Resolvent Positive Operator and Two-Parameter Abstract Cauchy Problem

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Abstract

We use the semigroup theory to study the homogeneous two-parameter abstract Cauchy problem 2-RACP

$$\begin{cases} \frac{\partial}{\partial t_i} u(t_1, t_2) = H_i u(t_1, t_2) \\ i = 1, 2 \quad t_i \in [0, a_i] \\ u(0, 0) = u_0, \quad u_0 \in E. \end{cases}$$

Where u is a function from $[0, a_1] \times [0, a_2]$ to ordered Banach space E whose positive cone is normal and generating and $a_1, a_2 > 0$, $H_i : D(H_i) \subseteq E \rightarrow E$, $i = 1, 2$, is a resolvent positive operator. We discuss the existence and uniqueness of solution of 2-RACP. In fact, we claim that if (H_1, H_2) is the generator of a two-parameter integrated semigroup $\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}$ and $u_0 \in \cap_{i=1}^2 D(H_i^2)$ then 2-RACP has a unique solution.

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1. Introduction

Let X be a Banach space, $B(X)$ is the Banach space of all bounded linear operators on X and $\mathbb{R}_+^n = \{(t_1, t_2, \dots, t_n) : t_i \geq 0, i = 1, 2, \dots, n\}$.

By an n-parameter semigroup of operators we mean a homomorphism $W : (\mathbb{R}_+^n, +) \rightarrow B(X)$ for which $W(0) = I$ and denote it by $(X, \mathbb{R}_+^n, W)$. Let

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now $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Trivially for $s \in \mathbb{R}^+$, the component $u_i(s) = W(se_i)$ of W defines a one-parameter semigroup of operators, $i = 1, 2, \dots, n$.

Also for each integers $0 \leq i, j \leq n$, the n -parameter semigroup property implies that, $u_i(s)u_j(s') = u_j(s')u_i(s)$. The n -parameter semigroup $(X, \mathbb{R}_+^n, W)$ is called strongly (respectively, uniformly) continuous if for each $i = 1, 2, \dots, n$, the one-parameter components $u_i(s) = W(se_i)$ are strongly (respectively, uniformly) continuous. One can prove that the n -parameter semigroup $(X, \mathbb{R}_+^n, W)$ is strongly continuous if and only if $\lim_{t \in \mathbb{R}_+^n, t \rightarrow 0} W(t)x = x$, for all $x \in X$, and it is uniformly continuous if and only if $\lim_{t \in \mathbb{R}_+^n, t \rightarrow 0} W(t) = I$. Consider a n -parameter semigroup of operators $(X, \mathbb{R}_+^n, W)$ and let $H_i, i = 1, 2, \dots, n$, be the infinitesimal generator of the component semigroup $\{u_i(t)\}_{t \geq 0}$ of W , $i = 1, 2, \dots, n$. We shall think of $(H_1, H_2, \dots, H_n)$ as the infinitesimal generator of $(X, \mathbb{R}_+^n, W)$.

N -parameter semigroups of operators introduced by Hille in 1944 and one can see some of their properties in [5], [6] and [7].

If W is a C_0 - n -parameter semigroup of operators then by the Hille-Yosida theorem, $H_i, i = 1, 2, \dots, n$, is a closed and densely defined operator.

In [3] one can see that if $(X, \mathbb{R}_+^n, W)$ is a C_0 - n -parameter semigroup of operators with the infinitesimal generator $(H_1, H_2, \dots, H_n)$ and $D(H_i) \subseteq X$ be the domain of $H_i, i = 1, 2, \dots, n$, then

- (a) There is $M \geq 1$ and $\omega_i \in \mathbb{R}, i = 1, 2, \dots, n$, such that $\|W(t_1, t_2, \dots, t_n)\| \leq Me^{\sum_{i=1}^n t_i \omega_i}$. So $\|W(t_1, \dots, t_n)\|$ is bounded in any compact subset of $\mathbb{R}_+^n$.
- (b) If $x \in D(H_i)$, so does $W(t)x$, for each $t \in \mathbb{R}_+^n$, and $H_i W(t)x = W(t)H_i x$, $i = 1, 2, \dots, n$.

