On Sensible Fuzzy $R$-Subgroups of Near-Rings with Respect to a $s$-Norm

Kyung Ho Kim

Department of Mathematics, Chungju National University
Chungju 380-702, Korea
ghkim@cjnu.ac.kr

Abstract

Using $s$-norm $S$, we introduce the notion of sensible fuzzy $R$-subgroups in near-rings, and some related properties are investigated.

Mathematics Subject Classification: 06F35, 03G25, 03E72.

Keywords: Near-rings, $s$-norm, sensible, sensible fuzzy $R$-subgrup.

1 Introduction

W. Liu [5] has studied fuzzy ideals of a ring, and many researchers are engaged in extending the concepts. S. Abou-Zaid [1] also introduced the concept of $R$-subgroups of a near-ring and since then the present author et al studied fuzzy $R$-subgroups of a near-ring. In this paper, we will redefine a fuzzy right (resp. left) $R$-subgroup of a near-ring $R$ with respect to a $s$-norm. Some fundamental properties concerning this notion are discussed.

2 Preliminary Notes

In this section we include some elementary aspects that are necessary for this paper.

By a near-ring we mean a non-empty set $R$ with two binary operations “$+$” and “$\cdot$” satisfying the following axioms:

(i) $(R, +)$ is a group,
(ii) $(R, \cdot)$ is a semigroup,
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word “near-ring” in stead of “left near-ring”. We denote
xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$. A two sided $R$-subgroup of a near-ring $R$ is a subset $H$ of $R$ such that

(i) $(H, +)$ is a subgroup of $(R, +),$
(ii) $RH \subset H,$
(iii) $HR \subset H.$

If $H$ satisfies (i) and (ii) then it is called a left $R$-subgroup of $R$. If $H$ satisfies (i) and (iii) then it is called a right $R$-subgroup of $R.$

We now review some fuzzy logic concepts. A fuzzy set $\mu$ in a set $R$ is a function $\mu : R \to [0, 1].$ Let $\text{Im}(\mu)$ denote the image set of $\mu.$ Let $\mu$ be a fuzzy set in a $R.$ For $t \in [0, 1],$ the set

$L(\mu; \alpha) := \{x \in R | \mu(x) \leq \alpha \}$

is called a lower level subset of $\mu.$

Let $R$ be a near-ring and let $\mu$ be a fuzzy set in $R.$ We say that $\mu$ is a fuzzy subnear-ring of $R$ if, for all $x, y \in R,$

(FS1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\},$
(FS2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}.$

If a fuzzy set $\mu$ in a near-ring $R$ satisfies the property (FS1) then $\mu(0) \geq \mu(x)$ for all $x \in R.$

**Definition 2.1.** [6] By a $s$-norm $S,$ we mean a function $S : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:

(S1) $S(x, 0) = x,$
(S2) $S(x, y) \leq S(x, z)$ if $y \leq z,$
(S3) $S(x, y) = S(y, x),$
(S4) $S(x, S(y, z)) = S(S(x, y), z),$

for all $x, y, z \in [0, 1].$

**Proposition 2.2.** Every $s$-norm $S$ has a useful property: $\max(\alpha, \beta) \leq S(\alpha, \beta)$ for all $\alpha, \beta \in [0, 1].$

**Definition 2.3.** Let $S$ be a $s$-norm. A fuzzy set $\mu$ in $R$ is said to satisfy sensible if $\text{Im}(\mu) \subseteq \Omega_S,$ where $\Omega_S := \{\alpha \in [0, 1] | S(\alpha, \alpha) = \alpha\}.$

**Definition 2.4.** [4] Let $(R, +, \cdot)$ be a near-ring. A fuzzy set $\mu$ in $R$ is called a fuzzy right (resp. left) $R$-subgroup of $R$ if

(FR1) $\mu$ is a fuzzy subgroup of $(R, +),$
(FR 2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R.$
Definition 2.5. [3] Let \((R, +, \cdot)\) be a near-ring. A fuzzy set \(\mu\) in \(R\) is called an anti fuzzy right (resp. left) \(R\)-subgroup of \(R\) if

(AF1) \(\mu(x - y) \leq \max\{\mu(x), \mu(y)\}, \forall x, y \in R\),
(AF2) \(\mu(xr) \leq \mu(x)\) (resp. \(\mu(rx) \leq \mu(x)\)), \(\forall r, x \in R\).

