Pairwise Semi Paracompact Spaces

Jamal M. Mustafa

Department of Mathematics
Al al-Bayt University
P.O. Box: 130095
Mafrq, Jordan
jjmmrr971@yahoo.com

Abstract

The purpose of the present paper is to study the concept of pairwise semi paracompactness in bitopological spaces. We give some characterizations of pairwise semi-paracompact spaces. We shall introduce two types of pairwise semi-paracompact spaces and then discuss the relation between them.

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1 Introduction.

The concept of bitopological spaces was initiated by Kelly [6]. A set X equipped with two topologies τ_1 and τ_2 is called a bitopological space denoted by (X, τ_1, τ_2) . *Semi-open* sets were studied by Levine [7], Crosseley [1], [2], Dorsett [3], [4] and Hanna and Dorsett [5].

Let (X, τ) be a topological space and A be a subset of X . We denote the closure of A (the interior of A respectively) by $Cl(A)$ ($Int(A)$ respectively).

A subset A of (X, τ) is called *semi-open* [7] if $A \subseteq Cl(Int(A))$. The complement of a *semi-open* set is called *semi-closed* [1]. The family of all *semi-open* sets in (X, τ) is denoted by $S(\tau)$.

Let A be a subset of a bitopological space (X, τ_1, τ_2) . The closure of A and the interior of A with respect to τ_i are denoted by $Cl_i(A)$ and $Int_i(A)$ respectively.

The intersection of all *semi-closed* sets containing A is called the *semi-closure* [1] of A , denoted by $sCl(A)$. The *semi-closure* of A with respect to $S(\tau_i)$ is denoted by $sCl_i(A)$.

2 Pairwise Semi Paracompact spaces.

In this section we define and study two types of pairwise semi-paracompact spaces namely p_1 - *semi-paracompact* and p_2 - *semi-paracompact* spaces.

Definition 2.1. A family $\tilde{A}$ of subsets of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ - *semi-open* if $\tilde{A} \subset S(\tau_1) \cup S(\tau_2)$. If, in addition $\tilde{A} \cap S(\tau_1) \neq \phi$ and $\tilde{A} \cap S(\tau_2) \neq \phi$ then $\tilde{A}$ is called *pairwise semi-open* (simply p - *semi-open*).

Definition 2.2. If $\tilde{U}$ and $\tilde{N}$ are $\tau_1\tau_2$ - *semi-open* covers of the bitopological space (X, τ_1, τ_2) , then $\tilde{U}$ is called a refinement of $\tilde{N}$ if each $U \in \tilde{U} \cap S(\tau_i)$ is contained in some $V \in \tilde{N} \cap S(\tau_i)$, $i = 1, 2$.

Definition 2.3. A family $\tilde{A}$ of subsets of a space (X, τ) is locally finite in $(X, S(\tau))$ if for each $x \in X$ there exists a *semi-open* set U such that $x \in U$ and U intersects at most finitely many elements of $\tilde{A}$.

Definition 2.4. A collection $\tilde{A}$ of subsets of the bitopological space (X, τ_1, τ_2) is called p_1 -locally finite if $\tilde{A} \cap S(\tau_i)$ is locally finite in $(X, S(\tau_i))$, $i = 1, 2$.

Definition 2.5. A collection $\tilde{A}$ of subsets of the bitopological space (X, τ_1, τ_2) is called p_2 -locally finite if $\tilde{A} \cap S(\tau_i)$ is locally finite in $(X, S(\tau_j))$, $i \neq j$; $i, j = 1, 2$.

Definition 2.6. A bitopological space (X, τ_1, τ_2) is called p_2 - *semi-paracompact* (p_1 - *semi-paracompact* respectively) if each p - *semi-open* cover of X has a p_2 -locally (p_1 -locally respectively) finite $\tau_1\tau_2$ - *semi-open* refinement.

Notice that p_1 - *semi-paracompactness* does not imply p_2 - *semi-paracompactness* and p_2 - *semi-paracompactness* does not imply p_1 - *semi-paracompactness* as shown in the following two examples:

Example 2.7. Let $X = R$ and τ_1, τ_2 be the topologies on X generated by the bases $\mathfrak{B}_1, \mathfrak{B}_2$ respectively, defined by

$$\mathfrak{B}_1 = \{\{x\} : x \geq 0\} \cup \{(-\infty, 0) - F : F \text{ is any finite set}\}, \text{ and}$$

$$\mathfrak{B}_2 = \{\{x\} : x < 0\} \cup \{[0, \infty) - F : F \text{ is any finite set}\}.$$

Then (X, τ_1, τ_2) is p_1 -semi-paracompact space but not p_2 -semi-paracompact for the p -semi-open cover $\{\{x\} : x \in X\}$ has no p_2 -locally finite $\tau_1\tau_2$ -semi-open refinement.