Also we have the Hille-Yosida theorem for n -parameter semigroups as follows ([6])

$(H_1, \dots, H_n)$ is the infinitesimal generator of a C_0 - n -parameter semigroup $\{W(t)\}_{t \in \mathbb{R}_+^n}$, satisfying $\|W(t_1, \dots, t_n)\| \leq M_0 e^{\sum_{i=1}^n t_i \omega_i}$, for some $M_0 \geq 1$ and $\omega_i > 0, i = 1, \dots, n$, if and only if

- (a) H_i is a closed densely defined operator, $i = 1, \dots, n$, and $R(\lambda', H_j)R(\lambda, H_i) = R(\lambda, H_i)R(\lambda', H_j)$, for all $\lambda > \omega_i$ and $\lambda' > \omega_j$ and integers $1 \leq i, j \leq n$.
- (b) For each $i = 1, \dots, n$, $[\omega_i, \infty) \subseteq \rho(H_i)$ (the resolvent set of H_i , see [9]) and there is $M \geq 1$, such that $\|R(\lambda, H_i)^k\| \leq \frac{M}{(Re\lambda - \omega_i)^k}, i = 1, \dots, n$, and $Re\lambda > \omega_i$.

Now we define one-parameter integrated semigroup which is introduced by W. Arendt in 1986 (see [1], [2]).

Definition 1.1 Let X be a Banach space. An integrated semigroup is a family $(S_t)_{t \geq 0}$ (sometimes we apply $\{S(t)\}_{t \geq 0}$ for this notion) of bounded linear operators S_t on X with the following properties:

- (a) $S_0 = 0$.

(b) $t \rightarrow S_t$ is strongly continuous.

$$(c) S_s S_t = \int_0^s (S_{r+t} - S_r) dr, \text{ for } s, t \geq 0.$$

A short calculation shows that (c) can also be written as $S_s S_t = \int_0^{s+t} S_r dr - \int_0^s S_r dr - \int_0^t S_r dr$, for all $s, t \geq 0$. A consequence of this is $S_s S_t = S_t S_s$, for all $s, t \geq 0$.

Definition 1.2 An integrated semigroup $(S_t)_{t \geq 0}$ is called exponentially bounded, if there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that $\|S_t\| \leq M e^{\omega t}$, for all $t \geq 0$. Moreover $(S_t)_{t \geq 0}$ is called non-degenerate if $S_t x = 0$, for all $t \geq 0$ implies that $x = 0$.

Let $\{S(t)\}_{t \geq 0}$ be an exponentially bounded integrated semigroup i.e. $\|S(t)\| \leq M e^{\omega t}$, for some constants $M \geq 0$ and $\omega \in \mathbb{R}$, and for all $t \geq 0$. Define $R(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S(t) dt$, for $\lambda > \omega$. Then $\ker R(\lambda)$ is independent of $\lambda > \omega$ (see [2] theorem 1.6.9). Hence $R(\lambda)$ is injective if and only if $\{S(t)\}_{t \geq 0}$ is non-degenerate. In this case there exists a unique operator A satisfying $(\omega, \infty) \subset \rho(A)$ such that $R(\lambda) = (\lambda - A)^{-1}$, for all $\lambda > \omega$. This operator is called the generator of $\{S(t)\}_{t \geq 0}$.

In the following theorem we characterize the generator of a one parameter integrated semigroup.

Theorem 1.3 Let A be a densely defined operator such that $(a, \infty) \subset \rho(A)$ for some $a \geq 0$. Let $M \geq 0$, $\omega \in (-\infty, a]$, then the following assertions are equivalent

- (i) A generates an integrated semigroup $\{S(t)\}_{t \geq 0}$ satisfying $\|S(t)\| \leq M e^{\omega t}$, for all $t \geq 0$.
- (ii) $\|(\lambda - \omega)^{k+1} [R(\lambda, A)/\lambda]^{(k)} / k!\| \leq M$, for all $\lambda > a$, $k = 0, 1, 2, \dots$.

Proof. See [2].□

We can extend the notion of one-parameter integrated semigroup to two-parameter integrated semigroup as follows. (see [8]).

Let $\{W(t, s)\}_{t, s \geq 0}$ be a C_0 -two-parameter semigroup on a Banach space X . We define a family $\{S(t_1, s_1)\}_{t_1, s_1 \geq 0}$ of bounded linear operators on X as follows:

$$S(t_1, s_1) = \int_0^{t_1} W(t, 0) dt + \int_0^{s_1} W(0, s) ds.$$

The family $\{S(t_1, s_1)\}_{t_1, s_1 \geq 0}$ have the following properties:

$$(a) S(0, 0) = 0.$$

(b) $(t_1, s_1) \rightarrow S(t_1, s_1)$ is strongly continuous.

$$(c) S(t_1, s_1) S(t_2, s_2) = \int_0^{t_1} S(t+t_2, 0) - S(t, 0) dt + S(t_1, 0) S(0, s_2)$$

$$\begin{aligned}
& +S(0, s_1)S(t_2, 0) + \int_0^{s_1} S(0, s_2 + s) - S(0, s)ds \\
& = [S(t_1, 0) + S(0, s_1)][S(t_2, 0) + S(0, s_2)].
\end{aligned}$$

(d) $S(t_1, 0)S(0, s_1) = S(0, s_1)S(t_1, 0)$.

