On Sensible Fuzzy R -Subgroups of Near-Rings with Respect to a s -Norm

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Abstract

Using s -norm S , we introduce the notion of sensible fuzzy R -subgroups in near-rings, and some related properties are investigated.

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1 Introduction

W. Liu [5] has studied fuzzy ideals of a ring, and many researchers are engaged in extending the concepts. S. Abou-Zaid [1] also introduced the concept of R -subgroups of a near-ring and since then the present author et al studied fuzzy R -subgroups of a near-ring. In this paper, we will redefine a fuzzy right (resp. left) R -subgroup of a near-ring R with respect to a s -norm. Some fundamental properties concerning this notion are discussed.

2 Preliminary Notes

In this section we include some elementary aspects that are necessary for this paper.

By a *near-ring* we mean a non-empty set R with two binary operations “+” and “ $\cdot$ ” satisfying the following axioms:

- (i) $(R, +)$ is a group,
- (ii) $(R, \cdot)$ is a semigroup,
- (iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word “near-ring” in stead of “left near-ring”. We denote

xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$. A *two sided R-subgroup* of a near-ring R is a subset H of R such that

- (i) $(H, +)$ is a subgroup of $(R, +)$,
- (ii) $RH \subset H$,
- (iii) $HR \subset H$.

If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R . If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R .

We now review some fuzzy logic concepts. A *fuzzy set* μ in a set R is a function $\mu : R \rightarrow [0, 1]$. Let $Im(\mu)$ denote the image set of μ . Let μ be a fuzzy set in a R . For $t \in [0, 1]$, the set

$$L(\mu; \alpha) := \{x \in R | \mu(x) \leq \alpha\}$$

is called a *lower level subset* of μ .

Let R be a near-ring and let μ be a fuzzy set in R . We say that μ is a *fuzzy subnear-ring* of R if, for all $x, y \in R$,

- (FS1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
- (FS2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$.

If a fuzzy set μ in a near-ring R satisfies the property (FS1) then $\mu(0) \geq \mu(x)$ for all $x \in R$.

Definition 2.1. [6] By a *s-norm* S , we mean a function $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

- (S1) $S(x, 0) = x$,
- (S2) $S(x, y) \leq S(x, z)$ if $y \leq z$,
- (S3) $S(x, y) = S(y, x)$,
- (S4) $S(x, S(y, z)) = S(S(x, y), z)$,

for all $x, y, z \in [0, 1]$.

Proposition 2.2. Every *s-norm* S has a useful property: $\max(\alpha, \beta) \leq S(\alpha, \beta)$ for all $\alpha, \beta \in [0, 1]$.

Definition 2.3. Let S be a *s-norm*. A fuzzy set μ in R is said to satisfy *sensible* if $Im(\mu) \subseteq \Omega_S$, where $\Omega_S := \{\alpha \in [0, 1] \mid S(\alpha, \alpha) = \alpha\}$.

Definition 2.4. [4] Let $(R, +, \cdot)$ be a near-ring. A fuzzy set μ in R is called a *fuzzy right* (resp. *left*) *R-subgroup* of R if

- (FR1) μ is a fuzzy subgroup of $(R, +)$,
- (FR 2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R$.

Definition 2.5. [3] Let $(R, +, \cdot)$ be a near-ring. A fuzzy set μ in R is called an *anti fuzzy right* (resp. *left*) R -subgroup of R if

$$(AF1) \mu(x - y) \leq \max\{\mu(x), \mu(y)\}, \forall x, y \in R,$$

$$(AF2) \mu(xr) \leq \mu(x) \text{ (resp. } \mu(rx) \leq \mu(x)), \forall r, x \in R.$$

3 On sensible fuzzy R -subgroups

In what follows, let R denote a near-ring unless otherwise specified.

Definition 3.1. Let S be a s -norm. A function $\mu : R \rightarrow [0, 1]$ is called a *fuzzy right* (resp. *left*) R -subgroup of R with respect to S if

$$(C1) \mu(x - y) \leq S(\mu(x), \mu(y)),$$

$$(C2) \mu(xr) \leq \mu(x) \text{ (resp. } \mu(rx) \leq \mu(x))$$

for all $r, x \in R$.

If a fuzzy R -subgroup μ of R with respect to S is sensible, we say that μ is a *sensible fuzzy R -subgroup* of R with respect to S .

