Common Fixed Point Theorems in Sequentially Compact Fuzzy Metric Spaces

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Abstract

In this paper, we give common fixed point theorems in sequentially compact fuzzy metric space $(X, M, *)$ for pairs of weakly compatible mappings and for sequences of self mappings.

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1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [4] in 1965. George and Veeramani [1] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [2]. In this paper we prove common fixed point theorems for pairs of weakly compatible self maps in a sequentially compact fuzzy metric space and also obtain two corollaries.

Definition 1.1 A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous *t*-norm if it satisfies the following conditions

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of continuous *t*-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t -norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X induced by the fuzzy metric M . A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$, for each $t > 0$. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.3 ([5]). Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t , for all x, y in X .

Lemma 1.4 (Proposition 1 of [3]). Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.

Definition 1.5 Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Now we give

Definition 1.6 $(X, M, *)$ is said to be sequentially compact fuzzy metric space if every sequence in X has a convergent sub sequence in it.

Let Φ be the set of all functions $\phi : [0, 1]^5 \longrightarrow [0, 1]$ such that

- (i) ϕ is non decreasing in all coordinates,
- (ii) $\phi(t_1, t_2, t_3, t_4, t_5)$ is continuous in t_4 and t_5 and
- (iii) $\phi(t, t, t, t, t) > t$ for every $t \in [0, 1]$.

2 Main Results

Hereafterwards, assume that $(X, M, *)$ be a sequentially compact fuzzy metric space with $t * t \geq t \quad \forall t \in [0, 1]$.

Theorem 2.1 *Let f, g, S and T be self-mappings on $(X, M, *)$ such that*

$$(1) f(X) \subseteq T(X) \text{ and } g(X) \subseteq S(X),$$

(2)

$$\begin{aligned} [1 + aM(Sx, Ty, t)] * M(fx, gy, t) \\ \geq b[M(Sx, Ty, t) * M(fx, Sx, t) * M(gy, Ty, t) * \\ M(fx, Ty, \alpha t) * M(gy, Sx, (2 - \alpha)t)] + \\ \phi \left(\begin{array}{c} M(Sx, Ty, t), M(fx, Sx, t), M(gy, Ty, t), \\ M(fx, Ty, \alpha t), M(gy, Sx, (2 - \alpha)t) \end{array} \right) \end{aligned}$$

for every $x, y \in X$, for all $t > 0$ and for every $\alpha \in (0, 2)$, where $\phi \in \Phi$ and $-1 \leq a \leq 0, 0 \leq b \leq 1$,

(3) the pairs (f, S) and (g, T) are weakly compatible,

(4) f and S are continuous or g and T are continuous.

Then f, g, S and T have a unique common fixed point p in X . Further p is the unique common fixed point of f and S and of g and T .

Proof. Suppose f and S are continuous.

For every $t > 0$, let $m = \sup \{ M(fx, Sx, t) : x \in X \}$.

Since f and S are continuous on sequentially compact fuzzy metric space, there exists $u \in X$ such that $m = M(fu, Su, t)$.

Since $f(X) \subseteq T(X)$, there exists $v \in X$ such that $fu = Tv$ (5)

Suppose $m < 1$.

Putting $x = u, y = v, \alpha = 1 - q_1, q_1 \in (0, 1)$ in (2) we have

$$\begin{aligned} [1 + aM(Su, Tv, t)] * M(fu, gv, t) \\ \geq b[M(Su, Tv, t) * M(fu, Su, t) * M(gv, Tv, t) * \\ M(fu, Tv, (1 - q_1)t) * M(gv, Su, (1 + q_1)t)] + \\ \phi \left(\begin{array}{c} M(Su, Tv, t), M(fu, Su, t), M(gv, Tv, t), \\ M(fu, Tv, (1 - q_1)t), M(gv, Su, (1 + q_1)t) \end{array} \right) \end{aligned}$$

$$\begin{aligned} [1 + am] * M(Tv, gv, t) &\geq b[m * m * M(gv, Tv, t) * 1 * m * M(gv, Tv, q_1t)] \\ &+ \phi \left(m, m, M(gv, Tv, t), 1, m * M(gv, Tv, q_1t) \right) \end{aligned}$$