Definition 1.4 Let X be a Banach space and consider the family $\{S(t, s)\}_{t,s \geq 0}$ of bounded linear operators on X . We say that $\{S(t, s)\}_{t,s \geq 0}$ is a two-parameter integrated semigroup of operators on X if it satisfies the conditions (a),(b),(c) and (d).

It obtains that if $\{S(t, s)\}_{t,s \geq 0}$ is a two-parameter integrated semigroup then both of the families $\{S(t, 0)\}_{t \geq 0}$ and $\{S(0, s)\}_{s \geq 0}$ are one-parameter integrated semigroups. Moreover from conditions (c) and (d) one can see that $S(t, s)$ commutes with $S(t', s')$, for all $t, t', s, s' \geq 0$.

We say that the two-parameter integrated semigroup $\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}$ is exponentially bounded if there exist $M \geq 0$ and $\omega_1, \omega_2 \in \mathbb{R}$ such that $\|S(t_1, t_2)\| \leq Me^{\omega_1 t_1 + \omega_2 t_2}$, for all $t_1, t_2 \geq 0$ and we say that $\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}$ is non-degenerate if both one-parameter integrated semigroups $\{S(t_1, 0)\}_{t_1 \geq 0}$ and $\{S(0, t_2)\}_{t_2 \geq 0}$ are non-degenerate.

We define the ordered pair (H, K) as the generator of $\{S(t, s)\}_{t,s \geq 0}$ if H is the generator of the integrated semigroup $\{S(t, 0)\}_{t \geq 0}$ and K is the generator of the integrated semigroup $\{S(0, s)\}_{s \geq 0}$. So we obtain H, K as follows

$$H = \lambda - R_\lambda^{-1} \quad \text{in which} \quad R_\lambda = \lambda \int_0^\infty e^{-\lambda t} S(t, 0) dt, \quad \text{for } \lambda \in \rho(H).$$

$$K = \mu - R_\mu^{-1} \quad \text{in which} \quad R_\mu = \mu \int_0^\infty e^{-\mu s} S(0, s) ds, \quad \text{for } \mu \in \rho(K).$$

Proposition 1.5 Let $H: D(H) \subseteq X \rightarrow X$ and $K: D(K) \subseteq X \rightarrow X$ be two densely-defined closed linear operators and there are $a, b \geq 0$ satisfying $(a, \infty) \subseteq \rho(H)$, $(b, \infty) \subseteq \rho(K)$ and let $M \geq 0$, $\omega_1 \in (-\infty, a]$ and $\omega_2 \in (-\infty, b]$ then the following assertions are equivalent:

(i) (H, K) generates a two-parameter integrated semigroup $\{S(t, s)\}_{t,s \geq 0}$ satisfying $\|S(t, s)\| \leq Me^{\omega_1 t + \omega_2 s}$, for all $t, s \geq 0$.

$$(ii) \quad \|(\lambda - \omega_1)^{k+1} \left[\frac{R(\lambda, H)}{\lambda} \right]^{(k)} / k! \| \leq M, \quad \text{for all } \lambda > a, \quad k = 0, 1, 2, \dots$$

$$\|(\mu - \omega_2)^{k+1} \left[\frac{R(\mu, K)}{\mu} \right]^{(k)} / k! \| \leq M, \quad \text{for all } \mu > b, \quad k = 0, 1, 2, \dots$$

$$R(\lambda, H)R(\mu, K) = R(\mu, K)R(\lambda, H), \quad \text{for all } \lambda > a, \quad \mu > b.$$

The first application of integrated semigroup was a result about resolvent positive operator that we explain some preliminaries about it here. One can

see more details in [1].

Definition 1.6 A cone C of vertex 0 is a subset of a vector space L such that for each $x \in C$ and $\lambda > 0$, $\lambda x \in C$. If in addition, C is convex, then C is called a convex cone of vertex 0. Thus a convex cone of vertex 0 is a subset of L such that

- (i) $C + C \subset C$
- (ii) $\lambda C \subset C$ for all $\lambda > 0$.

