Existence Results for Generalized Vector Variational-like Inequalities\textsuperscript{1}

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Abstract

In this paper, we consider a vector version of Minty’s Lemma and obtain existence theorems of solutions for two kinds of vector variational-like inequalities.

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1 Introduction

A vector variational inequality (for short, VVI) in a finite-dimensional Euclidean space was first introduced by Giannessi [1]. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Later on, many authors have investigated vector variational inequalities in abstract spaces, see [2, 3, 6, 7, 9, 13, 14, 19] and the references therein.

On the other hand, Minty’s lemma [5, 10] has been shown to be an important tool in the variational field including variational inequality problems,

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obstacle problems, confined plasmas, free boundary problems, and stochastic optimal control problems when the operator is monotone and the domain is convex.

In 1999, Lee et al. [16] obtained a vector version of Minty’s lemma using Nadler’s result [17], and with their result they considered two kinds of vector variational-like inequalities for set-valued mappings under certain new pseudomonotonicity and hemicontinuity conditions, respectively, different from the conditions that in [12, 15, 18]. In 2004, Khan et al. [9] also provided a vector version of Minty’s lemma and studied two new kinds of vector variational-like inequalities by the similar method of Lee et al. [16].

Inspired and motivated by the above research work, in this paper, a more general vector version of Minty’s lemma is obtained and two kinds of vector variational-like inequalities for set-valued mappings which are extensions of the corresponding vector variational-like inequalities in [9, 16] are considered. We show the existence of solutions to a kind of vector variational-like inequalities with set-valued mappings under certain pseudomonotonicity condition. By using the vector version of Minty’s lemma and the vector variational-like inequality, we prove the existence of solutions for another type of vector variational-like inequalities with compact-valued set-valued mappings under certain hemicontinuity condition. The results presented in this paper extend and unify corresponding results of [9,16].

2 Preliminaries

In this section, let’s recall the following definitions and lemmas.

Definition 2.1 Let $D$ be a subset of a topological vector space $X$. Then a set-valued mapping $F: D \rightarrow 2^X$ is called KKM-mapping if for each nonempty finite subset $N$ of $D$, $CoN \subset F(N)$, where $Co$ denotes the convex hull and $F(N) = \bigcup \{F(u) : u \in N\}$.

Definition 2.2 Let $X, Y$ be two Banach spaces and $L(X,Y)$ be a space of all linear and continuous operators of $X$ into $Y$. A bifunction $N(\cdot, \cdot): L(X,Y) \times L(X,Y) \rightarrow L(X,Y)$ is called continuous in the first argument if for any $u, v \in L(X,Y)$,

$$\|N(u, \cdot) - N(v, \cdot)\| \rightarrow 0 \text{ as } \|u - v\| \rightarrow 0,$$

where $\| \cdot \|$ denotes some norm in $L(X,Y)$.

In a similar way, we can define the continuity of $N$ in the second argument.

Lemma 2.1 [4] Let $D$ be an arbitrary nonempty subset of a Hausdorff topological vector space $X$. Let the set-valued mapping $F: D \rightarrow 2^X$ be a
KKM-mapping such that \( F(u) \) is closed for all \( u \in D \) and is compact for at least one \( u \in D \). Then
\[
\bigcap_{u \in D} F(u) \neq \emptyset.
\]

**Lemma 2.2** [17] Let \((X, \| \cdot \|)\) be a normed vector space and \( H \) be a Hausdorff metric on the collection \( C(X) \) of all closed and bounded subsets of \( X \), induced by a metric \( d \) in term of \( d(u, v) = \|u - v\| \), which is defined by
\[
H(A, B) = \max\{\sup_{u \in A} \inf_{v \in B} \|u - v\|, \sup_{v \in B} \inf_{u \in A} \|u - v\|\},
\]
for \( A \) and \( B \) in \( C(X) \). If \( A \) and \( B \) are compact sets in \( X \), then for each \( u \in A \), there exists \( v \in B \) such that
\[
\|u - v\| \leq H(A, B).
\]

**Lemma 2.3** (Minty’s lemma) Let \( X \) be a reflexive real Banach space, \( D \) a nonempty closed convex subset of \( X \) and \( X^* \) the dual of \( X \). Let \( T : D \to X^* \) be a monotone and hemicontinuous operator. Then the following are equivalent:
(a) there exists an \( u_0 \in D \) such that
\[
\langle T(u_0), v - u_0 \rangle \geq 0 \text{ for all } v \in D;
\]
(b) there exists an \( u_0 \in D \) such that
\[
\langle T(v), v - u_0 \rangle \geq 0 \text{ for all } v \in D.
\]

## 3 Main results

In this section, we state and prove the following generalized vector version Minty’s lemma under the conditions different from that in [9, 16].