Letting $q_1 \rightarrow 1$, we have

$$\begin{aligned} [1 + am] * M(Tv, gv, t) &\geq b[m * M(Tv, gv, t)] \\ &+ \phi \left(m, m, M(gv, Tv, t), 1, M(gv, Tv, t) * m \right) \end{aligned}$$

$$M(Tv, gv, t) \geq \phi \left(m, m, M(gv, Tv, t), 1, M(gv, Tv, t) * m \right)$$

If $m \geq M(gv, Tv, t) = r$ then using $t * t \geq t$ we have
 $r = M(gv, Tv, t) \geq \phi(r, r, r, r, r) > r$. It is a contradiction. Hence we have
 $m < M(gv, Tv, t)$ (6).

Since $g(X) \subseteq S(X)$, there exists $w \in X$ such that $gv = Sw$(7)

Now $M(fw, Sw, t) \leq m < 1$. From (2), with $\alpha = 1 + q_2$, $q_2 \in (0, 1)$ we have

$$\begin{aligned} [1 + aM(Sw, Tv, t)] * M(fw, gv, t) \\ \geq b[M(Sw, Tv, t) * M(fw, Sw, t) * M(gv, Tv, t) * \\ M(fw, Tv, (1 + q_2)t) * M(gv, Sw, (1 - q_2)t)] + \\ \phi \left(\begin{array}{c} M(Sw, Tv, t), M(fw, Sw, t), M(gv, Tv, t), \\ M(fw, Tv, (1 + q_2)t), M(gv, Sw, (1 - q_2)t) \end{array} \right) \end{aligned}$$

$$\begin{aligned} [1 + aM(gv, Tv, t)] * M(fw, Sw, t) \\ \geq b[M(gv, Tv, t) * M(fw, Sw, t) * M(gv, Tv, t) * \\ M(fw, Sw, t) * M(gv, Tv, q_2t) * 1] + \\ \phi \left(\begin{array}{c} M(gv, Tv, t), M(fw, Sw, t), M(gv, Tv, t), \\ M(fw, Sw, t) * M(gv, Tv, q_2t), 1 \end{array} \right) \end{aligned}$$

Letting $q_2 \rightarrow 1$, we have

$$\begin{aligned} [1 + aM(gv, Tv, t)] * M(fw, Sw, t) \\ \geq b[M(gv, Tv, t)] * M(fw, Sw, t) + \\ \phi \left(\begin{array}{c} M(gv, Tv, t), M(fw, Sw, t), M(gv, Tv, t) \\ M(gv, Tv, t) * M(fw, Sw, t), 1 \end{array} \right) \end{aligned}$$

$$M(fw, Sw, t) \geq \phi \left(\begin{array}{c} M(gv, Tv, t), M(fw, Sw, t), M(gv, Tv, t), \\ M(gv, Tv, t) * M(fw, Sw, t), 1 \end{array} \right).$$

If $M(gv, Tv, t) \geq M(fw, Sw, t) = s$ then

$$M(fw, Sw, t) \geq \phi(s, s, s, s, s) > s$$

It is a contradiction. Hence we have $M(gv, Tv, t) < M(fw, Sw, t)$ (8).

Now from definition of m and (8), (6) we have

$m \geq M(fw, Sw, t) > M(gv, Tv, t) > m$. It is a contradiction. Hence $m = 1$.

Thus $fu = Su$(9)

Suppose $M(gv, Tv, t) < 1$. Then from (2) with $\alpha = 1$ we have

$$\begin{aligned} [1 + aM(Su, Tv, t)] * M(fu, gv, t) \\ \geq b[M(Su, Tv, t) * M(fu, Su, t) * M(gv, Tv, t) * \\ M(fu, Tv, t) * M(gv, Su, t)] + \\ \phi \left(\begin{array}{c} M(Su, Tv, t), M(fu, Su, t), M(gv, Tv, t), \\ M(fu, Tv, t), M(gv, Su, t) \end{array} \right) \end{aligned}$$