Example 2.8. Let $X = R$ and τ_1, τ_2 be the topologies on X generated by the bases $\mathfrak{B}_1, \mathfrak{B}_2$ respectively, defined by

$$\mathfrak{B}_1 = \{\{x\} : x \leq 0\} \cup \{\{y\} \cup (-\infty, 0) - F : y \succ 0 \text{ and } F \text{ is any finite set}\},$$

and

$$\mathfrak{B}_2 = \{\{x\} : x \succ 0\} \cup \{[0, z) : z \succ 0\} \cup \{\{y\} \cup (0, \infty) - F : y < 0 \text{ and } F \text{ is any finite set}\}.$$

Then (X, τ_1, τ_2) is p_2 -semi-paracompact space but not p_1 -semi-paracompact for the p -semi-open cover $\{\{x\} : x \in X\}$ has no p_1 -locally finite $\tau_1\tau_2$ -semi-open refinement.

Definition 2.9. A subset F of a space (X, τ) is called τ -semi-paracompact relative to X if every semi-open cover of F by members of $S(\tau)$ has a locally finite (in X) refinement by members of $S(\tau)$.

Theorem 2.10. For the bitopological space (X, τ_1, τ_2) the following are equivalent:

- i) (X, τ_1, τ_2) is p_1 -semi-paracompact.
- ii) Each τ_1 -semi-closed proper subset of X is τ_2 -semi-paracompact relative to X , and each τ_2 -semi-closed proper subset of X is τ_1 -semi-paracompact relative to X .

Proof. (i) $\longrightarrow$ (ii). Let $F \neq \phi$ and $F \neq X$ be a τ_1 -semi-closed subset of X and $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_2 -semi-open cover of F . Then $\tilde{N} = \{U_\alpha : \alpha \in \Delta\} \cup \{X - F\}$ is a p -semi-open cover of X . But (X, τ_1, τ_2) is p_1 -semi-paracompact, then $\tilde{N}$ has a p_1 -locally finite $\tau_1\tau_2$ -semi-open refinement, say $\{U'_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$, where $U'_\alpha \in S(\tau_2)$ and $V_\beta \in S(\tau_1)$ for each $\alpha \in \Delta$ and $\beta \in \Gamma$. Thus $\{U'_\alpha : \alpha \in \Delta\}$ is a $S(\tau_2)$ -locally finite τ_2 -semi-open refinement of $\tilde{U}$ for F . Hence F is τ_2 -semi-paracompact relative to X . Similarly each τ_2 -semi-closed proper subset of X is τ_1 -semi-paracompact relative to X .

(ii) $\longrightarrow$ (i). Let $\tilde{U}$ be a p -semi-open cover of X , say $\tilde{U} = \{U_\beta : \beta \in \Gamma\} \cup \{V_\alpha : \alpha \in \Delta\}$ where $U_\beta \in S(\tau_1)$ and $V_\alpha \in S(\tau_2)$ for all $\beta \in \Gamma$ and all $\alpha \in \Delta$. Now we have two cases:

Case 1: If $\cup\{V_\alpha : \alpha \in \Delta\} = X$, choose $\beta_0 \in \Gamma$ such that $U_{\beta_0} \neq \phi$. Since $\{V_\alpha : \alpha \in \Delta\}$ is a τ_2 -semi-open cover of the τ_1 -semi-closed proper subset $X - U_{\beta_0}$, therefore it has a τ_2 -semi-open $S(\tau_2)$ -locally finite refinement $\{V'_\alpha : \alpha \in \Delta\}$. Hence $\{U_{\beta_0}\} \cup \{V'_\alpha : \alpha \in \Delta\}$ is a $\tau_1\tau_2$ -semi-open p_1 -locally finite refinement of $\tilde{U}$.