A cone $C \subset L$ satisfying (i) and (ii) is said to be generating if $L = C - C$. Also if C is a cone in topological vector space L then the dual cone C' of C is defined to be the set $\{f \in L' : f(x) \geq 0 \text{ if } x \in C\}$, where L' is the dual space of L .

A cone C in a topological vector space L is said to be normal if $\mathcal{U} = [\mathcal{U}]$, where $\mathcal{U}$ is a filter of neighborhoods of 0. Hence C is a normal cone if and only if there exists a base of C -saturated neighborhoods of 0.

From now we assume that E denotes a real ordered Banach space whose positive cone E_+ is generating and normal. For example, E may be a Banach lattice or the hermitian part of a C^* -algebra.

Definition 1.7 An operator A on E is called resolvent positive if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

Now let A be a resolvent positive operator. Define $s(A) = \inf\{\omega \in \mathbb{R} : (\omega, \infty) \subset \rho(A) \text{ and } R(\lambda, A) \geq 0 \text{ for all } \lambda > \omega\}$, $D(A)_+ = E_+ \cap D(A)$ and $D(A')_+ = E'_+ \cap D(A')$, where in the second definition we assume that A is densely defined and A' denotes the adjoint of A .

A subset C of E_+ is called cofinal in E_+ if for every $a \in E_+$ there exists $b \in C$ such that $a \leq b$.

If $\{T(t)\}_{t \geq 0}$ is a positive C_0 -semigroup with generator B , then the type (or growth bound) $\omega(B)$ is defined by $\omega(B) = \inf\{W \in \mathbb{R} : \text{there exists } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{Wt} \text{ for all } t \geq 0\}$. One always has $s(B) \leq \omega(B) < \infty$ (See [4] Chapter IV Proposition 2.2).

In the sequel, we state some conditions under which resolvent positive operator A is the infinitesimal generator of a C_0 -semigroup.

Theorem 1.8 Let A be a densely defined resolvent positive operator on E . If $D(A)_+$ is cofinal in E_+ or if $D(A')_+$ is cofinal in E'_+ , then A is the generator of a positive C_0 -semigroup. Moreover, $s(A) = \omega(A)$.

Proof. See [1] Theorem 2.2. $\square$

Corollary 1.9 Assume that $\text{int } E_+ \neq \emptyset$. If A is a densely defined resolvent positive operator on E , then A is the generator of a positive C_0 -semigroup and $s(A) = \omega(A)$.

Proof. See [1] Corollary 2.3. $\square$

Theorem 1.10 Let A be a densely defined resolvent positive operator on E . If there exist $\lambda_0 > s(A)$ and $c > 0$ such that $\|R(\lambda_0, A)a\| \geq c \|a\|$, $a \in E_+$,

then A is the generator of a positive C_0 -semigroup and $s(A) = \omega(A)$.

Proof. See [1] Theorem 2.5. $\square$

Theorem 1.11 Suppose that the norm is additive on the positive cone, that is $\|a + b\| = \|a\| + \|b\|$ for all $a, b \in E_+$ (for example, $E = L^1(X, \mu)$). Let A be a densely defined operator. The following assertions are equivalent:

- (i) A generates a positive C_0 -group (For this concept see [9]).
- (ii) A and $-A$ are resolvent positive and there exist $\lambda > \max\{s(A), \omega(A)\}$ and $c > 0$ such that $\|R(\lambda, \pm A)\| \geq c \|a\|$ for all $a \in E_+$.

Proof. See [1] Theorem 2.6. $\square$

The following theorem shows that a densely defined resolvent positive operator A is the infinitesimal generator of an increasing one-parameter integrated semigroup. This is based on the Hille-Yosida Theorem and can be applied when A has a dense domain.

Theorem 1.12 Let A be a densely defined resolvent positive operator. There exists a unique increasing integrated semigroup $\{S(t)\}_{t \geq 0}$ ($S(t) \leq S(s)$ for $0 \leq t \leq s$) of positive operators such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dS(t) \quad , \lambda > s(A).$$

Proof. See [1] Theorem 4.1. $\square$

In the previous theorem it is not necessary to assume that A has a dense domain if additional assumptions on the space are made.

Definition 1.13 We say that E is an ideal in E'' if for $a \in E$, $b \in E''$, $0 \leq b \leq a$ implies that $b \in E$. Here E is identified with a subspace of E'' (via evaluation) and $E''_+ \cap E = E_+$ (that is, E is an ordered subspace of E'').

