Resolvent Positive Operator and Two-Parameter Abstract Cauchy Problem

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Abstract

We use the semigroup theory to study the homogeneous two-parameter abstract Cauchy problem 2-RACP

\[
\begin{aligned}
\frac{\partial}{\partial t_i} u(t_1, t_2) &= H_i u(t_1, t_2) \\
i &= 1, 2 \\
t_i &\in [0, a_i] \\
u(0, 0) &= u_0, \quad u_0 \in E.
\end{aligned}
\]

Where \( u \) is a function from \([0, a_1] \times [0, a_2]\) to ordered Banach space \( E \) whose positive cone is normal and generating and \( a_1, a_2 > 0 \), \( H_i : D(H_i) \subseteq E \to E \), \( i = 1, 2 \), is a resolvent positive operator. We discuss the existence and uniqueness of solution of 2-RACP. In fact, we claim that if \((H_1, H_2)\) is the generator of a two-parameter integrated semigroup \( \{S(t_1, t_2)\}_{t_1, t_2 \geq 0} \) and \( u_0 \in \cap_{i=1}^2 D(H_i^2) \) then 2-RACP has a unique solution.

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1.Introduction

Let \( X \) be a Banach space, \( B(X) \) is the Banach space of all bounded linear operators on \( X \) and \( \mathbb{R}_+^n = \{(t_1, t_2, \ldots, t_n) : t_i \geq 0, \ i = 1, 2, \ldots, n\} \). By an n-parameter semigroup of operators we mean a homomorphism \( W: (\mathbb{R}_+^n, +) \to B(X) \) for which \( W(0) = I \) and denote it by \((X, \mathbb{R}_+^n, W)\). Let

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now \( \{e_i\}_{i=1}^n \) is the standard basis of \( \mathbb{R}^n \). Trivially for \( s \in \mathbb{R}^+ \), the component \( u_i(s) = W(se_i) \) of \( W \) defines a one-parameter semigroup of operators, \( i = 1, 2, \ldots, n \).

Also for each integers \( 0 \leq i, j \leq n \), the n-parameter semigroup property implies that, \( u_i(s)u_j(s') = u_j(s')u_i(s) \). The n-parameter semigroup \((X, \mathbb{R}_+^n, W)\) is called strongly (respectively, uniformly) continuous if for each \( i = 1, 2, \ldots, n \), the one-parameter components \( u_i(s) = W(se_i) \) are strongly (respectively, uniformly) continuous. One can prove that the n-parameter semigroup \((X, \mathbb{R}_+^n, W)\) is strongly continuous if and only if \( \lim_{t \in \mathbb{R}_+^n, t \rightarrow 0} W(t)x = x \), for all \( x \in X \), and it is uniformly continuous if and only if \( \lim_{t \in \mathbb{R}_+^n, t \rightarrow 0} W(t) = I \). Consider a n-parameter semigroup of operators \((X, \mathbb{R}_+^n, W)\) and let \( H_i, i = 1, 2, \ldots, n \), be the infinitesimal generator of the component semigroup \( \{u_i(t)\}_{t \geq 0} \) of \( W \), \( i = 1, 2, \ldots, n \). We shall think of \((H_1, H_2, \ldots, H_n)\) as the infinitesimal generator of \((X, \mathbb{R}_+^n, W)\).

N-parameter semigroups of operators introduced by Hille in 1944 and one can see some of their properties in [5], [6] and [7].

If \( W \) is a \( C_0 \)-n-parameter semigroup of operators then by the Hille-Yosida theorem, \( H_i, i = 1, 2, \ldots, n \), is a closed and densely defined operator.

In [3] one can see that if \((X, \mathbb{R}_+^n, W)\) is a \( C_0 \)-n-parameter semigroup of operators with the infinitesimal generator \((H_1, H_2, \ldots, H_n)\) and \( D(H_i) \subseteq X \) be the domain of \( H_i, i = 1, 2, \ldots, n \), then

(a) There is \( M \geq 1 \) and \( \omega_i \in \mathbb{R}, i = 1, 2, \ldots, n \), such that \( ||W(t_1, t_2, \ldots, t_n)|| \leq Me^{\sum_{i=1}^n t_i \omega_i} \). So \( ||W(t_1, \ldots, t_n)|| \) is bounded in any compact subset of \( \mathbb{R}_+^n \).

(b) If \( x \in D(H_i) \), so does \( W(t)x \), for each \( t \in \mathbb{R}_+^n \), and \( H_iW(t)x = W(t)H_ix, i = 1, 2, \ldots, n \).