Case 2: If $\cup\{V_\alpha : \alpha \in \Delta\} \neq X$, then $F = X - \cup\{V_\alpha : \alpha \in \Delta\}$ is a τ_2 -semi-closed proper subset of X , and $F \subseteq \cup\{U_\beta : \beta \in \Gamma\}$. Hence there is a $S(\tau_1)$ -locally finite τ_1 -semi-open refinement of $\{U_\beta : \beta \in \Gamma\}$, say $\{U'_\beta : \beta \in \Gamma\}$. If $\cup\{U'_\beta : \beta \in \Gamma\} = X$ there is nothing to prove. But if $\cup\{U'_\beta : \beta \in \Gamma\} \neq X$ then $X - \cup\{U'_\beta : \beta \in \Gamma\}$ is a τ_1 -semi-closed proper subset contained in $\cup\{V_\alpha : \alpha \in \Delta\}$, so there is a τ_2 -locally finite τ_2 -semi-open refinement $\{V'_\alpha : \alpha \in \Delta\}$ of $\{V_\alpha : \alpha \in \Delta\}$. Therefore $\{U'_\beta : \beta \in \Gamma\} \cup \{V'_\alpha : \alpha \in \Delta\}$ is a $\tau_1\tau_2$ -semi-open p_1 -locally finite refinement of $\tilde{U}$. Hence the result.

Similarly, we can prove the following theorem:

Theorem 2.11. For the bitopological space (X, τ_1, τ_2) the following are equivalent:

- i) (X, τ_1, τ_2) is p_2 -semi-paracompact.
- ii) If F is any τ_i -semi-closed proper subset of X then any τ_j -semi-open cover of F has a τ_j -semi-open $S(\tau_i)$ -locally finite (in X) refinement, $i \neq j; i, j = 1, 2$.

Theorem 2.12. For the bitopological space (X, τ_1, τ_2) the following are equivalent:

- i) (X, τ_1, τ_2) is p_1 -semi-paracompact and p_2 -semi-paracompact.
- ii) Every p -semi-open cover of X has a $S(\tau_1)$ -locally finite $\tau_1\tau_2$ -semi-open refinement and every p -semi-open cover of X has a $S(\tau_2)$ -locally finite $\tau_1\tau_2$ -semi-open refinement.

Proof. (i) $\longrightarrow$ (ii). Let $\tilde{U}$ be a p -semi-open cover of X . Then there exist $U \in \tilde{U} \cap S(\tau_1)$ and $V \in \tilde{U} \cap S(\tau_2)$ such that $U \neq \phi$ and $V \neq \phi$. Since (X, τ_1, τ_2) is p_2 -semi-paracompact, there exists a p_2 -locally finite $\tau_1\tau_2$ -semi-open refinement $\tilde{N}$ of $\tilde{U}$. Since (X, τ_1, τ_2) is p_1 -semi-paracompact, the p -semi-open cover $\tilde{N}$ (add U to the cover $\tilde{N}$ if $\tilde{N} \cap S(\tau_1) = \phi$ and add V to the cover $\tilde{N}$ if $\tilde{N} \cap S(\tau_2) = \phi$) has a p_1 -locally finite $\tau_1\tau_2$ -semi-open refinement $\tilde{O}$. Observe that $(\tilde{N} \cap S(\tau_1)) \cup (\tilde{O} \cap S(\tau_2))$ is a $S(\tau_2)$ -locally finite $\tau_1\tau_2$ -semi-open refinement (replace $\tilde{N} \cap S(\tau_1)$ by $\{U\}$ if $\tilde{N} \cap S(\tau_1) = \phi$) of $\tilde{U}$ and $(\tilde{N} \cap S(\tau_2)) \cup (\tilde{O} \cap S(\tau_1))$ is a $S(\tau_1)$ -locally finite $\tau_1\tau_2$ -semi-open refinement (replace $\tilde{N} \cap S(\tau_2)$ by $\{V\}$ if $\tilde{N} \cap S(\tau_2) = \phi$) of $\tilde{U}$.

By a similar technique used above we can prove that (ii) $\longrightarrow$ (i).

Definition 2.13. In a bitopological space (X, τ_1, τ_2) , $S(\tau_1)$ is said to be *semi-regular* with respect to $S(\tau_2)$ if, for each point x in X and each τ_1 - *semi-closed* set F such that $x \notin F$, there are a τ_1 - *semi-open* set U and a τ_2 - *semi-open* set V such that $x \in U, F \subseteq V$ and $U \cap V = \phi$.

(X, τ_1, τ_2) is *p-semi-regular* if $S(\tau_1)$ is *semi-regular* with respect to $S(\tau_2)$ and vice versa.

Definition 2.14. ([8]). A bitopological space (X, τ_1, τ_2) is said to be *p-semi-Lindelöf* if each *p-semi-open* cover of X has a countable subcover.

Definition 2.15. ([8]). A subset A of a bitopological space (X, τ_1, τ_2) is said to be *p-semi-Lindelöf* relative to (X, τ_1, τ_2) if every *p-semi-open* cover of A , by elements in $S(\tau_1) \cup S(\tau_2)$, has a countable subcover.