Theorem 1.14 Suppose that E is an ideal in E'' . Let A be a resolvent positive operator. Then there exists a unique increasing integrated semigroup $\{S(t)\}_{t \geq 0}$ of operators on E such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dS(t) \quad , \lambda > s(A).$$

Proof. See [1] Theorem 5.7. $\square$

By the result of the two preceding theorems, resolvent positive operator A generates a unique increasing integrated semigroup $\{S(t)\}_{t \geq 0}$ when either A is densely defined or E is an ideal in E'' .

Note that the integrated semigroup generated by a resolvent positive operator is non-degenerate and increasing. The following theorem shows that conversely, every non-degenerate, increasing integrated semigroup is generated by a resolvent positive operator.

Theorem 1.15 Let $\{S(t)\}_t \geq 0$ be a non-degenerate, increasing integrated

semigroup, then there exists a unique resolvent positive operator A such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} dS(t) \quad , \lambda > s(A).$$

Moreover $D(A)$ is given by

$$\{a \in E; \text{there exists a (necessarily unique) } b \in E$$

$$\text{such that } S(t)a = ta + \int_0^t S(r)bdr \text{ for all } t \geq 0\}$$

and so $A(a) = b$.

Proof. See [1] Theorem 6.8. $\square$

We terminate this section with the homogeneous abstract Cauchy problem associated to the resolvent positive operator.

Theorem 1.16 Assume that A in (ACP)

$$\begin{cases} \frac{du(t)}{dt} = Au(t) & t \in (0, T] \\ u(0) = x \end{cases}$$

is a resolvent positive operator and either $D(A)$ is dense or E is an ideal in E'' . For every $x \in D(A^2)$ there exists a unique solution of (ACP). Furthermore, denote by $\{S(t)\}_{t \geq 0}$ the integrated semigroup generated by A . Then $u(t) = \frac{\partial}{\partial t} S(t)x = AS(t)x + x$ for all $t \geq 0$. Moreover if $x \geq 0$ then $u(t) \geq 0$ for all $t \geq 0$.

Proof. See [1] Theorem 7.1. $\square$

2. Resolvent Positive Operator and Two-Parameter Abstract Cauchy Problem

We introduced resolvent positive operator and we stated some theorems about conditions that a resolvent positive operator generates a C_0 -semigroup. We can extend Theorem 1.8 for a pair of resolvent positive operators (H_1, H_2) . Throughout this section E is a real ordered Banach space with generating and normal positive cone.

Theorem 2.1 Let (H_1, H_2) be a pair of densely defined resolvent positive operators on E . If $D(H_i)_+$ is cofinal in E_+ or $D(H'_i)_+$ is cofinal in E'_+ , $i = 1, 2$ and also there exists $\omega \in \mathbb{R}$ such that $[\omega, \infty) \subseteq \bigcap_{i=1}^2 \rho(H_i)$ and $R(\lambda, H_1)R(\lambda', H_2) = R(\lambda', H_2)R(\lambda, H_1)$ for all $\lambda, \lambda' \geq \omega$, then (H_1, H_2) is the generator of a positive C_0 -two-parameter semigroup and $s(H_i) = \omega(H_i)$, $i = 1, 2$.

Proof. It is obvious by Hille-Yosida theorem for n -parameter semigroup and Theorem 1.8. $\square$

Similarly by using this additional condition (commutativity condition of

resolvent operators) we can extend Corollary 1.9 and Theorems 1.10 and 1.11.

Also we can extend Theorems 1.12 and 1.14 for a pair of resolvent positive operators as follows

Theorem 2.2 Suppose that H_1 and H_2 are two densely defined operators or E is an ideal in E'' . If H_1, H_2 are resolvent positive operators and there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \bigcap_{i=1}^2 \rho(H_i)$ and $R(\lambda, H_1)R(\lambda', H_2) = R(\lambda', H_2)R(\lambda, H_1)$ for all $\lambda, \lambda' \geq \omega$, then there exists a unique two-parameter integrated semigroup $\{S(t, s)\}_{t,s \geq 0}$ such that $\{S(t, 0)\}_{t \geq 0}$ and $\{S(0, s)\}_{s \geq 0}$ are increasing one-parameter integrated semigroups and

$$R(\lambda, H_1) = \int_0^\infty e^{-\lambda t} dS(t, 0) \quad \text{for } \lambda > s(H_1)$$

and

$$R(\lambda', H_2) = \int_0^\infty e^{-\lambda' s} dS(0, s) \quad \text{for all } \lambda' > s(H_2).$$