Example 3.2. Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then $(R, +, \cdot)$ is a near-ring. We define a fuzzy set μ in R by $\mu(c) = \mu(d) = 0.7$, $\mu(b) = 0.5$ and $\mu(a) = 0.3$. Let S_m be a s -norm defined by

$$S_m(\alpha, \beta) = \min(\alpha + \beta, 1) \text{ for all } \alpha, \beta \in [0, 1].$$

Routine calculations give that μ is a fuzzy right R -subgroup of R with respect to S_m .

Proposition 3.3. Let S be a s -norm on $[0, 1]$. If μ is a sensible fuzzy R -subgroup of R with respect to S , then we have

$$\mu(0) \leq \mu(x)$$

for all $x \in R$.

Proof. For every $x \in R$, we have

$$\mu(0) = \mu(x - x) \leq S(\mu(x), \mu(x)) = \mu(x).$$

This completes the proof. $\square$

Proposition 3.4. *Let S be a s -norm. If μ is a sensible fuzzy R -subgroup of R with respect to S , then the set*

$$R_\mu = \{x \in X \mid \mu(x) = \mu(0)\}$$

is an R -subgroup of a near-ring R .

Proof. Let S be a s -norm and let $x, y \in R_\mu$. Then $\mu(x) = \mu(y) = \mu(0)$. Since μ is a sensible fuzzy R -subgroup of R with respect to S , it follows that

$$\mu(x - y) \leq S(\mu(x), \mu(y)) = S(\mu(0), \mu(0)) = \mu(0)$$

so that $\mu(x - y) = \mu(0)$. Thus $x - y \in R_\mu$. Let $x \in R_\mu$ and $r \in R$. Then $\mu(xr) \leq \mu(x) = \mu(0)$, and so $xr \in R_\mu$. This proves the proposition. $\square$

Proposition 3.5. *Let S be a s -norm. Every sensible fuzzy R -subgroup of R with respect to S is an anti fuzzy R -subgroup of R .*

Proof. Let μ be a sensible fuzzy R -subgroup of R with respect to S , then

$$\mu(x - y) \leq S(\mu(x), \mu(y))$$

for all $x, y \in R$. Since μ is a sensible, we have

$$\begin{aligned} \max(\mu(x), \mu(y)) &= S(\max(\mu(x), \mu(y)), (\max(\mu(x), \mu(y)))) \\ &\geq S(\mu(x), \mu(y)) \geq \mu(x - y) \end{aligned}$$

by using (S2) and (S3). It follows that $\mu(x - y) \leq S(\mu(x), \mu(y)) = \max(\mu(x), \mu(y))$. Clearly $\mu(xr) \leq \mu(x)$ (resp. $\mu(rx) \leq \mu(x)$) for all $r, x \in R$. So, μ is an anti fuzzy R -subgroup of R . $\square$

Proposition 3.6. *Let S be a s -norm and let μ be a fuzzy subset of a near-ring R . If $L(\mu; \alpha)$ is a right (resp. left) R -subgroup of R for all $\alpha \in \text{Im}(\mu)$, then μ is a fuzzy right (resp. left) R -subgroup with respect to S .*

Proof. Assume that $L(\mu; \alpha)$ is a right (resp. left) R -subgroup of R for all $\alpha \in \text{Im}(\mu)$. Then $0 \in L(\mu; \alpha)$ for all $\alpha \in \text{Im}(\mu)$. Hence $\mu(0) \leq \alpha$ for all $\alpha \in \text{Im}(\mu)$. Let $x, y \in R$ be such that $\mu(x) = \alpha$ and $\mu(y) = \beta$ for some $\alpha, \beta \in \text{Im}(\mu)$. Without loss of generality we may assume that $\alpha \geq \beta$. Then $\mu(y) = \beta \leq \alpha$, and so $x, y \in R_\mu^\alpha$. Since $L(\mu; \alpha)$ is a right (resp. left) R -subgroup of R , it follows that $x - y \in L(\mu; \alpha)$ and $xr \in L(\mu; \alpha)$ (resp. $rx \in L(\mu; \alpha)$) for all $r \in R$. Therefore

$$\mu(x - y) \leq \alpha = \max\{\alpha, \beta\} = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$$

and $\mu(xr) \leq \alpha = \mu(x)$ (resp. $\mu(rx) \leq \alpha = \mu(x)$). This completes the proof. $\square$

The converse of the Proposition 3.6 is not true in general as following Example.