Also we have the Hille-Yosida theorem for n-parameter semigroups as follows ([6])

\[(H_1, \ldots, H_n) \] is the infinitesimal generator of a \( C_0 \)-n-parameter semigroup \( \{W(t)\}_{t \in \mathbb{R}_+^n} \), satisfying \( ||W(t_1, \ldots, t_n)|| \leq M_0 e^{\sum_{i=1}^n t_i \omega_i} \), for some \( M_0 \geq 1 \) and \( \omega_i > 0, i = 1, \ldots, n \), if and only if

(a) \( H_i \) is a closed densely defined operator, \( i = 1, \ldots, n \), and \( R(\lambda', H_j)R(\lambda, H_i) = R(\lambda, H_i)R(\lambda', H_j) \), for all \( \lambda > \omega_i \) and \( \lambda' > \omega_j \) and integers \( 1 \leq i, j \leq n \).

(b) For each \( i = 1, \ldots, n \), \( [\omega_i, \infty) \subseteq \rho(H_i) \) (the resolvent set of \( H_i \), see [9]) and there is \( M \geq 1 \), such that

\[ ||R(\lambda, H_i)^k|| \leq \frac{M}{(Re\lambda - \omega_i)^k}, i = 1, \ldots, n \], and \( Re\lambda > \omega_i \).

Now we define one-parameter integrated semigroup which is introduced by W. Arendt in 1986 (see [1], [2]).

**Definition 1.1** Let \( X \) be a Banach space. An integrated semigroup is a family \( \{S(t)\}_{t \geq 0} \) (sometimes we apply \( \{S(t)\}_{t \geq 0} \) for this notion) of bounded linear operators \( S_t \) on \( X \) with the following properties:

(a) \( S_0 = 0 \).
(b) \( t \to S_t \) is strongly continuous.

(c) \( S_s S_t = \int_0^s (S_{r+t} - S_r) dr \), for \( s, t \geq 0 \).

A short calculation shows that (c) can also be written as \( S_s S_t = \int_0^{s+t} S_r dr - \int_0^s S_r dr - \int_0^t S_r dr \), for all \( s, t \geq 0 \). A consequence of this is \( S_s S_t = S_t S_s \), for all \( s, t \geq 0 \).

**Definition 1.2** An integrated semigroup \( (S_t)_{t \geq 0} \) is called exponentially bounded, if there exist constants \( M \geq 0 \) and \( \omega \in \mathbb{R} \) such that \( \| S_t \| \leq Me^{\omega t} \), for all \( t \geq 0 \). Moreover \( (S_t)_{t \geq 0} \) is called non-degenerate if \( S_t x = 0 \), for all \( t \geq 0 \) implies that \( x = 0 \).

Let \( \{S(t)\}_{t \geq 0} \) be an exponentially bounded integrated semigroup i.e. \( \| S(t) \| \leq Me^{\omega t} \), for some constants \( M \geq 0 \) and \( \omega \in \mathbb{R} \), and for all \( t \geq 0 \). Define \( R(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S(t) dt \), for \( \lambda > \omega \). Then \( \ker R(\lambda) \) is independent of \( \lambda > \omega \) (see [2] theorem 1.6.9). Hence \( R(\lambda) \) is injective if and only if \( \{S(t)\}_{t \geq 0} \) is non-degenerate. In this case there exists a unique operator \( A \) satisfying \( (\omega, \infty) \subset \rho(A) \) such that \( R(\lambda) = (\lambda - A)^{-1} \), for all \( \lambda > \omega \). This operator is called the generator of \( \{S(t)\}_{t \geq 0} \).

In the following theorem we characterize the generator of a one parameter integrated semigroup.

**Theorem 1.3** Let \( A \) be a densely defined operator such that \( (a, \infty) \subset \rho(A) \) for some \( a \geq 0 \). Let \( M \geq 0 \), \( \omega \in (-\infty, a] \), then the following assertions are equivalent

(i) \( A \) generates an integrated semigroup \( \{S(t)\}_{t \geq 0} \) satisfying \( \| S(t) \| \leq Me^{\omega t} \), for all \( t \geq 0 \).

(ii) \( \| (\lambda - \omega)^k [R(\lambda, A)/\lambda]^{(k)}/k! \| \leq M \), for all \( \lambda > a \), \( k = 0, 1, 2, \ldots \).

