms. Let J be the closed ideal of A generated by $w(I)$. We define $\tilde{w} : A\hat{\otimes}_{\mathfrak{A}}A = A\hat{\otimes}A/I \longrightarrow \dfrac{A}{J}$ by

$$\tilde{w}(a \otimes b + I) = ab + J \qquad (a, b \in A).$$

This extends to an element $\tilde{w} \in L(A\hat{\otimes}_{\mathfrak{A}}A, \dfrac{A}{J})$ and both $\tilde{w}$ and its dual conjugate $\tilde{w}^{**} \in L((A\hat{\otimes}_{\mathfrak{A}}A)^{**}, \dfrac{A^{**}}{J^{\perp\perp}})$ are A-$\mathfrak{A}$-module homomorphisms.
Also $\dfrac{A}{J}$ and $A\hat{\otimes}_{\mathfrak{A}}A$ are Banach $\dfrac{A}{J} - \mathfrak{A}-$module and $\tilde{w}$ and $\tilde{w}^{**}$ are $\dfrac{A}{J} - \mathfrak{A}-$module homomorphisms.

Definition 2.2. A bounded net $\{e_\alpha\}$ in $A\widehat{\otimes}_{\mathfrak{A}}A$ is called a module approximate diagonal for A if for each $a \in A$,

$$\lim_\alpha [e_\alpha \cdot (a + J) - (a + J) \cdot e_\alpha] = 0, \lim_\alpha \tilde{w}(e_\alpha)(a + J) = a + J.$$

An element $\tilde{M} \in (A\widehat{\otimes}_{\mathfrak{A}}A)^{**}$ is called a module virtual diagonal for A if for each $a \in A$,

$$(\tilde{w}^{**}\tilde{M}) \cdot (a + J) = a + J, \quad (a + J) \cdot \tilde{M} = \tilde{M} \cdot (a + J).$$

If $\{e_\alpha\}$ be a module approximate diagonal for A then $\{\tilde{w}(e_\alpha)\}$ is a bounded approximate identity for $\frac{A}{J}$.

The following results are proved as in the classical case $[A]$.

Theorem 2.3. The following are equivalent:

(i) A is module amenable and $\frac{A}{J}$ has a bounded approximate identity .

(ii) A has a module approximate diagonal.

(iii) A has a module virtual diagonal .

Proposition 2.4. Let $\frac{A}{J}$ be a commutative Banach $\mathfrak{A}$-module. if A is module amenable then $\frac{A}{J}$ has a bounded approximate identity.

2. Characterization of module amenability

If X, Y and Z are Banach $\frac{A}{J}-\mathfrak{A}-$modules and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ are $\frac{A}{J} - \mathfrak{A}-$ module homomorphisms, Then the sequence

$$\Sigma : 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is exact if f is one -to-one ,$Im\ g = Z$, and $Im\ f = ker\ g$. The exact sequence Σ is admissible if there is a $\mathfrak{A}-$module homomorphism $F : Y \longrightarrow X$ such $Ff = I$ on X. The exact sequence Σ splits if there is a $\frac{A}{J} - \mathfrak{A}-$ modules homomorphism $F : Y \longrightarrow X$ such that $Ff = I$ on X.

If K is the kernel of $\tilde{w}$, and $\frac{A}{J}$ has a bounded approximate identity , then the sequence

$$\tilde{\Pi} : 0 \longrightarrow K \xrightarrow{i} A\widehat{\otimes}_{\mathfrak{A}}A \xrightarrow{\tilde{\omega}} \frac{A}{J} \longrightarrow 0$$

is exact as a sequence of Banach $\frac{A}{J}$-$\mathfrak{A}$-modules, and the same is true for the dual sequence

$$\tilde{\Pi}^* : 0 \longrightarrow \left(\frac{A}{J}\right)^* \xrightarrow{\tilde{\omega}^*} (A\widehat{\otimes}_{\mathfrak{A}}A)^* \xrightarrow{i^*} K^* \longrightarrow 0$$

Proposition 3.1. Let $\Sigma : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be a short exact sequence of Banach $\frac{A}{J} - \mathfrak{A}$-modules. If there exists a $\mathfrak{A}$ $-$ module morphism $F : Y \longrightarrow X$ satisfying $Ff = I$ on X then there exists a unique $\mathfrak{A}$-module morphism $G : Z \longrightarrow Y$ satisfying $gG = I$ on Y and conversely. moreover F is an $\frac{A}{J} - \mathfrak{A}-$module morphism if and only if G is.

Proof. is similar to [l]. ■

Lemma 3.2. Let $\frac{A}{J}$ and $A\widehat{\otimes}_{\mathfrak{A}}A$ are commutative Banach $\mathfrak{A}$-modules. If $\frac{A}{J}$ has an identity , the sequence $\tilde{\Pi}, \tilde{\Pi}^*$ are admissible . If $\frac{A}{J}$ has a bounded approximate identity , $\tilde{\Pi}^*$ is admissible.