Example 3.7. In Example 3.2, if S_m is a s -norm defined by

$$S_m(\alpha, \beta) = \min\{\alpha + \beta, 1\}$$

for each $\alpha, \beta \in [0, 1]$ and μ is a fuzzy right R -subgroup with respect to S_m given by

$$\mu(a) = 0.2, \mu(b) = 0.4, \mu(c) = 0.6, \mu(d) = 0.8,$$

then $L(\mu; 0.6) = \{a, b, c\}$ is not a right R -subgroup of R .

Theorem 3.8. Let S be a s -norm and let H be a right (resp. left) R -subgroup of a near-ring R . Then there exists a fuzzy right (resp. left) R -subgroup μ with respect to S such that $L(\mu; \alpha)$ for some $t \in (0, 1]$.

Proof. Define $\mu : R \rightarrow [0, 1]$ by

$$\mu(x) := \begin{cases} t & \text{if } x \in H, \\ 0 & \text{otherwise,} \end{cases}$$

where t is a fixed number in $(0, 1]$. Let $x, y \in R$. If $x \in R \setminus H$ or $y \in R \setminus H$, then $\mu(x) = 0$ or $\mu(y) = 0$ and so

$$\mu(x - y) \leq 0 = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y)).$$

Assume that $x, y \in H$. Then $x - y \in H$ since H is a subgroup of $(R, +)$. Hence

$$\mu(x - y) = t = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y)).$$

Now if $x \in R \setminus H$, then $\mu(xr) \leq 0 = \mu(x)$ (resp. $\mu(rx) \leq 0 = \mu(x)$) for all $r \in R$. If $x \in H$, then $xr \in H$ (resp. $rx \in H$) which imply that $\mu(xr) = t = \mu(x)$ (resp. $\mu(rx) = t = \mu(x)$) for all $r \in R$. Therefore μ is a fuzzy right (resp. left) R -subgroup with respect to S . It is clear that $L(\mu; \alpha) = H$. $\square$

Theorem 3.9. Let S be a s -norm and let μ be a fuzzy set in a near-ring R with $\text{Im}(\mu) = \{t_1, t_2, \dots, t_n\}$, where $t_i < t_j$ whenever $i > j$. Suppose that there exists a chain of right (resp. left) R -subgroups of R :

$$H_0 \subset H_1 \subset \dots \subset H_n = R$$

such that $\mu(H_k^*) = t_k$, where $H_k^* = H_k \setminus H_{k-1}$, $H_{-1} = \emptyset$ for $k = 0, 1, \dots, n$. Then μ is a fuzzy right (resp. left) R -subgroup with respect to S .

Proof. Let $x, y \in R$. If x and y belong to the same H_k^* , then $\mu(x) = \mu(y) = t_k$ and $x - y \in H_k$. Hence $\mu(x - y) \leq t_k = \max\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$. Assume that $x \in H_i^*$ and $y \in H_j^*$ for every $i \neq j$. Without loss of generality, we may assume that $i > j$. Then $\mu(x) = t_i < t_j = \mu(y)$ and $x - y \in H_i$. It follows that $\mu(x - y) \leq t_i = \min\{\mu(x), \mu(y)\} \leq S(\mu(x), \mu(y))$. For any $r \in R$ and $x \in H_k^*$, we have $xr \in H_k$ (resp. $rx \in H_k$) and so $\mu(xr) \leq t_k = \mu(x)$ (resp. $\mu(rx) \leq t_k = \mu(x)$). Therefore μ is a fuzzy right (resp. left) R -subgroup with respect to S . $\square$

Theorem 3.10. For a right (resp. left) R -subgroup H of R let μ be a fuzzy set in R given by

$$\mu(x) := \begin{cases} \alpha & \text{if } x \in H \\ \beta & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in [0, 1]$ with $\alpha < \beta$. Then μ is a fuzzy right (resp. left) R -subgroup with respect to S , where S_m is the s -norm in Example 3.7. In particular, if $\alpha = 1$ and $\beta = 0$, then μ is sensible.