**Proof.** See [2]. \( \Box \)

We can extend the notion of one-parameter integrated semigroup to two-parameter integrated semigroup as follows. (see [8]).

Let \( \{W(t, s)\}_{t, s \geq 0} \) be a \( C_0 \)-two-parameter semigroup on a Banach space \( X \). We define a family \( \{S(t_1, s_1)\}_{t_1, s_1 \geq 0} \) of bounded linear operators on \( X \) as follows:

\[
S(t_1, s_1) = \int_0^{t_1} W(t, 0) dt + \int_0^{s_1} W(0, s) ds.
\]

The family \( \{S(t_1, s_1)\}_{t_1, s_1 \geq 0} \) have the following properties:

(a) \( S(0, 0) = 0 \).

(b) \((t_1, s_1) \to S(t_1, s_1)\) is strongly continuous.

(c) \( S(t_1, s_1) S(t_2, s_2) = \int_0^{t_1} S(t_2, t_1, s) dt + S(t_1, s_1) S(0, s_2) \)
+S(0, s_1)S(t_2, 0) + \int_0^{t_1} S(0, s_2 + s) - S(0, s)ds
= [S(t_1, 0) + S(0, s_1)][S(t_2, 0) + S(0, s_2)].

(d) \(S(t_1, 0)S(0, s_1) = S(0, s_1)S(t_1, 0)\).

**Definition 1.4** Let \(X\) be a Banach space and consider the family \(\{S(t, s)\}_{t, s \geq 0}\) of bounded linear operators on \(X\). We say that \(\{S(t, s)\}_{t, s \geq 0}\) is a two-parameter integrated semigroup of operators on \(X\) if it satisfies the conditions (a), (b), (c) and (d).

It obtains that if \(\{S(t, s)\}_{t, s \geq 0}\) is a two-parameter integrated semigroup then both of the families \(\{S(t, 0)\}_{t \geq 0}\) and \(\{S(0, s)\}_{s \geq 0}\) are one-parameter integrated semigroups. Moreover from conditions (c) and (d) one can see that \(S(t, s)\) commutes with \(S(t', s')\), for all \(t, t', s, s' \geq 0\).

We say that the two-parameter integrated semigroup \(\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}\) is exponentially bounded if there exist \(M \geq 0\) and \(\omega_1, \omega_2 \in \mathbb{R}\) such that \(\|S(t_1, t_2)\| \leq Me^{\omega_1 t_1 + \omega_2 t_2}\), for all \(t_1, t_2 \geq 0\) and we say that \(\{S(t_1, t_2)\}_{t_1, t_2 \geq 0}\) is non-degenerate if both one-parameter integrated semigroups \(\{S(t, 0)\}_{t \geq 0}\) and \(\{S(0, t)\}_{t \geq 0}\) are non-degenerate.

We define the ordered pair \((H, K)\) as the generator of \(\{S(t, s)\}_{t, s \geq 0}\) if \(H\) is the generator of the integrated semigroup \(\{S(t, 0)\}_{t \geq 0}\) and \(K\) is the generator of the integrated semigroup \(\{S(0, s)\}_{s \geq 0}\). So we obtain \(H, K\) as follows

\[H = \lambda - R_{\lambda}^{-1}\] in which \(R_{\lambda} = \lambda \int_0^\infty e^{-\lambda t} S(t, 0)dt, \ \text{for} \ \lambda \in \rho(H)\).

\[K = \mu - R_{\mu}^{-1}\] in which \(R_{\mu} = \mu \int_0^\infty e^{-\mu s} S(0, s)ds, \ \text{for} \ \mu \in \rho(K)\).

**Proposition 1.5** Let \(H: D(H) \subseteq X \rightarrow X\) and \(K: D(K) \subseteq X \rightarrow X\) be two densely-defined closed linear operators and there are \(a, b \geq 0\) satisfying \((a, \infty) \subseteq \rho(H), (b, \infty) \subseteq \rho(K)\) and let \(M \geq 0, \omega_1 \in (-\infty, a]\) and \(\omega_2 \in (-\infty, b]\) then the following assertions are equivalent:

(i) \((H, K)\) generates a two-parameter integrated semigroup \(\{S(t, s)\}_{t, s \geq 0}\) satisfying \(\|S(t, s)\| \leq Me^{\omega_1 t + \omega_2 s}\), for all \(t, s \geq 0\).