**Theorem 3.1** Let \( X \) and \( Y \) be real Banach spaces, \( D \) be a nonempty convex subset of \( X \), and \( \{C(u) : u \in D\} \) be a family of closed convex solid cone of \( Y \). Let \( S, T : D \to 2^{L(X,Y)} \) be nonempty compact-valued set-valued mappings such that for any \( u, v \in D, \) \( H(S(u + \lambda(v - u)), S(u)) \to 0 \) and \( H(T(u + \lambda(v - u)), T(u)) \to 0 \) as \( \lambda \to 0^+ \), where \( H \) is a Hausdorff metric defined on \( L(X,Y) \) and \( \eta : D \times D \to D \) an operator, suppose that the following conditions hold:
(i) \( N : L(X,Y) \times L(X,Y) \to L(X,Y) \) is continuous in the first and in the second arguments, respectively;
(ii) \( h : D \to Y \) is a continuous mapping ;
(iii) \( \langle N(p, q), \eta(v, v) \rangle \in C(u) \), for each \( v, u \in D \) and \( p \in S(v) \), \( q \in T(v) \);
(iv) the operator \( u \mapsto \eta(v, u) \) of \( D \) into \( X \) is continuous for each \( v \in D \);
(v) the operator \( u \mapsto \langle N(p, q), \eta(u, v) \rangle + h(u) - h(v) \) of \( D \) into \( Y \) is affine for each \( v \in D \), \( p \in S(v) \) and \( q \in T(v) \);
(vi) for each \( u, v \in D \), there exist \( s \in S(u) \), \( t \in T(u) \) such that
\[
\langle N(s, t), \eta(v, u) \rangle + h(v) - h(u) \notin -\text{int } C(u)
\]
implies
\[
\langle N(p, q), \eta(u, v) \rangle + h(u) - h(v) \notin \text{int } C(u)
\]
for any \( p \in S(v) \) and \( q \in T(v) \). Then the following are equivalent:
(a) there exists an \( u_0 \in D \) such that for each \( v \in D \) there exist \( s \in S(u_0) \) and \( t \in T(u_0) \) such that
\[
\langle N(s, t), \eta(v, u_0) \rangle + h(v) - h(u_0) \notin -\text{int } C(u_0);
\]
(b) there exists an \( u_0 \in D \) such that
\[
\langle N(p, q), \eta(u_0, v) \rangle + h(u_0) - h(v) \notin \text{int } C(u_0);
\]
for all \( v \in D \), \( p \in S(v) \) and \( q \in T(v) \).

**Proof** Suppose that there exists an \( u_0 \in D \) such that for each \( v \in D \), there exist \( s \in S(u_0) \), \( t \in T(u_0) \) satisfying
\[
\langle N(s, t), \eta(v, u_0) \rangle + h(v) - h(u_0) \notin -\text{int } C(u_0).
\]
Then it follows from condition (vi) that (b) holds.

Conversely, suppose that there exists an \( u_0 \in D \) such that
\[
\langle N(p, q), \eta(u_0, v) \rangle + h(u_0) - h(v) \notin \text{int } C(u_0)
\]
for all \( v \in D \), \( p \in S(v) \) and \( q \in T(v) \). For any arbitrary \( v \in D \), letting \( v_\lambda = \lambda v + (1 - \lambda)u_0 \), \( 0 < \lambda < 1 \), we have \( v_\lambda \in D \) by the convexity of \( D \). Hence for all \( p_\lambda \in S(v_\lambda) \), \( q_\lambda \in T(v_\lambda) \)
\[
\langle N(p_\lambda, q_\lambda), \eta(u_0, v_\lambda) \rangle + h(u_0) - h(v_\lambda) \notin \text{int } C(u_0). \quad (3.1)
\]
By the affinity of the operator
\[
u \mapsto \langle N(p, q), \eta(u, v) \rangle + h(u) - h(v),
\]
we have
\[
\langle N(p_\lambda, q_\lambda), \eta(v_\lambda, v_\lambda) \rangle + h(v_\lambda) - h(v_\lambda)
\]
\[
= \langle N(p_\lambda, q_\lambda), \eta(\lambda v + (1 - \lambda)u_0, v_\lambda) \rangle + h(\lambda v + (1 - \lambda)u_0) - h(v_\lambda)
\]
\[
= \lambda \langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + \lambda h(v) - \lambda h(v_\lambda)
\]
\[
+ (1 - \lambda) \langle N(p_\lambda, q_\lambda), \eta(u_0, v_\lambda) \rangle + (1 - \lambda) h(u_0) - (1 - \lambda) h(v_\lambda).
\]
Hence
\[ \langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + h(v) - h(v_\lambda) \notin -int C(u_0). \] (3.2)