Proof. Let $x, y \in R$. If $x, y \in H$ then $x - y \in H$ and so

$$S_m(\mu(x), \mu(y)) = \min(\alpha + \alpha, 1) \geq \alpha = \mu(x - y).$$

Assume that $x \in H$ and $y \notin H$ (or, $x \notin H$ and $y \in H$). Then $\mu(x) = \alpha > \beta = \mu(y)$ (or, $\mu(x) = \beta < \alpha = \mu(y)$). It follows that

$$S_m(\mu(x), \mu(y)) = \min(\alpha + \beta, 1) \geq \beta \geq \mu(x - y).$$

If $x \notin H$ and $y \notin H$, then $\mu(x) = \beta = \mu(y)$ and so

$$S_m(\mu(x), \mu(y)) = \min(\beta + \beta, 1) \geq \beta \geq \mu(x - y).$$

Also, let $x, r \in R$. If $x \in H$, then $xr \in H$ (resp. $rx \in H$). It follows that $\mu(xr) = \alpha = \mu(x)$ (resp. $\mu(rx) = \alpha = \mu(x)$). If $x \notin H$, then $\mu(x) = \beta$ and so $\mu(xr) \leq \beta = \mu(x)$ (resp. $\mu(rx) \leq \beta = \mu(x)$). Hence μ is a fuzzy right (resp. left) R -subgroup with respect to S . Obviously μ is sensible when $\alpha = 1$ and $\beta = 0$. $\square$

Theorem 3.11. Let μ be a fuzzy right (resp. left) R -subgroup with respect to of S and let $\alpha \in [0, 1]$. Then

- (i) if $\alpha = 0$ then $L(\mu; \alpha)$ is either empty or a right (resp. left) R -subgroup of R .
- (ii) if $T = \max$, then $L(\mu; \alpha)$ is either empty or a right (resp. left) R -subgroup of R .

Proof. (i) Assume that $\alpha = 0$ and let $x, y \in L(\mu; \alpha)$. Then $\mu(x) \leq \alpha = 0$ and $\mu(y) \leq \alpha = 0$. It follows that $\mu(x - y) \leq S(\mu(x), \mu(y)) \leq S(0, 0) = 0$ so that $x - y \in L(\mu; \alpha)$. Also, let $x, r \in R$ and $x \in L(\mu; \alpha)$. Then $\mu(xr) \leq \mu(x) \leq \alpha = 0$ (resp. $\mu(rx) \leq \mu(x) \leq \alpha = 0$). So, we have $xr \in L(\mu; \alpha)$. Hence $L(\mu; \alpha)$ is a right (resp. left) R -subgroup of R when $\alpha = 0$.

(ii) Assume that $S = \max$ and let $x, y \in L(\mu; \alpha)$. Then

$$\mu(x - y) \leq S(\mu(x), \mu(y)) = \max(\mu(x), \mu(y)) \leq \max(\alpha, \alpha) = \alpha$$

for all $\alpha \in [0, 1]$. Hence $x - y \in L(\mu; \alpha)$. Also, let $x, r \in R$ and $x \in L(\mu; \alpha)$. Then $\mu(xr) \leq \mu(x) \leq \alpha$ (resp. $\mu(rx) \leq \mu(x) \leq \alpha$). Hence $xr \in L(\mu; \alpha)$ and so $L(\mu; \alpha)$ is a right (resp. left) R -subgroup of R . $\square$

Theorem 3.12. *Let S be a s -norm and let μ be an imaginable fuzzy set in R . If each non-empty lower level set $L(\mu; \alpha)$ of μ is a right (resp. left) R -subgroup of R , then μ is an imaginable fuzzy right (resp. left) R -subgroup with respect to S of R .*

Proof. Suppose each non-empty lower level set $L(\mu; \alpha)$ of μ is a right (resp. left) R -subgroup of R . Then we first show that

$$\mu(x - y) \leq \max(\mu(x), \mu(y)) \text{ for all } x, y \in R.$$

In fact, if not then there exist $x_0, y_0 \in R$ such that $\mu(x_0 - y_0) > \max(\mu(x_0), \mu(y_0))$. Taking

$$\alpha_0 := \frac{1}{2}(\mu(x_0 - y_0) + \max(\mu(x_0), \mu(y_0))),$$

we get $\mu(x_0 - y_0) > \alpha_0 > \max(\mu(x_0), \mu(y_0))$ and thus $x_0, y_0 \in R_\mu^\alpha$ and $x_0 - y_0 \notin L(\mu; \alpha)$. This is a contradiction. Hence

$$\mu(x - y) \leq \max(\mu(x), \mu(y)) \leq S(\mu(x), \mu(y))$$

for all $x, y \in R$. Now if (C2) is not true, then $\mu(x_0 r_0) > \mu(x_0)$ (resp. $\mu(r_0 x_0) > \mu(x_0)$) for some $x_0, r_0 \in R$. Taking $s_1 := \frac{1}{2}(\mu(x_0 r_0) + \mu(x_0))$ (resp. $s_2 := \frac{1}{2}(\mu(r_0 x_0) + \mu(x_0))$), then $0 < \mu(x_0) \leq s_1$ and $\mu(x_0 r_0) > s_1$ (resp. $0 < \mu(x_0) \leq s_2$ and $\mu(r_0 x_0) > s_2$). Hence $x_0 \in L(\mu; s_1)$ and $x_0 r_0 \notin L(\mu; s_1)$ (resp. $x_0 \in L(\mu; s_2)$ and $r_0 x_0 \notin L(\mu; s_2)$), a contradiction. This completes the proof. $\square$