(ii) \(\| (\lambda - \omega_1)^{k+1} \left[ \frac{R(\lambda, H)}{\lambda} \right]^{(k)} / k! \| \leq M, \ \text{for} \ \lambda > a, k = 0, 1, 2, \ldots\)

\(\| (\mu - \omega_2)^{k+1} \left[ \frac{R(\mu, K)}{\mu} \right]^{(k)} / k! \| \leq M, \ \text{for} \ \mu > b, k = 0, 1, 2, \ldots\)

\(R(\lambda, H)R(\mu, K) = R(\mu, K)R(\lambda, H), \ \text{for} \ \lambda > a, \ \mu > b\).

The first application of integrated semigroup was a result about resolvent positive operator that we explain some preliminaries about it here. One can
see more details in [1].

**Definition 1.6** A cone $C$ of vertex 0 is a subset of a vector space $L$ such that for each $x \in C$ and $\lambda > 0$, $\lambda x \in C$. If in addition, $C$ is convex, then $C$ is called a convex cone of vertex 0. Thus a convex cone of vertex 0 is a subset of $L$ such that

(i) $C + C \subset C$
(ii) $\lambda C \subset C$ for all $\lambda > 0$.

A cone $C \subset L$ satisfying (i) and (ii) is said to be generating if $L = C - C$. Also if $C$ is a cone in topological vector space $L$ then the dual cone $C'$ of $C$ is defined to be the set $\{f \in L' : f(x) \geq 0$ if $x \in C\}$, where $L'$ is the dual space of $L$.

A cone $C$ in a topological vector space $L$ is said to be normal if $U = [U]$, where $U$ is a filter of neighborhoods of 0. Hence $C$ is a normal cone if and only if there exists a base of $C$-saturated neighborhoods of 0.

From now we assume that $E$ denotes a real ordered Banach space whose positive cone $E_+$ is generating and normal. For example, $E$ may be a Banach lattice or the hermitian part of a $C^*$-algebra.

**Definition 1.7** An operator $A$ on $E$ is called resolvent positive if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

Now let $A$ be a resolvent positive operator. Define $s(A) = \inf\{\omega \in \mathbb{R} : (\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega\}$, $D(A)_+ = E_+ \cap D(A)$ and $D(A'_+)_+ = E'_+ \cap D(A')$, where in the second definition we assume that $A$ is densely defined and $A'$ denotes the adjoint of $A$.

A subset $C$ of $E_+$ is called cofinal in $E_+$ if for every $a \in E_+$ there exists $b \in C$ such that $a \leq b$.

If $\{T(t)\}_{t \geq 0}$ is a positive $C_0$-semigroup with generator $B$, then the type (or growth bound) $\omega(B)$ is defined by $\omega(B) = \inf\{W \in \mathbb{R} : \text{there exists } M \geq 1 \text{ such that } \| T(t) \| \leq M e^{W t} \text{ for all } t \geq 0\}$. One always has $s(B) \leq \omega(B) < \infty$ (See [4] Chapter IV Proposition 2.2).

In the sequel, we state some conditions under which resolvent positive operator $A$ is the infinitesimal generator of a $C_0$-semigroup.

**Theorem 1.8** Let $A$ be a densely defined resolvent positive operator on $E$. If $D(A)_+$ is cofinal in $E_+$ or if $D(A'_+)_+$ is cofinal in $E'_+$, then $A$ is the generator of a positive $C_0$-semigroup. Moreover, $s(A) = \omega(A)$.

**Proof.** See [1] Theorem 2.2. □

**Corollary 1.9** Assume that $\text{int } E_+ \neq \emptyset$. If $A$ is a densely defined resolvent positive operator on $E$, then $A$ is the generator of a positive $C_0$-semigroup and $s(A) = \omega(A)$.

**Proof.** See [1] Corollary 2.3. □

**Theorem 1.10** Let $A$ be a densely defined resolvent positive operator on $E$. If there exist $\lambda_0 > s(A)$ and $c > 0$ such that $\| R(\lambda_0, A)a \| \geq c \| a \|$, $a \in E_+$,
then $A$ is the generator of a positive $C_0$–semigroup and $s(A) = \omega(A)$.

**Proof.** See [1] Theorem 2.5.\)

**Theorem 1.11** Suppose that the norm is additive on the positive cone, that is $\|a + b\| = \|a\| + \|b\|$ for all $a, b \in E_+$ (for example, $E = L^1(X, \mu)$). Let $A$ be a densely defined operator. The following assertions are equivalent:

(i) $A$ generates a positive $C_0$–group (For this concept see [9]).