3 On sensible fuzzy \(R\)-subgroups

In what follows, let \(R\) denote a near-ring unless otherwise specified.

Definition 3.1. Let \(S\) be a \(s\)-norm. A function \(\mu : R \rightarrow [0, 1]\) is called a fuzzy right (resp. left) \(R\)-subgroup of \(R\) with respect to \(S\) if

(C1) \(\mu(x - y) \leq S(\mu(x), \mu(y))\),
(C2) \(\mu(xr) \leq \mu(x)\) (resp. \(\mu(rx) \leq \mu(x)\))

for all \(r, x \in R\).

If a fuzzy \(R\)-subgroup \(\mu\) of \(R\) with respect to \(S\) is sensible, we say that \(\mu\) is a sensible fuzzy \(R\)-subgroup of \(R\) with respect to \(S\).

Example 3.2. Let \(R = \{a, b, c, d\}\) be a set with two binary operations as follows:

\[
\begin{array}{cccc}
+ & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & a & d & c \\
c & c & d & b & a \\
d & d & c & a & b \\
\end{array}
\quad
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & b & a & a & a \\
c & c & a & a & a \\
d & d & a & b & b \\
\end{array}
\]

Then \((R, +, \cdot)\) is a near-ring. We define a fuzzy set \(\mu\) in \(R\) by \(\mu(c) = \mu(d) = 0.7, \mu(b) = 0.5\) and \(\mu(a) = 0.3\). Let \(S_m\) be a \(s\)-norm defined by

\[S_m(\alpha, \beta) = \min(\alpha + \beta, 1)\text{ for all }\alpha, \beta \in [0, 1].\]

Routine calculations give that \(\mu\) is a fuzzy right \(R\)-subgroup of \(R\) with respect to \(S_m\).

Proposition 3.3. Let \(S\) be a \(s\)-norm on \([0, 1]\). If \(\mu\) is a sensible fuzzy \(R\)-subgroup of \(R\) with respect to \(S\), then we have

\[\mu(0) \leq \mu(x)\]

for all \(x \in R\).

Proof. For every \(x \in R\), we have

\[\mu(0) = \mu(x - x) \leq S(\mu(x), \mu(x)) = \mu(x).\]

This completes the proof. \(\square\)
Proposition 3.4. Let $S$ be a s-norm. If $\mu$ is a sensible fuzzy $R$-subgroup of $R$ with respect to $S$, then the set

$$R_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$$

is an $R$-subgroup of a near-ring $R$.

Proof. Let $S$ be a s-norm and let $x, y \in R_\mu$. Then $\mu(x) = \mu(y) = \mu(0)$. Since $\mu$ is a sensible fuzzy $R$-subgroup of $R$ with respect to $S$, it follows that

$$\mu(x - y) \leq S(\mu(x), \mu(y)) = S(\mu(0), \mu(0)) = \mu(0)$$

so that $\mu(x - y) = \mu(0)$. Thus $x - y \in R_\mu$. Let $x \in R_\mu$ and $r \in R$. Then $\mu(xr) \leq \mu(x) = \mu(0)$, and so $xr \in R_\mu$. This proves the proposition. 

Proposition 3.5. Let $S$ be a s-norm. Every sensible fuzzy $R$-subgroup of $R$ with respect to $S$ is an anti fuzzy $R$-subgroup of $R$.

Proof. Let $\mu$ be a sensible fuzzy $R$-subgroup of $R$ with respect to $S$, then

$$\mu(x - y) \leq S(\mu(x), \mu(y))$$

for all $x, y \in R$. Since $\mu$ is a sensible, we have

$$\max(\mu(x), \mu(y)) = S(\max(\mu(x), \mu(y)), (\max(\mu(x), \mu(y))) \geq S(\mu(x), \mu(y)) \geq \max(\mu(x), \mu(y))$$

by using (S2) and (S3). It follows that $\mu(x - y) \leq S(\mu(x), \mu(y)) = \max(\mu(x), \mu(y))$. Clearly $\mu(xr) \leq \mu(x)$ (resp. $\mu(rx) \leq \mu(x)$) for all $r, x \in R$. So, $\mu$ is an anti fuzzy $R$-subgroup of $R$. 

