A Vector Lyapunov Approach to
the Stability Problem for the n-Population
Continuous Time Replicator Dynamics

Zvi Retchkiman Königsberg

Instituto Politécnico Nacional, CIC
Mineria 17-2, Col. Escandon, Mexico D.F 11800, Mexico
mzvi@cic.ipn.mx

Abstract

In this paper the stability problem for the n-population continuous
time replicator dynamics using vector Lyapunov methods is addressed. After introducing the evolutionary stable strategy concept and proving that it is equivalent to being a strict Nash equilibrium, the n-population continuous time replicator dynamics equation is presented. Finally, it is shown that every strict Nash equilibrium is asymptotically stable in the associated dynamics via Lyapunov methods.

Mathematics Subject Classification: 91A22, 34D20, 93D05

1 Preliminaries

Definition 1.1 A continuous function \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is said to belong to class \( \mathcal{K} \) if it is strictly increasing, \( \alpha(0) = 0 \) and \( \alpha(r) \rightarrow \infty \) as \( r \rightarrow \infty \).

Consider the differential system

\[
\frac{dx}{dt} = f(t, x)
\]

where \( f : C(\mathbb{R}_+ \times \mathbb{R}^n) \rightarrow \mathbb{R}^n \) \((x(t, t_0) = x_0)\). Suppose that \( f \) is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions \( x(t) = x(t, t_0, x_0) \) of system 1. Then, the next result takes care of its stability.

Theorem 1.2 [1] Let \( V : \mathbb{R}_+ \times B(\rho) \subset \mathbb{R}^n \rightarrow \mathbb{R}_+^+ \) \((B(\rho) = \{ x \in \mathbb{R}^n : \| x \| < \rho \})\) be a continuously differentiable vector Lyapunov function, such that
\[ V_0(t, x) = \sum_{i=1}^{\mathcal{u}} V_i(t, x) \]
satisfies
\[ \varphi_1(\|x\|_2) \leq V_0(t, x) \leq \varphi_2(\|x\|_2), \]
for \( \varphi_1, \varphi_2 \in K \) and, the differential inequality
\[ \dot{V}(x, t) \leq h[t, V(x, t)] \]
holds for all \( (t, x) \in \mathbb{R}_+ \times B(\rho) \subset \mathbb{R}^n \) where \( h : \mathbb{R}_+ \times \mathbb{R}_+^\mathbf{s} \rightarrow \mathbb{R}^\mathbf{s} \) is a continuously differentiable function such that \( h(t, 0) \equiv 0 \) and \( h(t, V) \) is quasimonotone non-decreasing in \( V \) (for all \( V \in \mathbb{R}_+^\mathbf{s} \) and \( t \in \mathbb{R}_+ \)). Let \( \omega(t; w_0, t_0) \) be the solution of the comparison system

\[ \dot{\omega}(t) = h(\omega(t), t), \quad \omega(t_0) = \omega_0 \geq 0. \]

Then,
\[ \dot{V}(t, x(t)) \leq \omega(t; \omega_0, t_0), \quad t \geq t_0 \]
provided that \( V(t_0, x_0) \leq \omega_0 \). Therefore, the stability properties of the trivial solution \( \omega = 0 \) of the comparison system imply the corresponding stability properties of \( x \).

**Corollary 1.3** In Theorem (1.2):

i). If \( h(t, \omega) \equiv 0 \) we get stability.

ii). If \( h(t, \omega) = -\alpha \omega, \alpha > 0 \), we get asymptotic stability.

**Remark 1.4** We encourage those readers not familiar with game theory, its basic concepts and mathematical notations, to see [2] and [3].

**Definition 1.5** Let \( \Theta \) be the polyhedron of mixed strategies profiles, a strategy profile \( x \in \Theta \) is said to be a strict Nash equilibrium if \( \{x\} = B(x) \) the best reply function.

**Definition 1.6** Let \( u_i : \Theta \rightarrow \mathbb{R}_+ \), \( i \in I = 1, 2, ..., n \) be the utility function. A strategy profile \( x \in \Theta \) is evolutionary stable if for every strategy \( y \neq x \) there exists some \( \epsilon_y \in (0, 1) \) such that for all \( \epsilon \in (0, \epsilon_y) \) and with \( w = \epsilon y + (1 - \epsilon)x \)
\[ u_i(x_i, w_{-i}) > u_i(y_i, w_{-i}) \] for some \( i \in I \) \hspace{1cm} (2)

The next result’s proof is much in the flavor of the one provided in [2].

**Proposition 1.7**  A strategy profile \( x \in \Theta \) is evolutionary stable if and only if \( x \) is a strict Nash equilibrium.