Definition 2.16. Let (X, τ_1, τ_2) be a bitopological space. A subset D of X is called *p-semi-dense* if $sCl_1 D = sCl_2 D = X$ and for every *p-semi-open* (in X) p_2 -locally finite (in X) countable cover $\tilde{U}$ of D we have
 $sCl_1 \cup \{U : U \in \tilde{U} \cap S(\tau_1)\} \subseteq sCl_2 \cup \{U : U \in \tilde{U} \cap S(\tau_1)\}$ or
 $sCl_2 \cup \{U : U \in \tilde{U} \cap S(\tau_2)\} \subseteq sCl_1 \cup \{U : U \in \tilde{U} \cap S(\tau_2)\}.$

Theorem 2.17. Let (X, τ_1, τ_2) be a p_2 -*semi-paracompact* *p-semi-regular* space having a *p-semi-dense* subset A where A is *p-semi-Lindelöf* relative to X . Then X is *p-semi-Lindelöf*.

Proof. Let $\tilde{N}$ be a *p-semi-open* cover of X . Then there exist at least two non-empty sets $V_0, U_0 \in \tilde{N}$ such that $V_0 \in S(\tau_1)$ and $U_0 \in S(\tau_2)$. Since (X, τ_1, τ_2) is *p-semi-regular* and p_2 -*semi-paracompact*, $\tilde{N}$ has a p_2 -locally finite $\tau_1\tau_2$ -*semi-open* refinement $\tilde{U}$ such that for each $U \in \tilde{U} \cap S(\tau_1)$, $sCl_2 U$ is contained in some $V \in \tilde{N} \cap S(\tau_1)$ and for each $U \in \tilde{U} \cap S(\tau_2)$, $sCl_1 U$ is contained in some $V \in \tilde{N} \cap S(\tau_2)$. Now $\tilde{U}$ is *p-semi-open* cover of A by *semi-open* sets in $S(\tau_1) \cup S(\tau_2)$ (add V_0 to $\tilde{U}$ if $\tilde{U} \cap S(\tau_1) = \phi$ and add U_0 to $\tilde{U}$ if $\tilde{U} \cap S(\tau_2) = \phi$). Since A is *p-semi-Lindelöf* relative to X , there exists a countable subcover $\tilde{U}'$ of $\tilde{U}$ which covers A . We may assume that $\tilde{U}'$ is *p-semi-open*. But A is *p-semi-dense* in X , so we have two cases to consider:

Case 1. $sCl_1 \cup \{U : U \in \tilde{U}' \cap S(\tau_1)\} \subseteq sCl_2 \cup \{U : U \in \tilde{U}' \cap S(\tau_1)\}$: In this case we have:

$$\begin{aligned} X &= sCl_1 A \\ &\subseteq sCl_1 \cup \{U : U \in \tilde{U}'\} \\ &= [sCl_1 \cup \{U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [sCl_1 \cup \{U : U \in \tilde{U}' \cap S(\tau_2)\}] \\ &\subseteq [sCl_2 \cup \{U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [sCl_1 \cup \{U : U \in \tilde{U}' \cap S(\tau_2)\}] \end{aligned}$$

$$= [\cup\{sCl_2U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [\cup\{sCl_1U : U \in \tilde{U}' \cap S(\tau_2)\}].$$

Case 2. $sCl_2 \cup \{U : U \in \tilde{U}' \cap S(\tau_2)\} \subseteq sCl_1 \cup \{U : U \in \tilde{U}' \cap S(\tau_2)\}$: In this case we have :

$$\begin{aligned} X &= sCl_2A \\ &\subseteq sCl_2 \cup \{U : U \in \tilde{U}'\} \\ &= [sCl_2 \cup \{U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [sCl_2 \cup \{U : U \in \tilde{U}' \cap S(\tau_2)\}] \\ &\subseteq [sCl_2 \cup \{U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [sCl_1 \cup \{U : U \in \tilde{U}' \cap S(\tau_2)\}] \\ &= [\cup\{sCl_2U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [\cup\{sCl_1U : U \in \tilde{U}' \cap S(\tau_2)\}]. \end{aligned}$$

Now in each case, we get

$$X \subseteq [\cup\{sCl_2U : U \in \tilde{U}' \cap S(\tau_1)\}] \cup [\cup\{sCl_1U : U \in \tilde{U}' \cap S(\tau_2)\}].$$

For each $U \in \tilde{U}' \cap S(\tau_i)$ choose one element $V_U \in \tilde{N}$ such that $sCl_i U \subseteq V_U$ ($i \neq j; i, j = 1, 2$). Then $\{V_U : U \in \tilde{U}'\}$ is a countable subcover of $\tilde{N}$ for X . Hence X is p -semi-Lindelöf.

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