Proof. Let $\frac{A}{J}$ has an identity $e+J$, $\theta(a+J) = a \otimes e + I$ is a $\mathfrak{A}-$module morphism and a right inverse for $\tilde{w}$, and θ^* is the required left inverse for $\tilde{w}^*$ as $\tilde{w}\theta = I$ implies that $\theta^*\tilde{w}^* = I^*$ on $(\frac{A}{J})^*$. If $\frac{A}{J}$ has a bounded approximate identity $\{e_\alpha + J\}$, let $u \in (A\widehat{\otimes}_{\mathfrak{A}}A)^{**}$ be a weak* limit point of $\{e_\alpha \otimes e_\alpha + I\}$. Define $\sigma : (A\widehat{\otimes}_{\mathfrak{A}}A)^* \to (\frac{A}{J})^*$ by

$$\langle a+J, \sigma(g)\rangle = \langle (a+J)\cdot g, u\rangle \; a \in A, g \in (A\widehat{\otimes}_{\mathfrak{A}}A)^* \; .$$

σ is a $\mathfrak{A}-$module morphism and a left inverse for $\tilde{w}^*$. ■

Theorem 3.3. Let $\frac{A}{J}$ and $A\widehat{\otimes}_{\mathfrak{A}}A$ are commutative Banach $\mathfrak{A}$-modules . The Banach algebra A is module amenable if and only if

(i) $\frac{A}{J}$ has a bounded approximate identity , and
(ii) The exact sequence $\tilde{\Pi}^*$ splits.

Proof. If A is module amenable , then $\frac{A}{J}$ has a bounded approximate identity . Let $\tilde{M}$ be a module virtual diagonal for A . For $f \in (A\widehat{\otimes}_{\mathfrak{A}}A)^*, a \in A$ define $\langle \theta(f), a+J\rangle = \langle \tilde{M}, f\cdot(a+J)\rangle \cdot \theta$ is a $\frac{A}{J} - \mathfrak{A}-$module morphism from $(A\widehat{\otimes}_{\mathfrak{A}}A)^*$ in to $(\frac{A}{J})^*$ and $\theta o \tilde{w}^* = I$ on $(\frac{A}{J})^*$. Conversely, Suppose that $\frac{A}{J}$ has a bounded approximate identity $\{e_\alpha + J\}$ and θ is an $\frac{A}{J} - \mathfrak{A}-$module morphism with $\theta\tilde{w}^* = I$, on $(\frac{A}{J})^*$. Suppose that $\{e_\alpha \otimes e_\alpha + I\}$ converges weak* to $u \in (A\widehat{\otimes}_{\mathfrak{A}}A)^{**}$. Set $\tilde{M} = \theta^*\tilde{w}^{**}\mathfrak{A}$. Then $\tilde{M}$ is a module virtual diagonal for A. ■

Theorem 3.4. Let A be a module amenable Banach algebra , and let $\Sigma : 0 \longrightarrow X^* \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an admissible short exact sequence of commutative Banach $\frac{A}{J} - \mathfrak{A}-$module with X^* a dual $\frac{A}{J} - \mathfrak{A}-$module . Then Σ splits.

Proof. Since Σ is admissible , there exists $\tilde{G} \in L_{\mathfrak{A}}(Z, Y)$ satisfying $g\tilde{G} = I$ on Z . Define $D(a) = (a + J) \cdot \tilde{G} - \tilde{G}(a + J)$. Then D is a module derivation from A to the bimodule $L_{\mathfrak{A}}(Z, Y)$ .Moreover , for $z \in Z$ $g(Da(z)) = g[(a + J) \cdot \tilde{G} - \tilde{G} \cdot (a + J)](z) = (a + J) \cdot z - (a + J) \cdot z = 0$. Therefore $D(A) \subset L_{\mathfrak{A}}(Z, ker\ g) = L_{\mathfrak{A}}(Z, Im\ f)$. Hence $h = f^{-1}D$ is a module derivation from A to the commutative Banach A$-\mathfrak{A}-$module $L_{\mathfrak{A}}(Z, X^*) = (Z\widehat{\otimes}_{\mathfrak{A}}X)^*$ since A is module amenable , there exists $Q \in L_{\mathfrak{A}}(Z, X^*)$ satisfying $D(a) = (a + J) \cdot \tilde{G} - \tilde{G} \cdot (a + J) = (a + J) \cdot fQ - fQ \cdot (a + J)$. If $G = \tilde{G} - fQ$ then $G \in L_{\mathfrak{A}}(Z, Y)$ and $(a+J)\cdot G = G\cdot(a+J)$ also $G((a+J)\cdot z) = [G\cdot(a+J)](z) = [(a + J) \cdot G](z) = (a + J) \cdot G(z)$ Therefore G is a $\frac{A}{J}-$module morphism and $(gG)(z) = g\tilde{G}(z) - (gfQ)(z) = g\tilde{G}(z) = z$ Therefore G is a right inverse for g , and consequently the sequence Σ splits. ■

Proposition 3.5. Let A be an module amenable , unital Banach algebra . Then any two sided ideal of codimension one such that is a $\mathfrak{A}-$submodule is module amenable.

Proof. Let M be such an ideal and an $\mathfrak{A}-$submodule . for $a \in A$ there exists $\alpha \in \mathfrak{A}, m \in M$ such that $a = m + \alpha 1$ .Let X be a commutative Banach M$-\mathfrak{A}$-module. Now is clearly a commutative A-$\mathfrak{A}$-module if we set $a \cdot x = m \cdot x + \alpha x$, $x \cdot a = x \cdot m + \alpha x$ $(a \in A, x \in X)$. If $D : m \rightarrow X'$ is a module derivation , D may be extended as a module derivation from A to X' by setting $D(1) = 0$ .The module amenability of A then implies the module amenability of M. ■

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