In fact, suppose to the contrary that
\[ \langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + h(v) - h(v_\lambda) \in -int C(u_0). \]

Since \(-int C(u_0)\) is a convex cone,
\[ \lambda \langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + \lambda h(v) - \lambda h(v_\lambda) \in -int C(u_0). \]

It follows from the condition (iii) that
\[
\begin{align*}
(1 - \lambda) \langle N(p_\lambda, q_\lambda), \eta(u_0, v_\lambda) \rangle &+ (1 - \lambda)(h(u_0) - h(v_\lambda)) \\
&= \langle N(p_\lambda, q_\lambda), \eta(v_\lambda, v_\lambda) \rangle - \lambda \langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle - \lambda (h(v) + h(v_\lambda)) \\
&\in C(u_0) - (-int C(u_0)) \\
&= int C(u_0).
\end{align*}
\]

Thus
\[
\langle N(p_\lambda, q_\lambda), \eta(u_0, v_\lambda) \rangle + h(u_0) - h(v_\lambda) \in int C(u_0),
\]
which contradicts (3.1). Hence
\[ \langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + h(v) - h(v_\lambda) \notin -int C(u_0). \]

Since \(S(v_\lambda), S(u_0)\) are compact, by Lemma 2.2, for each \(p_\lambda \in S(v_\lambda)\), we can find \(s_\lambda \in S(u_0)\), such that
\[ \|p_\lambda - s_\lambda\| \leq H(S(v_\lambda), S(u_0)). \]

Since \(S(u_0)\) is compact, without loss of generality, we may assume that \(s_\lambda \to s \in S(u_0)\), as \(\lambda \to 0^+\). Moreover, we have
\[ \|p_\lambda - s\| \leq \|p_\lambda - s_\lambda\| + \|s_\lambda - s\| \]
\[ \leq H(S(v_\lambda), S(u_0)) + \|s_\lambda - s\|, \]
since \(H(S(v_\lambda), S(u_0)) \to 0\) as \(\lambda \to 0^+\), then \(p_\lambda \to s\) as \(\lambda \to 0^+\). Using the same argument, we have \(q_\lambda \to t\) as \(\lambda \to 0^+\), where \(q_\lambda \in T(v_\lambda)\) and \(t \in T(u_0)\). By the condition (ii) and (iv), we get that \(h(v_\lambda) \to h(u_0)\) and \(\eta(v, v_\lambda) \to \eta(v, u_0)\) as \(\lambda \to 0^+\).

Moreover, we have
\[
\begin{align*}
&\|\langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + h(v) - h(v_\lambda) - \langle N(s, t), \eta(v, u_0) \rangle - h(v) + h(u_0)\| \\
\leq &\|\langle N(p_\lambda, q_\lambda) - N(s, t), \eta(v, v_\lambda) \rangle\| + \|h(v_\lambda) - h(u_0)\| \\
+ &\|\langle N(s, t), \eta(v, v_\lambda) - \eta(v, u_0) \rangle\| \\
\leq &\|N(p_\lambda, q_\lambda) - N(s, t)\|\|\eta(v, v_\lambda)\| + \|h(v_\lambda) - h(u_0)\| \\
+ &\|N(s, t)\|\|\eta(v, v_\lambda) - \eta(v, u_0)\| \\
\leq &\left(\|N(p_\lambda, q_\lambda) - N(s, q_\lambda)\| + \|N(s, p_\lambda) - N(s, t)\|\right)\|\eta(v, v_\lambda)\| \\
+ &\|h(v_\lambda) - h(u_0)\| + \|N(s, t)\|\|\eta(v, v_\lambda) - \eta(v, u_0)\|. 
\end{align*}
\]
Since \( \{\eta(v, v_\lambda)\} \) and \( \{h(v_\lambda)\} \) are bounded and \( p_\lambda \to s, q_\lambda \to t \) as \( \lambda \to 0^+ \), then
\[
\langle N(p_\lambda, q_\lambda), \eta(v, v_\lambda) \rangle + h(v) - h(v_\lambda) \to \langle N(s, t), \eta(v, u_0) \rangle + h(v) - h(u_0),
\]
as \( \lambda \to 0^+ \). It follows from (3.2) and the closedness of \( Y \setminus (\text{int} C(u_0)) \) that
\[
\langle N(s, t), \eta(v, u_0) \rangle + h(v) - h(u_0) \notin -\text{int}C(u_0)
\]
for all \( v \in D \). This completes the proof.