**Proof 1.8**  First assume that \( x \) is a strict Nash equilibrium then \( u_i(x_i, x_{-i}) > u_i(y_i, y_{-i}) \) for every \( i \), by continuity taking \( \epsilon \) small we get that

\[ u_i(x_i, \epsilon y_{-i} + (1 - \epsilon) x_{-i}) > u_i(y_i, \epsilon y_{-i} + (1 - \epsilon) x_{-i}). \]

Now let us prove the converse: Take \( i \in I \) arbitrary and \( y_i \in B_i(x) \), \( y_i \neq x_i \Rightarrow u_i(y_i, x_{-i}) > u_i(x_i, x_{-i}) \) and set \( y_j = x_j \) for \( j \neq i \) (i.e., \( y_i \in B_i(x) \) for all \( i \in I \)) \Rightarrow u_i(y_i, x_{-i}) > u_i(x_i, x_{-i}) \) for all \( i \in I \) (\( \blacklozenge \)). Since \( x \in \Theta \) is evolutionary stable, for some \( i \in I \)

\[ \epsilon u_i(x_i, y_{-i}) + (1 - \epsilon) u_i(x_i, x_{-i}) \]

\[ > \epsilon u_i(y_i, y_{-i}) + (1 - \epsilon) u_i(y_i, x_{-i}) \]

\[ \Rightarrow u_i(x_i, y_{-i}) \geq u_i(y_i, y_{-i}) \]

i.e., \( x_i \in B_i(y) \) for some \( i \) but \( y_i \in B_i(x) \) for all \( i \) therefore, \( y_i = x_i \) is a Nash equilibrium and \( \{x_i\} = B_i(x) \) and since \( i \in I = \{1, 2, \ldots, n\} \) was arbitrary we conclude that \( \{x\} = B(x) \) and therefore \( x \) is a strict Nash equilibrium.\( \blacksquare \)

Next, the \( n \)-population continuous time replicator dynamics is presented. Unlike the single population setting, there are two versions of the continuous time \( n \)-population replicator dynamics. We will deal with the one suggested by Taylor [4] which has the form:

\[ \dot{x}_{ih} = \left[ u_i(e_i^h, x_{-i}) - u_i(x) \right] x_i \] \hspace{1cm} (3)

with \( x \in \Theta \), pure strategy \( h \) and \( i \in I = \{1, 2, \ldots, n\} \), which by standard Lipshitz arguments has a unique solution.
2 Main Result

Theorem 2.1 Every \( x \in \Theta \), strict Nash equilibrium is asymptotically stable in the n-population continuous time replicator dynamics.

Proof 2.2 Let \( x \in \Theta \), be a strict Nash equilibrium then, \( x \) is a vertex of \( \Theta \) i.e., \( x_i = e_i^{h_i} \) and \( u_i(e_i^{h_i}, x_{-i}) > u_i(z_i, x_{-i}) \) for all \( i \in I = \{1, 2, ..., n\} \) and \( z_i \neq x_i \). By continuity this implies that \( u_i(y_i, y_{-i}) > u_i(z_i, y_{-i}) \) in a \( \mathcal{N}(x) \cap \Theta \). Now take the neighborhood as small as needed in such away that it contains no other vertex of \( \Theta \) i.e., the set of strategies \( y_{-i} \) is included in the set of strategies \( x_{-i} \). Therefore, from the definition of strict Nash equilibrium with the \( x_{-i} \mid y_{-i} \), we get that \( u_i(e_i^{h_i}, y_{-i}) > u_i(y_i, y_{-i}) = u_i(y) \) for all \( y \neq x \), \( y \in \mathcal{N}(x) \cap \Theta \) and all \( i \in I \). Now, define as our vector Lyapunov function \( V(y) = [V_1(y), V_2(y), ..., V_n(y)]^T \); where \( V_i(y) = \sum x_i \log \frac{x_i}{y_i}, 1 \leq i \leq n \) are relative entropy functions defined in a specific neighborhood of \( x \) which without loss of generality will be taken equal to \( \mathcal{N}(x) \). Then, applying theorem 1.2 in \( \mathcal{N}(x) \cap \Theta \), we can verify that all the its conditions are satisfied and that,

\[
\dot{V}(y) = - \left[ u_1(e_1^{h_1} - y, y_{-1}), u_2(e_2^{h_2} - y, y_{-2}), ..., u_n(e_n^{h_n} - y, y_{-n}) \right]^T < 0^T
\]

implying that \( x \) is asymptotically stable in the n-population continuous time replicator dynamics. \( \blacksquare \)

Remark 2.3 Converse of (2.1) is also true [2]. Therefore, being a strictly Nash equilibrium is equivalent to asymptotic stability.

3 Conclusions

Asymptotic stability for the continuous time n-population replicator dynamics was shown to hold, in a more natural way than other previous scalar approaches by means of employing vector Lyapunov functions.

References


Received: April 3, 2007