$$\begin{aligned} [1 + a1] * M(Tv, gv, t) \geq b[1 * 1 * M(gv, Tv, t) * 1 * M(gv, Tv, t)] + \\ \phi \left(1, 1, M(gv, Tv, t), 1, M(gv, Tv, t) \right) \end{aligned}$$

$$M(Tv, gv, t) \geq \phi(1, 1, M(gv, Tv, t), 1, M(gv, Tv, t)) > M(gv, Tv, t).$$

It is a contradiction. Hence $gv = Tv \dots (10)$

Thus $Su = fu = Tv = gv = p$, say $\dots (11)$

Since the pair (f, S) is weakly compatible we have

$$fp = ffu = fSu = Sfu = SSu = Sp \dots (12).$$

Suppose $r = M(p, fp, t) < 1$. From (2) with $\alpha = 1, x = p, y = v$ we have

$$\begin{aligned} [1 + aM(Sp, p, t)] * M(fp, p, t) \\ \geq b[M(Sp, p, t) * 1 * 1 * M(fp, p, t) * M(p, Sp, t)] + \\ \phi(M(Sp, p, t), 1, 1, M(fp, p, t), M(p, Sp, t)) \end{aligned}$$

$$M(fp, p, t) \geq \phi(r, r, r, r, r) > r.$$

It is a contradiction.

Hence $fp = p$. Thus $Sp = fp = p \dots (13)$

Since the pair (g, T) is weakly compatible we have $gp = Tp$. Using (2) with $\alpha = 1, x = u, y = p$ we can show that $gp = p$. Thus $Tp = p = gp \dots (14)$.

Hence p is a common fixed point of f, g, S and T .

Suppose p_0 is another common fixed point of f, g, S and T . Using (2) with $\alpha = 1, x = p, y = p_0$ we can show that $p_0 = p$. Thus p is the unique common fixed point of f, g, S and T .

Suppose p_1 is another common fixed point of f and S . Using (2) with $\alpha = 1, x = p_1, y = p$ we can show that $p_1 = p$. Thus p is the unique common fixed point of f and S . Similarly we can show that p is the unique common fixed point of g and T .

Similarly the theorem holds when g and T are continuous.

Let $\mathbf{N}$ denote the set of all natural numbers.

Corollary 2.2. Let $\{f_i\}, \{g_i\}, \{S_i\}$ and $\{T_i\}$ for every $i \in \mathbf{N}$ be self-mappings on $(X, M, *)$ such that

- (1) there exist $i_0, j_0, m_0, n_0 \in \mathbf{N}$ such that $f_{i_0}(X) \subseteq T_{n_0}(X)$ and $g_{j_0}(X) \subseteq S_{m_0}(X)$,
 (2)

$$M(f_i x, g_j y, t) \geq \phi \left(\begin{array}{c} M(S_m x, T_n y, t), M(f_i x, S_m x, t), M(g_j y, T_n y, t), \\ M(f_i x, T_n y, \alpha t), M(g_j y, S_m x, (2 - \alpha)t) \end{array} \right)$$

for every $x, y \in X$, for all $t > 0$, for every $i, j, m, n \in \mathbf{N}$ and for every $\alpha \in (0, 2)$, where $\phi \in \Phi$,

- (3) the pairs (f_{i_0}, S_{m_0}) and (g_{j_0}, T_{n_0}) are weakly compatible,
 (4) f_{i_0} and S_{m_0} are continuous or g_{j_0} and T_{n_0} are continuous,
 (5) $S_m S_{m_0} = S_{m_0} S_m$, $S_m f_{i_0} = f_{i_0} S_m$ for all $m \in \mathbf{N}$ and $T_n T_{n_0} = T_{n_0} T_n$,
 $T_n g_{j_0} = g_{j_0} T_n$ for all $n \in \mathbf{N}$.

Then the sequences of maps $\{f_i\}, \{g_j\}, \{S_m\}$ and $\{T_n\}$ have a unique common fixed point in X .