Now we will apply our vector version of Minty’s lemma (Theorem 3.1) to prove the existence of solution for a vector variational-like inequality.

**Theorem 3.2** Let \( X \) and \( Y \) be two real Banach spaces, \( D \) be a nonempty compact convex subset of \( X \) and \( \{C(u) : u \in D\} \) be a family of closed convex solid cone of \( Y \) such that for any \( u \in D \), \( C(u) \neq Y \). Let \( S, T : D \to 2^{L(X,Y)} \) be set-valued mappings, \( \eta : D \times D \to D \) an operator and \( W : D \to 2^Y \) a set-valued mapping, defined by \( W(u) = Y \setminus (\text{int} C(u)) \), such that the graph \( Gr(W) \) is closed in \( X \times Y \). Suppose that (i)-(iii), (v), (vi) hold and the following condition is satisfied:

(iv): the operator \( u \to \eta(u, v) \) is continuous for each \( v \in D \).

Then there exists an \( u_0 \in D \) such that \( u_0 \in D \) is a solution of the following generalized vector variational-like inequality (denoted by GVVLI-1):
\[
\langle N(p, q), \eta(u_0, v) \rangle + h(u_0) - h(v) \notin \text{int}C(u_0)
\]
for all \( p \in S(v) \) and \( q \in T(v) \).

Further, if \( S \) and \( T \) are nonempty compact-valued set-valued mappings satisfying the condition (iv) in Theorem 3.1 and the following condition: \( H(S(u + \lambda(v - u)), S(u)) \to 0 \) and \( H(T(u + \lambda(v - u)), T(u)) \to 0 \) as \( \lambda \to 0^+ \), where \( H \) is a Hausdorff metric defined on \( L(X,Y) \). Then there exists a \( u_0 \in D \) such that \( u_0 \in D \) is a solution of the following generalized vector variational-like inequality (denoted by GVVLI-2): find \( u_0 \in D \) such that for each \( v \in D \), there exist \( s \in S(u_0), t \in T(u_0) \) such that
\[
\langle N(s, t), \eta(v, u_0) \rangle + h(v) - h(u_0) \notin -\text{int}C(u_0).
\]

**Proof** Define a set-valued mapping \( F_1 : D \to 2^D \) by
\[
F_1(v) = \{ u \in D : \text{there exist } s \in S(u) \text{ and } t \in T(u) \text{ such that } \langle N(s, t), \eta(v, u) \rangle + h(v) - h(u) \notin -\text{int}C(u) \}
\]
for each \( v \in D \). Then \( F_1(v) \) is nonempty for each \( v \in D \), since \( v \in F_1(v) \). Note that \( F_1 \) is a KKM mapping on \( D \). In fact, suppose that \( M = \{u_1, u_2, \ldots, u_n\} \subset D, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0, i = 1, 2, \ldots, n \) and \( u = \sum_{i=1}^{n} \alpha_i u_i \notin \bigcup_{i=1}^{n} F_1(u_i) \). Then for any \( s \in S(u), t \in T(u) \),
\[
\langle N(s, t), \eta(u_i, u) \rangle + h(u_i) - h(u) \in -\text{int}C(u)
\]
By the condition (iii)

\[ \langle N(s, t), \eta(u_j, \sum_{i=1}^{n} \alpha_i u_i) \rangle + h(u_j) - h(\sum_{i=1}^{n} \alpha_i u_i) \in -intC(u) \]