Proposition 3.6. Let $S$ be a s-norm and let $\mu$ be a fuzzy subset of a near-ring $R$. If $L(\mu; \alpha)$ is a right (resp. left) $R$-subgroup of $R$ for all $t \in \text{Im}(\mu)$, then $\mu$ is a fuzzy right (resp. left) $R$-subgroup with respect to $S$.

Proof. Assume that $L(\mu; \alpha)$ is a right (resp. left) $R$-subgroup of $R$ for all $\alpha \in \text{Im}(\mu)$. Then $0 \in L(\mu; \alpha)$ for all $\alpha \in \text{Im}(\mu)$. Hence $\mu(0) \leq \alpha$ for all $\alpha \in \text{Im}(\mu)$. Let $x, y \in R$ be such that $\mu(x) = \alpha$ and $\mu(y) = \beta$ for some $\alpha, \beta \in \text{Im}(\mu)$. Without loss of generality we may assume that $\alpha \geq \beta$. Then $\mu(y) = \beta \leq \alpha$, and so $x, y \in R_\mu^\alpha$. Since $L(\mu; \alpha)$ is a right (resp. left) $R$-subgroup of $R$, it follows that $x - y \in L(\mu; \alpha)$ and $xr \in L(\mu; \alpha)$ (resp. $rx \in L(\mu; \alpha)$) for all $r \in R$. Therefore

$$\mu(x - y) \leq \alpha = \max\{\alpha, \beta\} = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$$

and $\mu(xr) \leq \alpha = \mu(x)$ (resp. $\mu(rx) \leq \alpha = \mu(x)$). This completes the proof. 

\[\Box\]
The converse of the Proposition 3.6 is not true in general as following Example.

**Example 3.7.** In Example 3.2, if $S_m$ is a $s$-norm defined by

$$S_m(α, β) = \min\{α + β, 1\}$$

for each $α, β \in [0, 1]$ and $μ$ is a fuzzy right $R$-subgroup with respect to $S_m$ given by

$$μ(a) = 0.2, μ(b) = 0.4, μ(c) = 0.6, μ(d) = 0.8,$$

then $L(μ; 0.6) = \{a, b, c\}$ is not a right $R$-subgroup of $R$.

**Theorem 3.8.** Let $S$ be a $s$-norm and let $H$ be a right (resp. left) $R$-subgroup of a near-ring $R$. Then there exists a fuzzy right (resp. left) $R$-subgroup $μ$ with respect to $S$ such that $L(μ; α)$ for some $t \in (0, 1]$.

**Proof.** Define $μ : R \to [0, 1]$ by

$$μ(x) := \begin{cases} 
t & \text{if } x \in H, \\
0 & \text{otherwise},
\end{cases}$$

where $t$ is a fixed number in $(0, 1]$. Let $x, y \in R$. If $x \in R \setminus H$ or $y \in R \setminus H$, then $μ(x) = 0$ or $μ(y) = 0$ and so

$$μ(x − y) ≤ 0 = \max\{μ(x), μ(y)\} ≤ S(μ(x), μ(y)).$$

Assume that $x, y \in H$. Then $x − y \in H$ since $H$ is a subgroup of $(R, +)$. Hence

$$μ(x − y) = t = \max\{μ(x), μ(y)\} ≤ S(μ(x), μ(y)).$$

Now if $x \in R \setminus H$, then $μ(xr) ≤ 0 = μ(x)$ (resp. $μ(rx) ≤ 0 = μ(x)$) for all $r \in R$. If $x \in H$, then $xr ∈ H$ (resp. $rx ∈ H$) which imply that $μ(xr) = t = μ(x)$ (resp. $μ(rx) = t = μ(x)$) for all $r \in R$. Therefore $μ$ is a fuzzy right (resp. left) $R$-subgroup with respect to $S$. It is clear that $L(μ; α) = H$. □