Theorem 3.13. *Let S be a s -norm satisfying $S(\alpha, \alpha) = \alpha$ for all $\alpha \in [0, 1]$. If a fuzzy set μ in R is a fuzzy right (resp. left) R -subgroup of R with respect to S , then the non-empty lower level set $L(\mu; \alpha)$ of μ is an R -subgroup of R for all $\alpha \in [0, 1]$.*

Proof. Let $x, y \in L(\mu; \alpha)$ for all $\alpha \in [0, 1]$. Using (S2) and (S3), we have

$$\mu(x - y) \leq S(\mu(x), \mu(y)) \leq S(\mu(x), \alpha) = S(\alpha, \mu(x)) \leq S(\alpha, \alpha) = \alpha,$$

which means that $x - y \in L(\mu; \alpha)$. Let $x \in L(\mu; \alpha)$ and $r \in R$. Then we have $\mu(xr) \leq \mu(x) \leq \alpha$ (resp. $\mu(rx) \leq \mu(x) \leq \alpha$), and so $xr \in L(\mu; \alpha)$ (resp. $rx \in L(\mu; \alpha)$). This proves the theorem. $\square$

If μ is a fuzzy set in R and θ is a mapping from R into itself, we define a mapping $\mu[\theta] : X \rightarrow [0, 1]$ by $\mu[\theta](x) = \mu(\theta(x))$ for all $x \in R$.

Theorem 3.14. *If μ is a fuzzy right (resp. left) R -subgroup of R with respect to S and θ is an endomorphism of R , then $\mu[\theta]$ is a fuzzy right (resp. left) R -subgroup of R with respect to S .*

Proof. For any $x, y \in R$, we have

$$\begin{aligned}\mu[\theta](x - y) &= \mu(\theta(x - y)) = \mu(\theta(x) - \theta(y)) \\ &\leq S(\mu(\theta(x)), \mu(\theta(y))) = S(\mu[\theta](x), \mu[\theta](y)).\end{aligned}$$

Also, let $x, r \in R$. Then

$$\mu[\theta](xr) = \mu(\theta(xr)) = \mu(\theta(x)\theta(r)) \leq \mu(\theta(x)) = \mu[\theta](x)$$

Similarly, $\mu[\theta](rx) \geq \mu[\theta](x)$. This completes the proof. $\square$

Let f be a mapping defined on R . If ν is a fuzzy set in $f(R)$ then the fuzzy set $f^{-1}(\nu)$ in R defined by $[f^{-1}(\nu)](x) = \nu(f(x))$ for all $x \in R$ is called the *preimage* of ν under f .

Theorem 3.15. *An onto homomorphic preimage of a fuzzy right (resp. left) R -subgroup with respect to S is a fuzzy right (resp. left) R -subgroup with respect to S .*

Proof. Let $f : R \rightarrow R'$ be an onto homomorphism of near-ring with sup property and let ν be a fuzzy right (resp. left) R' -subgroup of R' with respect to S . Then

$$\begin{aligned}[f^{-1}(\nu)](x - y) &= \nu(f(x - y)) = \nu(f(x) - f(y)) \\ &\leq S(\nu(f(x)), \nu(f(y))) \\ &= S([f^{-1}(\nu)](x), [f^{-1}(\nu)](y))\end{aligned}$$

for all $x, y \in R$. Also, let $x, r \in R$. Then we have

$$[f^{-1}(\nu)](xr) = \nu(f(xr)) = \nu(f(x)f(r)) \leq \nu(f(x)) = [f^{-1}(\nu)](x)$$

Similarly, $[f^{-1}(\nu)](rx) \leq [f^{-1}(\nu)](x)$. Hence $f^{-1}(\nu)$ is a fuzzy right (resp. left) R -subgroup of R with respect to S . $\square$

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