A Vector Lyapunov Approach to the Stability Problem for the n-Population Continuous Time Replicator Dynamics

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Abstract

In this paper the stability problem for the n-population continuous time replicator dynamics using vector Lyapunov methods is addressed. After introducing the evolutionary stable strategy concept and proving that it is equivalent to being a strict Nash equilibrium, the n-population continuous time replicator dynamics equation is presented. Finally, it is shown that every strict Nash equilibrium is asymptotically stable in the associated dynamics via Lyapunov methods.

Mathematics Subject Classification: 91A22, 34D20, 93D05

1 Preliminaries

Definition 1.1 *A continuous function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing, $\alpha(0) = 0$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.*

Consider the differential system

$$dx/dt = f(t, x) \tag{1}$$

where $f : C(\mathbb{R}_+ \times \mathbb{R}^n) \rightarrow \mathbb{R}^n$ ($x(t, t_0) = x_0$). Suppose that f is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $x(t) = x(t, t_0, x_0)$ of system 1. Then, the next result takes care of its stability.

Theorem 1.2 [1] *Let $V : \mathbb{R}_+ \times B(\rho) \subset \mathbb{R}^n \rightarrow \mathbb{R}_+^s$ ($B(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$) be a continuously differentiable vector Lyapunov function, such that*

$$V_0(t, x) = \sum_{i=1}^s V_i(t, x)$$

satisfies

$$\varphi_1(\|x\|_2) \leq V_0(t, x) \leq \varphi_2(\|x\|_2),$$

for $\varphi_1, \varphi_2 \in K$ and , the differential inequality

$$\dot{V}((x, t) \leq h[t, V(x, t)]$$

holds for all $(t, x) \in \mathfrak{R}_+ \times B(\rho) \subset \mathfrak{R}^n$ where $h : \mathfrak{R}_+ \times \mathfrak{R}_+^s \rightarrow \mathfrak{R}^s$ is a continuously differentiable function such that $h(t, 0) \equiv 0$ and $h(t, V)$ is quasimonotone non-decreasing in V (for all $V \in \mathfrak{R}_+^s$ and $t \in \mathfrak{R}_+$). Let $\omega(t; w_0, t_0)$ be the solution of the comparison system

$$\dot{\omega}(t) = h(\omega(t), t), \omega(t_0) = \omega_0 \geq 0.$$

Then,

$$\dot{V}(t, x(t)) \leq \omega(t; \omega_0, t_0), t \geq t_0$$

provided that $V(t_0, x_0) \leq \omega_0$. Therefore, the stability properties of the trivial solution $\omega = 0$ of the comparison system imply the corresponding stability properties of x .

Corollary 1.3 *In Theorem (1.2):*

- i). *If $h(t, \omega) \equiv 0$ we get stability.*
- ii). *If $h(t, \omega) = -\alpha\omega$, $\alpha > 0$, we get asymptotic stability.*

Remark 1.4 *We encourage those readers not familiar with game theory, its basic concepts and mathematical notations, to see [2] and [3].*

Definition 1.5 *Let Θ be the polyhedron of mixed strategies profiles, a strategy profile $x \in \Theta$ is said to be a strict Nash equilibrium if $\{x\} = B(x)$ the best reply function.*

Definition 1.6 *Let $u_i : \Theta \rightarrow \mathfrak{R}_+$, $i \in I = 1, 2, \dots, n$ be the utility function. A strategy profile $x \in \Theta$ is evolutionary stable if for every strategy $y \neq x$ there exists some $\epsilon_y \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_y)$ and with $w = \epsilon y + (1 - \epsilon)x$*

$$u_i(x_i, w_{-i}) > u_i(y_i, w_{-i}) \text{ for some } i \in I \quad (2)$$

The next result's proof is much in the flavor of the one provided in [2].