**Theorem 3.9.** Let $S$ be a $s$-norm and let $μ$ be a fuzzy set in a near-ring $R$ with $\text{Im}(μ) = \{t_1, t_2, \ldots, t_n\}$, where $t_i < t_j$ whenever $i > j$. Suppose that there exists a chain of right (resp. left) $R$-subgroups of $R$: $H_0 ⊂ H_1 ⊂ \cdots ⊂ H_n = R$

such that $μ(H_k^*) = t_k$, where $H_k^* = H_k \setminus H_{k−1}$, $H_{−1} = \emptyset$ for $k = 0, 1, \ldots, n$. Then $μ$ is a fuzzy right (resp. left) $R$-subgroup with respect to $S$. 
Proof. Let \( x, y \in R \). If \( x \) and \( y \) belong to the same \( H_k^* \), then \( \mu(x) = \mu(y) = t_k \) and \( x - y \in H_k \). Hence \( \mu(x - y) \leq t_k = \max \{ \mu(x), \mu(y) \} \leq S(\mu(x), \mu(y)) \). Assume that \( x \in H_k^* \) and \( y \in H_j^* \) for every \( i \neq j \). Without loss of generality, we may assume that \( i > j \). Then \( \mu(x) = t_i < t_j = \mu(y) \) and \( x - y \in H_i \). It follows that \( \mu(x - y) \leq t_i = \min \{ \mu(x), \mu(y) \} \leq S(\mu(x), \mu(y)) \). For any \( r \in R \) and \( x \in H_k^* \), we have \( xr \in H_k \) (resp. \( rx \in H_k \)) and so \( \mu(xr) \leq t_k = \mu(x) \) (resp. \( \mu(rx) \leq t_k = \mu(x) \)). Therefore \( \mu \) is a fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \).

\[ \square \]

**Theorem 3.10.** For a right (resp. left) \( R \)-subgroup \( H \) of \( R \) let \( \mu \) be a fuzzy set in \( R \) given by

\[
\mu(x) := \begin{cases} 
\alpha & \text{if } x \in H \\
\beta & \text{otherwise}
\end{cases}
\]

for all \( \alpha, \beta \in [0, 1] \) with \( \alpha < \beta \). Then \( \mu \) is a fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \), where \( S_m \) is the \( s \)-norm in Example 3.7. In particular, if \( \alpha = 1 \) and \( \beta = 0 \), then \( \mu \) is sensible.

Proof. Let \( x, y \in H \). If \( x, y \in H \) then \( x - y \in H \) and so

\[
S_m(\mu(x), \mu(y)) = \min(\alpha + \alpha, 1) \geq \alpha = \mu(x - y).
\]

Assume that \( x \in H \) and \( y \notin H \) (or, \( x \notin H \) and \( y \in H \)). Then \( \mu(x) = \alpha > \beta = \mu(y) \) (or, \( \mu(x) = \beta < \alpha = \mu(y) \)). It follows that

\[
S_m(\mu(x), \mu(y)) = \min(\alpha + \beta, 1) \geq \beta \geq \mu(x - y).
\]

If \( x \notin H \) and \( y \notin H \), then \( \mu(x) = \beta = \mu(y) \) and so

\[
S_m(\mu(x), \mu(y)) = \min(\beta + \beta, 1) \geq \beta \geq \mu(x - y).
\]

Also, let \( x, r \in R \). If \( x \in H \), then \( xr \in H \) (resp. \( rx \in H \)). It follows that \( \mu(xr) = \alpha = \mu(x) \) (resp. \( \mu(rx) = \alpha = \mu(x) \)). If \( x \notin H \), then \( \mu(x) = \beta \) and so \( \mu(xr) \leq \beta = \mu(x) \) (resp. \( \mu(rx) \leq \beta = \mu(x) \)). Hence \( \mu \) is a fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \). Obviously \( \mu \) is sensible when \( \alpha = 1 \) and \( \beta = 0 \).

\[ \square \]

**Theorem 3.11.** Let \( \mu \) be a fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \) and let \( \alpha \in [0, 1] \). Then

(i) if \( \alpha = 0 \) then \( L(\mu; \alpha) \) is either empty or a right (resp. left) \( R \)-subgroup of \( R \).