(ii) $A$ and $-A$ are resolvent positive and there exist $\lambda > \max\{s(A), \omega(A)\}$ and $c > 0$ such that $\|R(\lambda, \pm A)\| \geq c \|a\|$ for all $a \in E_+$.

**Proof.** See [1] Theorem 2.6.\)

The following theorem shows that a densely defined resolvent positive operator $A$ is the infinitesimal generator of an increasing one-parameter integrated semigroup. This is based on the Hille-Yosida Theorem and can be applied when $A$ has a dense domain.

**Theorem 1.12** Let $A$ be a densely defined resolvent positive operator. There exists a unique increasing integrated semigroup $\{S(t)\}_{t \geq 0}$ of positive operators such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}dS(t), \lambda > s(A).$$

**Proof.** See [1] Theorem 4.1.\)

In the previous theorem it is not necessary to assume that $A$ has a dense domain if additional assumptions on the space are made.

**Definition 1.13** We say that $E$ is an ideal in $E''$ if for $a \in E$, $b \in E''$, $0 \leq b \leq a$ implies that $b \in E$. Here $E$ is identified with a subspace of $E''$ (via evaluation) and $E''_+ \cap E = E_+$ (that is, $E$ is an ordered subspace of $E''$).

**Theorem 1.14** Suppose that $E$ is an ideal in $E''$. Let $A$ be a resolvent positive operator. Then there exists a unique increasing integrated semigroup $\{S(t)\}_{t \geq 0}$ of operators on $E$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}dS(t), \lambda > s(A).$$

**Proof.** See [1] Theorem 5.7.\)

By the result of the two preceding theorems, resolvent positive operator $A$ generates a unique increasing integrated semigroup $\{S(t)\}_{t \geq 0}$ when either $A$ is densely defined or $E$ is an ideal in $E''$.

Note that the integrated semigroup generated by a resolvent positive operator is non-degenerate and increasing. The following theorem shows that conversely, every non-degenerate, increasing integrated semigroup is generated by a resolvent positive operator.

**Theorem 1.15** Let $\{S(t)\}_{t \geq 0}$ be a non-degenerate, increasing integrated
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there exists a unique resolvent positive operator $A$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t}dS(t), \lambda > s(A).$$

Moreover $D(A)$ is given by

$$\{a \in E; \text{there exists a (necessarily unique) } b \in E \text{ such that } S(t)a = ta + \int_0^t S(r)bdr \text{ for all } t \geq 0\}$$

and so $A(a) = b$.


We terminate this section with the homogeneous abstract Cauchy problem associated to the resolvent positive operator.

**Theorem 1.16** Assume that $A$ in (ACP)

$$\begin{cases} \frac{du(t)}{dt} = Au(t) & t \in (0, T] \\ u(0) = x \end{cases}$$

is a resolvent positive operator and either $D(A)$ is dense or $E$ is an ideal in $E''$. For every $x \in D(A^2)$ there exists a unique solution of (ACP). Furthermore, denote by $\{S(t)\}_{t \geq 0}$ the integrated semigroup generated by $A$. Then $u(t) = \frac{d}{dt}S(t)x = AS(t)x + x$ for all $t \geq 0$. Moreover if $x \geq 0$ then $u(t) \geq 0$ for all $t \geq 0$.


2. Resolvent Positive Operator and Two-Parameter Abstract Cauchy Problem

We introduced resolvent positive operator and we stated some theorems about conditions that a resolvent positive operator generates a $C_0-$semigroup. We can extend Theorem 1.8 for a pair of resolvent positive operators $(H_1, H_2)$. Throughout this section $E$ is a real ordered Banach space with generating and normal positive cone.

**Theorem 2.1** Let $(H_1, H_2)$ be a pair of densely defined resolvent positive operators on $E$. If $D(H_i)_+$ is cofinal in $E_+$ or $D(H'_i)_+$ is cofinal in $E'_+$, $i = 1, 2$ and also there exists $\omega \in R$ such that $[\omega, \infty) \subseteq \bigcap_{i=1}^{2} \rho(H_i)$ and $R(\lambda, H_1)R(\lambda', H_2) = R(\lambda', H_2)R(\lambda, H_1)$ for all $\lambda, \lambda' \geq \omega$, then $(H_1, H_2)$ is the generator of a positive $C_0-$two-parameter semigroup and $s(H_i) = \omega(H_i)$, $i = 1, 2$.