Proposition 1.7 *A strategy profile $x \in \Theta$ is evolutionary stable if and only if x is a strict Nash equilibrium.*

Proof 1.8 *First assume that x is a strict Nash equilibrium then $u_i(x_i, x_{-i}) > u_i(y_i, x_{-i})$ for every i , by continuity taking ϵ small we get that $u_i(x_i, \underbrace{\epsilon y_{-i} + (1 - \epsilon)x_{-i}}_{w_{-i}}) > u_i(y_i, \underbrace{\epsilon y_{-i} + (1 - \epsilon)x_{-i}}_{w_{-i}})$.*

Now let us prove the converse: Take $i \in I$ arbitrary and $y_i \in B_i(x)$, $y_i \neq x_i \Rightarrow u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ and set $y_j = x_j$ for $j \neq i$ (i.e., $y_i \in B_i(x)$ for all $i \in I \Rightarrow u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ for all $i \in I$ ($\diamond$)). Since $x \in \Theta$ is evolutionary stable, for some $i \in I$

$$\begin{aligned} & \epsilon u_i(x_i, y_{-i}) + (1 - \epsilon)u_i(x_i, x_{-i}) \\ & > \epsilon u_i(y_i, y_{-i}) + (1 - \epsilon)u_i(y_i, x_{-i}) \stackrel{(\diamond)}{\geq} \epsilon u_i(y_i, y_{-i}) + (1 - \epsilon)u_i(x_i, x_{-i}) \\ & \Rightarrow u_i(x_i, y_{-i}) \geq u_i(y_i, y_{-i}) \end{aligned}$$

i.e., $x_i \in B_i(y)$ for some i but $y_i \in B_i(x)$ for all i therefore, $y_i = x_i$ is a Nash equilibrium and $\{x_i\} = B_i(x)$ and since $i \in I = \{1, 2, \dots, n\}$ was arbitrary we conclude that $\{x\} = B(x)$ and therefore x is a strict Nash equilibrium. ■

Next, the n-population continuous time replicator dynamics is presented. Unlike the single population setting, there are two versions of the continuous time n-population replicator dynamics. We will deal with the one suggested by Taylor [4] which has the form:

$$\dot{x}_{ih} = [u_i(e_i^h, x_{-i}) - u_i(x)] x_i \quad (3)$$

with $x \in \Theta$, pure strategy h and $i \in I = \{1, 2, \dots, n\}$, which by standard Lipschitz arguments has a unique solution.

2 Main Result

Theorem 2.1 *Every $x \in \Theta$, strict Nash equilibrium is asymptotically stable in the n -population continuous time replicator dynamics*

Proof 2.2 *Let $x \in \Theta$, be a strict Nash equilibrium then, x is a vertex of Θ i.e., $x_i = e_i^{h_i}$ and $u_i(e_i^{h_i}, x_{-i}) > u_i(z_i, x_{-i})$ for all $i \in I = \{1, 2, \dots, n\}$ and $z_i \neq x_i$. By continuity this implies that $u_i(y_i, y_{-i}) > u_i(z_i, y_{-i})$ in a $N(x) \cap \Theta$. Now take the neighborhood as small as needed in such away that it contains no other vertex of Θ i.e., the set of strategies y_{-i} is included in the set of strategies x_{-i} . Therefore, from the definition of strict Nash equilibrium with the $x_{-i} | y_{-i}$, we get that $u_i(e_i^{h_i}, y_{-i}) > u_i(y_i, y_{-i}) = u_i(y)$ for all $y \neq x$, $y \in N(x) \cap \Theta$ and all $i \in I$. Now, define as our vector Lyapunov function $V(y) = [V_1(y), V_2(y), \dots, V_n(y)]^T$; where $V_i(y) = \sum_i x_i \log \frac{x_i}{y_i}$, $1 \leq i \leq n$ are relative entropy functions defined in a specific neighborhood of x which without loss of generality will be taken equal to $N(x)$. Then, applying theorem 1.2 in $N(x) \cap \Theta$, we can verify that all the its conditions are satisfied and that,*

$$\dot{V}(y) = -[u_1(e_1^{h_1} - y, y_{-1}), u_2(e_2^{h_2} - y, y_{-2}), \dots, u_{|I|}(e_n^{h_n} - y, y_{-n})]^T < 0^T$$

implying that x is asymptotically stable in the n -population continuous time replicator dynamics. ■

Remark 2.3 *Converse of (2.1) is also true [2]. Therefore, being a strictly Nash equilibrium is equivalent to asymptotic stability.*

3 Conclusions

Asymptotic stability for the continuous time n -population replicator dynamics was shown to hold, in a more natural way than other previous scalar approaches by means of employing vector Lyapunov functions.

References

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Received: April 3, 2007