Proof. By Theorems 1.12 and 1.14 there exist increasing one-parameter integrated semigroups $\{S_t\}_{t \geq 0}$ and $\{T_s\}_{s \geq 0}$ such that

$$R(\lambda, H_1) = \int_0^\infty e^{-\lambda t} dS(t) \quad \text{for } \lambda > s(H_1)$$

and

$$R(\lambda', H_2) = \int_0^\infty e^{-\lambda' s} dT(s) \quad \text{for } \lambda' > s(H_2).$$

Now consider the family $S(t, s) = S_t + T_s$ for $s, t \geq 0$. From Proposition 1.5 it yields that $\{S(t, s)\}_{t,s \geq 0}$ is a two-parameter integrated semigroup (see [8] for more details). $\square$

Theorem 2.3 Let $\{S(t, s)\}_{t,s \geq 0}$ be a non-degenerate, two-parameter integrated semigroup such that both of one-parameter integrated semigroups $\{S(t, 0)\}_{t \geq 0}$ and $\{S(0, s)\}_{s \geq 0}$ are increasing. Then there exists a unique pair of resolvent positive operators (H_1, H_2) such that

$$R(\lambda, H_1) = \int_0^\infty e^{-\lambda t} dS(t, 0) \quad \text{for } \lambda > s(H_1)$$

and

$$R(\lambda', H_2) = \int_0^\infty e^{-\lambda' s} dS(0, s) \quad \text{for } \lambda' > s(H_2).$$

Moreover $D(H_1)$ is given by

$$D(H_1) = \{a \in E : \exists! b \in E \text{ such that } S(t, 0)a = ta + \int_0^t S(r, 0)b \, dr \text{ for all } t \geq 0\}$$

where notation $\exists!$ means "there exists a unique". In this case $H_1(a) = b$. Similarly $D(H_2)$ is defined by

$$D(H_2) = \{a' \in E : \exists! b' \in E$$

$$\text{such that } S(0, s)a' = sa' + \int_0^s S(0, p)b' dp \text{ for all } s \geq 0\}$$

and $H_2(a') = b'$.

Proof. It is easily proved by Theorem 1.15. $\square$

In the following theorem we discuss the two-parameter abstract Cauchy problem associated to resolvent positive operators.

Theorem 2.4 Consider 2-RACP

$$\begin{cases} \frac{\partial}{\partial t_i} u(t_1, t_2) = H_i u(t_1, t_2) & i = 1, 2, \quad t_i \in [0, a_i] \\ u(0, 0) = u_0 \end{cases} \quad (1)$$

Where u is a function from $[0, a_1] \times [0, a_2]$ to ordered Banach space E , $a_1, a_2 > 0$, $H_i : D(H_i) \subseteq E \rightarrow E$, $i = 1, 2$, is a resolvent positive operator and $u_0 \in E$.

By a solution of 2-RACP we mean a continuous function $u : [0, a_1] \times [0, a_2] \rightarrow E$ in which u has continuous partial derivatives and $u(t_1, t_2) \in \bigcap_{i=1}^2 D(H_i)$ and u satisfies in 2-RACP.

If either H_1 and H_2 are densely defined or E is an ideal in E'' then for every $u_0 \in \bigcap_{i=1}^2 D(H_i^2)$ there exists a unique solution of the Cauchy problem 2-RACP. Furthermore, denote by $\{S(t, s)\}_{t, s \geq 0}$ the two-parameter integrated semigroup generated by (H_1, H_2) then $u(t_1, t_2) = \frac{\partial}{\partial t_1} S(t_1, 0) \frac{\partial}{\partial t_2} S(0, t_2) u_0$ for all $t_1, t_2 \geq 0$.

Moreover, if $u_0 \geq 0$, then $u(t_1, t_2) \geq 0$ for all $t_1, t_2 \geq 0$.

Proof. The proof of the existence and uniqueness of solutions is completely similar to the proof of Theorem 4.1 of [8] with the difference that $u_0 \in \bigcap_{i=1}^2 D(H_i^2) \subseteq \bigcap_{i=1}^2 D(\widetilde{H}_i)$. Also by Proposition 1.16 one can see that if $u_0 \geq 0$, then $u(t_1, t_2) \geq 0$ for all $t_1, t_2 \geq 0$. $\square$

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