(ii) if \( T = \max \), then \( L(\mu; \alpha) \) is either empty or a right (resp. left) \( R \)-subgroup of \( R \).
Proof. (i) Assume that \( \alpha = 0 \) and let \( x, y \in L(\mu; \alpha) \). Then \( \mu(x) \leq \alpha = 0 \) and \( \mu(y) \leq \alpha = 0 \). It follows that \( \mu(x - y) \leq S(\mu(x), \mu(y)) \leq S(0, 0) = 0 \) so that \( x - y \in L(\mu; \alpha) \). Also, let \( x, r \in R \) and \( x \in L(\mu; \alpha) \). Then \( \mu(xr) \leq \mu(x) \leq \alpha = 0 \) (resp. \( \mu(rx) \leq \mu(x) \leq \alpha = 0 \)). So, we have \( xr \in L(\mu; \alpha) \). Hence \( L(\mu; \alpha) \) is a right (resp. left) \( R \)-subgroup of \( R \) when \( \alpha = 0 \).

(ii) Assume that \( S = \max \) and let \( x, y \in L(\mu; \alpha) \). Then

\[
\mu(x - y) \leq S(\mu(x), \mu(y)) = \max(\mu(x), \mu(y)) \leq \max(\alpha, \alpha) = \alpha
\]

for all \( \alpha \in [0, 1] \). Hence \( x - y \in L(\mu; \alpha) \). Also, let \( x, r \in R \) and \( x \in L(\mu; \alpha) \). Then \( \mu(xr) \leq \mu(x) \leq \alpha \) (resp. \( \mu(rx) \leq \mu(x) \leq \alpha \)). Hence \( xr \in L(\mu; \alpha) \) and so \( L(\mu; \alpha) \) is a right (resp. left) \( R \)-subgroup of \( R \). \( \square \)

Theorem 3.12. Let \( S \) be a \( s \)-norm and let \( \mu \) be an imaginable fuzzy set in \( R \). If each non-empty lower level set \( L(\mu; \alpha) \) of \( \mu \) is a right (resp. left) \( R \)-subgroup of \( R \), then \( \mu \) is an imaginable fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \) of \( R \).

Proof. Suppose each non-empty lower level set \( L(\mu; \alpha) \) of \( \mu \) is a right (resp. left) \( R \)-subgroup of \( R \). Then we first show that

\[
\mu(x - y) \leq \max(\mu(x), \mu(y)) \text{ for all } x, y \in R.
\]

In fact, if not then there exist \( x_0, y_0 \in R \) such that \( \mu(x_0 - y_0) > \max(\mu(x_0), \mu(y_0)) \).

Taking

\[
\alpha_0 := \frac{1}{2}(\mu(x_0 - y_0) + \max(\mu(x_0), \mu(y_0))),
\]

we get \( \mu(x_0 - y_0) > \alpha_0 > \max(\mu(x_0), \mu(y_0)) \) and thus \( x_0, y_0 \in R^a \) and \( x_0 - y_0 \notin L(\mu; \alpha) \). This is a contradiction. Hence

\[
\mu(x - y) \leq \max(\mu(x), \mu(y)) \leq S(\mu(x), \mu(y))
\]

for all \( x, y \in R \). Now if (C2) is not true, then \( \mu(x_0 r_0) > \mu(x_0) \) (resp. \( \mu(r_0 x_0) > \mu(x_0) \)) for some \( x_0, r_0 \in R \). Taking \( s_1 := \frac{1}{2}(\mu(x_0 r_0) + \mu(x_0)) \) (resp. \( s_2 := \frac{1}{2}(\mu(r_0 x_0) + \mu(x_0)) \)), then \( 0 < \mu(x_0) \leq s_1 \) and \( \mu(x_0 r_0) > s_1 \) (resp. \( 0 < \mu(x_0) \leq s_2 \) and \( \mu(r_0 x_0) > s_2 \)). Hence \( x_0 \in L(\mu; s_1) \) and \( x_0 r_0 \notin L(\mu; s_1) \) (resp. \( x_0 \in L(\mu; s_2) \) and \( r_0 x_0 \notin L(\mu; s_2) \), a contradiction. This completes the proof. \( \square \)

Theorem 3.13. Let \( S \) be a \( s \)-norm satisfying \( S(\alpha, \alpha) = \alpha \) for all \( \alpha \in [0, 1] \).