Proof. From Theorem 2.1 with $a = 0, b = 0$, the maps $f_{i_0}, g_{j_0}, S_{m_0}$, and T_{n_0} have a unique common fixed point $p \in X$. Further p is the unique common fixed point of f_{i_0} and S_{m_0} and p is the unique common fixed point of g_{j_0} and T_{n_0} . Suppose there exists $i \in \mathbf{N}$ such that $i \neq i_0$ with $f_i(p) \neq p$. Then from (2) with $j = j_0, m = m_0, n = n_0, \alpha = 1$ and $x = y = p$ we have

$$\begin{aligned} M(f_i p, p, t) &= M(f_i p, g_{j_0} p, t) \\ &\geq \phi \left(\begin{array}{c} M(S_{m_0} p, T_{n_0} p, t), M(f_i p, S_{m_0} p, t), M(g_{j_0} p, T_{n_0} p, t), \\ M(f_i p, T_{n_0} p, t), M(g_{j_0} p, S_{m_0} p, t) \end{array} \right) \\ &= \phi(1, M(f_i p, p, t), 1, M(f_i p, p, t), 1) \\ &\geq \phi(r, r, r, r, r) > r \end{aligned}$$

where $r = M(f_i p, p, t)$. It is a contradiction.

Hence $f_i(p) = p$ for all $i \in \mathbf{N}$.

Similarly we can show that $g_j(p) = p$ for all $j \in \mathbf{N}$.

Now from (5), $S_m(p) = S_m(S_{m_0} p) = S_{m_0}(S_m p)$ and $S_m(p) = S_m(f_{i_0} p) = f_{i_0}(S_m p)$. Thus $S_m(p)$ are common fixed points of f_{i_0} and S_{m_0} for each m . But p is the unique common fixed point of f_{i_0} and S_{m_0} . Therefore $S_m(p) = p$ for all m . Similarly we can show that $T_n(p) = p$ for all n . The uniqueness of p follows from (2).

Corollary 2.3. Let f_1, f_2, g_1, g_2, S and T be self-mappings on $(X, M, *)$ such that

- (1) $f_1 f_2(X) \subseteq T(X)$ and $g_1 g_2(X) \subseteq S(X)$,

(2)

$$M(f_1 f_2 x, g_1 g_2 y, t) \geq \phi \left(\begin{array}{l} M(Sx, Ty, t), M(f_1 f_2 x, Sx, t), M(g_1 g_2 y, Ty, t), \\ M(f_1 f_2 x, Ty, \alpha t), M(g_1 g_2 y, Sx, (2 - \alpha)t) \end{array} \right)$$

for every $x, y \in X$, for all $t > 0$ and for every $\alpha \in (0, 2)$, where $\phi \in \Phi$,

(3) the pairs $(f_1 f_2, S)$ and $(g_1 g_2, T)$ are weakly compatible,

(4) $f_1 f_2$ and S are continuous or $g_1 g_2$ and T are continuous.

(5) $f_1 f_2 = f_2 f_1, S f_2 = f_2 S, g_1 g_2 = g_2 g_1, T g_2 = g_2 T$.

Then f_1, f_2, g_1, g_2, S and T have a unique common fixed point in X .

Proof. From Theorem 2.1 with $a = 0, b = 0$, the maps $f_1 f_2, g_1 g_2, S$ and T have a unique common fixed point $p \in X$. Suppose $f_2(p) \neq p$.

From (2) with $x = f_2 p, y = p, \alpha = 1$ we have

$$M(f_1 f_2 f_2 p, g_1 g_2 p, t) \geq \phi \left(\begin{array}{l} M(S f_2 p, T p, t), M(f_1 f_2 f_2 p, S f_2 p, t), M(g_1 g_2 p, T p, t), \\ M(f_1 f_2 f_2 p, T p, t), M(g_1 g_2 p, S f_2 p, t) \end{array} \right)$$

$$M(f_2 p, p, t) \geq \phi \left(\begin{array}{l} M(f_2 p, p, t), M(f_2 p, f_2 p, t), M(p, p, t), \\ M(f_2 p, p, t), M(p, f_2 p, t) \end{array} \right)$$

$$> M(f_2 p, p, t)$$

Hence $f_2 p = p$. Now $f_1 p = f_1 f_2 p = p$. Similarly we can show that $g_1 p = p = g_2 p$. Uniqueness of p follows from (2).

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