for each \( j = 1, 2, \cdots, n \). By the affinity of the operator

\[ u \mapsto \langle N(s, t), \eta(u, v) \rangle + h(u) - h(v) , \]

it follows that

\[ \langle N(s, t), \eta(\sum_{j=1}^{n} \alpha_j u_j, \sum_{i=1}^{n} \alpha_i u_i) \rangle + h(\sum_{j=1}^{n} \alpha_j u_j) - h(\sum_{i=1}^{n} \alpha_i u_i) \]

\[ = \sum_{j=1}^{n} \alpha_j [\langle N(s, t), \eta(u_j, \sum_{i=1}^{n} \alpha_i u_i) \rangle + h(u_j)] - h(\sum_{i=1}^{n} \alpha_i u_i) \]

\[ \in -intC(u) . \]

By the condition (iii)

\[ \langle N(s, t), \eta(u, u) \rangle \in C(u) \cap (-intC(u)) , \]

and hence \( 0 \in intC(u) \), which contradicts \( C(u) \neq Y \). Therefore, \( F_1 \) is a KKM-mapping on \( D \).

Now, let us define another set-valued mapping \( F_2 : D \to 2^D \) by

\[ F_2(v) = \{ u \in D : \forall p \in S(v), \forall q \in T(v) \ such \ that \ \langle N(p, q), \eta(u, v) \rangle + h(u) - h(v) \notin intC(u) \} , \]

then by the condition (vi), \( F_1(v) \subset F_2(v) \) for each \( v \in D \). Therefore, \( F_2 \) is also a KKM-mapping on \( D \). Next, we show that for any \( v \in D \), \( F_2(v) \) is closed. Indeed, let \( \{ u_n \} \) be a sequence in \( F_2(v) \) such that \( u_n \to u_0 \in D \), since \( u_n \in F_2(v) \) for all \( n \), \( \forall p \in S(v) \) and \( q \in T(v) \) such that

\[ \langle N(p, q), \eta(u_n, v) \rangle + h(u_n) - h(v) \notin intC(u_n) . \quad (3.3) \]

By (ii) and (iv), for all \( p \in S(v), q \in T(v) \), we have

\[ \langle N(p, q), \eta(u_n, v) \rangle + h(u_n) - h(v) \to \langle N(p, q), \eta(u_0, v) \rangle + h(u_0) - h(v) \]

as \( n \to \infty \). From (3.3) and the closedness of \( Gr(W) \),

\[ \langle N(p, q), \eta(u_0, v) \rangle + h(u_0) - h(v) \notin intC(u_0) . \]

Therefore \( u_0 \in F_2(v) \) and so \( F_2(v) \) is closed. Since \( D \) is compact, so is \( F_2(v) \) for all \( v \in D \). Hence, by the KKM-Fan theorem

\[ \bigcap_{v \in D} F_2(v) \neq \emptyset , \]
then there exists an \( u_0 \in D \) such that
\[
\langle N(p,q), \eta(u_0,v) \rangle + h(u_0) - h(v) \notin \text{int} C(u_0)
\]
for any \( v \in D, p \in S(v) \) and \( q \in T(v) \).

Let \( S \) and \( T \) are nonempty compact-valued set-valued mappings such that for any \( u, v \in D, H(S(u + \lambda(v - u)), S(u)) \to 0, H(T(u + \lambda(v - u)), T(u)) \to 0 \) as \( \lambda \to 0^+ \) and the operator \( u \mapsto \eta(v, u) \) of \( D \) into \( X \) is continuous for each \( v \in D \), and \( h : D \to Y \) be lower semicontinuous, then it follows from Theorem 3.1 that there exists an \( u_0 \in D \) such that for each \( v \in D \), \( \exists s \in S(u_0) \) and \( t \in T(u_0) \) such that
\[
\langle N(s,t), \eta(v,u_0) \rangle + h(v) - h(u_0) \notin -\text{int} C(u_0).
\]

This completes the proof of Theorem 3.2.

**Remark 3.1** From Theorem 3.1, it is easy to see that the set of solutions for the GVVLI-1 is \( \cap_{v \in D} F_2(v) = \cap_{v \in D} F_1(v) \), which is nonempty closed and compact, so is that for the GVVLI-2.

**Remark 3.2** Theorems 3.1 and 3.2 generalize the corresponding results in [9,16].

In the following, we give another existence of solutions to GVVLI-1 and GVVLI-2 without the compactness of \( D \).