If a fuzzy set \( \mu \) in \( R \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \) with respect to \( S \), then the non-empty lower level set \( L(\mu; \alpha) \) of \( \mu \) is an \( R \)-subgroup of \( R \) for all \( \alpha \in [0, 1] \).

Proof. Let \( x, y \in L(\mu; \alpha) \) for all \( \alpha \in [0, 1] \). Using (S2) and (S3), we have

\[
\mu(x - y) \leq S(\mu(x), \mu(y)) \leq S(\mu(x), \alpha) = S(\alpha, \mu(x)) \leq S(\alpha, \alpha = \alpha),
\]
Proof. For any \( R \to S \) and \( x, y \in R \) we have \( \mu(xr) \leq \mu(x) \leq \alpha \) (resp. \( \mu(rx) \leq \mu(x) \leq \alpha \)), and so \( xr \in L(\mu; \alpha) \) (resp. \( rx \in L(\mu; \alpha) \)). This proves the theorem.

If \( \mu \) is a fuzzy set in \( R \) and \( \theta \) is a mapping from \( R \) into itself, we define a mapping \( \mu[\theta] : X \to [0, 1] \) by \( \mu[\theta](x) = \mu(\theta(x)) \) for all \( x \in R \).

**Theorem 3.14.** If \( \mu \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \) with respect to \( S \) and \( \theta \) is an endomorphism of \( R \), then \( \mu[\theta] \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \) with respect to \( S \).

*Proof.* For any \( x, y \in R \), we have
\[
\mu[\theta](x - y) = \mu(\theta(x - y)) = \mu(\theta(x) - \theta(y)) \\
\leq S(\mu(\theta(x)), \mu(\theta(y))) = S(\mu[\theta](x), \mu[\theta](y)).
\]
Also, let \( x, r \in R \). Then
\[
\mu[\theta](xr) = \mu(\theta(xr)) = \mu(\theta(x)\theta(r)) \leq \mu(\theta(x)) = \mu[\theta](x).
\]
Similarly, \( \mu[\theta](rx) \geq \mu[\theta](x) \). This completes the proof.

Let \( f \) be a mapping defined on \( R \). If \( \nu \) is a fuzzy set in \( f(R) \) then the fuzzy set \( f^{-1}(\nu) \) in \( R \) defined by \( [f^{-1}(\nu)](x) = \nu(f(x)) \) for all \( x \in R \) is called the preimage of \( \nu \) under \( f \).

**Theorem 3.15.** An onto homomorphic preimage of a fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \) is a fuzzy right (resp. left) \( R \)-subgroup with respect to \( S \).

*Proof.* Let \( f : R \to R' \) be an onto homomorphism of near-ring with sup property and let \( \nu \) be a fuzzy right (resp. left) \( R' \)-subgroup of \( R' \) with respect to \( S \). Then
\[
[f^{-1}(\nu)](x - y) = \nu(f(x) - f(y)) \\
\leq S(\nu(f(x)), \nu(f(y))) \\
= S([f^{-1}(\nu)](x), [f^{-1}(\nu)](y))
\]
for all \( x, y \in R \). Also, let \( x, r \in R \). Then we have
\[
[f^{-1}(\nu)](xr) = \nu(f(xr)) = \nu(f(x)f(r)) \leq \nu(f(x)) = [f^{-1}(\nu)](x)
\]
Similarly, \( [f^{-1}(\nu)](rx) \leq [f^{-1}(\nu)](x) \). Hence \( f^{-1}(\nu) \) is a fuzzy right (resp. left) \( R \)-subgroup of \( R \) with respect to \( S \).

**Acknowledgements.** The research on which this paper is based was supported by a grant from Chungju National University Research Foundation, 2007.
References


Received: April 6, 2007