**Theorem 3.3** Let \( X, Y, C \) and \( W \) be as in Theorem 3.2 and let \( D \) be a nonempty, closed and convex subset of \( X \). Let \( N : L(X,Y) \times L(X,Y) \to L(X,Y), S, T : D \to 2^{L(X,Y)}, \eta : D \times D \to D \) and \( h : D \to Y \). Assume that conditions (i)-(iii), (iv), (v), (vi) hold and the following coercive condition on \( D \) is satisfied:

(vii) there exists a compact subset \( K \) of \( D \) and \( u_0 \in D \backslash K \) such that \( \exists s \in S(u_0), t \in T(u_0) \) and
\[
\langle N(s,t), \eta(v,u_0) \rangle + h(v) - h(u) \in -\text{int} C(u),
\]
for all \( v \in K \). Then the GVVLI-1 has a solution.

Further, if \( S \) and \( T \) are nonempty compact-valued set-valued mappings satisfying the condition (iv) and the following conditions: \( H(S(u + \lambda(v - u)), S(u)) \to 0 \) and \( H(T(u + \lambda(v - u)), T(u)) \to 0 \) as \( \lambda \to 0^+ \), where \( H \) is a Hausdorff metric defined on \( L(X,Y) \). Then there exists a \( u_0 \in D \), such that \( u_0 \) is a solution of the GVVLI-2.

**Proof.** Let
\[
G_1(v) = \{ u \in K : \exists s \in S(u), t \in T(u) \text{ such that } \langle N(s,t), \eta(v,u) \rangle + h(v) - h(u) \notin -\text{int} C(u) \},
\]
\[ G_2(v) = \{ u \in K : \forall p \in S(v), \forall q \in T(v) \text{ such that } \langle N(p, q), \eta(u, v) \rangle + h(u) - h(v) \notin \text{int } C(u) \}, \]

for each \( v \in D \). Now we shall prove \( G_1(v) \neq \emptyset \) for each \( v \in D \). It follows from condition (iii) that \( v \in G_1(v) \neq \emptyset \) for each \( v \in K \). On the other hand, for each fixed \( z \in D \setminus K \), let \( D_z = \overline{\text{co}} \{ K \cup \{ z \} \} \), where \( \overline{\text{co}} \) denotes the closed convex hull of a set. Since \( K \) is compact, we know that \( D_z \) is also compact. For each \( v \in D_z \), let \( G(v) = \{ u \in D_z : \exists s \in S(u), t \in T(u) \text{ such that } \langle N(s, t), \eta(v, u) \rangle + h(v) - h(u) \notin -\text{int } C(u) \} \). It follows from the proof in Theorem 2.1, we know that there exists \( u_1 \in D_z \) and \( s_1 \in S(u_1), t_1 \in T(u_1) \) such that

\[ \langle N(s_1, t_1), \eta(v, u_1) \rangle + h(v) - h(u_1) \notin -\text{int } C(u_1) \] (3.5)

for all \( v \in D_z \). Moreover, we assert that \( u_1 \in K \). In fact, if \( u_1 \in D_z \setminus K \subseteq D \setminus K \), it follows from (vii) that \( \exists s_1 \in S(u), t_1 \in T(u) \) and

\[ \langle N(s_1, t_1), \eta(v, u_1) \rangle + h(v) - h(u_1) \in -\text{int } C(u_1), \]

which contradicts (3.5). So \( u_1 \in K \) and this implies \( G_1(v) \neq \emptyset \) for each \( v \in D_z \). Especially, \( G_1(z) \neq \emptyset \). Since \( z \in D \setminus K \) is arbitrary, we have \( G_1(v) \neq \emptyset \), for all \( v \in D \setminus K \). From the proof in Theorem 3.1, we can obtain that \( G_1(v) \) is a KKM-mapping and \( G_1(v) \subseteq G_2(v) \) for all \( v \in D \) in view of condition (vi). We also know \( G_2(v) \) is closed for all \( v \in D \). In order to obtain our result, we only need to prove that there exists an \( v_0 \in D \) such that \( G_2(v_0) \) is compact. Since \( K \) is compact, \( G_2(v_0) \subseteq K \). Then, \( G_2(v_0) \) is also compact. The rest of the proof is the same as that in Theorem 3.2, therefore is omitted. This completes the proof.

References


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