**Proof.** It is obvious by Hille-Yosida theorem for n-parameter semigroup and Theorem 1.8.

Similarly by using this additional condition (commutativity condition of
resolvent operators) we can extend Corollary 1.9 and Theorems 1.10 and 1.11.

Also we can extend Theorems 1.12 and 1.14 for a pair of resolvent positive operators as follows.

**Theorem 2.2** Suppose that \( H_1 \) and \( H_2 \) are two densely defined operators or \( E \) is an ideal in \( E'' \). If \( H_1, H_2 \) are resolvent positive operators and there exists \( \omega \in \mathbb{R} \) such that \((\omega, \infty) \subseteq \bigcap_{i=1}^{2} \rho(H_i)\) and \( R(\lambda, H_1)R(\lambda', H_2) = R(\lambda', H_2)R(\lambda, H_1) \) for all \( \lambda, \lambda' \geq \omega \), then there exists a unique two-parameter integrated semigroup \( \{S(t, s)\}_{t, s \geq 0} \) such that \( \{S(t, 0)\}_{t \geq 0} \) and \( \{S(0, s)\}_{s \geq 0} \) are increasing one-parameter integrated semigroups and

\[
R(\lambda, H_1) = \int_{0}^{\infty} e^{-\lambda t}dS(t, 0) \quad \text{for} \quad \lambda > s(H_1)
\]

and

\[
R(\lambda', H_2) = \int_{0}^{\infty} e^{-\lambda' s}dS(0, s) \quad \text{for} \quad \lambda' > s(H_2).
\]

**Proof.** By Theorems 1.12 and 1.14 there exist increasing one-parameter integrated semigroups \( \{S_t\}_{t \geq 0} \) and \( \{T_s\}_{s \geq 0} \) such that

\[
R(\lambda, H_1) = \int_{0}^{\infty} e^{-\lambda t}dS(t) \quad \text{for} \quad \lambda > s(H_1)
\]

and

\[
R(\lambda', H_2) = \int_{0}^{\infty} e^{-\lambda' t}dS(t) \quad \text{for} \quad \lambda' > s(H_2).
\]

Now consider the family \( S(t, s) = S_t + T_s \) for \( s, t \geq 0 \). From Proposition 1.5 it yields that \( \{S(t, s)\}_{t, s \geq 0} \) is a two-parameter integrated semigroup (see [8] for more details).

**Theorem 2.3** Let \( \{S(t, s)\}_{t, s \geq 0} \) be a non-degenerate, two-parameter integrated semigroup such that both of one-parameter integrated semigroups \( \{S(t, 0)\}_{t \geq 0} \) and \( \{S(0, s)\}_{s \geq 0} \) are increasing. Then there exists a unique pair of resolvent positive operators \( (H_1, H_2) \) such that

\[
R(\lambda, H_1) = \int_{0}^{\infty} e^{-\lambda t}dS(t, 0) \quad \text{for} \quad \lambda > s(H_1)
\]

and

\[
R(\lambda', H_2) = \int_{0}^{\infty} e^{-\lambda' t}dS(t, 0) \quad \text{for} \quad \lambda' > s(H_2).
\]

Moreover \( D(H_1) \) is given by

\[
D(H_1) = \{a \in E : \exists ! b \in E \text{ such that } S(t, 0)a = ta + \int_{0}^{t} S(r, 0)b \, dr \text{ for all } t \geq 0\}.
\]
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where notation \( \exists! \) means "there exists a unique". In this case \( H_1(a) = b \). Similarly \( D(H_2) \) is defined by

\[
D(H_2) = \{ a' \in E : \exists! b' \in E \text{ such that } S(0, s)a' = sa' + \int_0^s S(0, p)b' \, dp \text{ for all } s \geq 0 \}
\]

and \( H_2(a') = b' \).

**Proof.** It is easily proved by Theorem 1.15. \( \square \)

In the following theorem we discuss the two-parameter abstract Cauchy problem associated to resolvent positive operators.

**Theorem 2.4** Consider 2-RACP

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t_i} u(t_1, t_2) = H_i u(t_1, t_2) \quad i = 1, 2, \quad t_i \in [0, a_i] \\
u(0, 0) = u_0
\end{array} \right.
\]  

(1)

Where \( u \) is a function from \([0, a_1] \times [0, a_2] \) to ordered Banach space \( E \), \( a_1, a_2 > 0, H_i : D(H_i) \subseteq E \rightarrow E, i = 1, 2, \) is a resolvent positive operator and \( u_0 \in E \).

By a solution of 2-RACP we mean a continuous function \( u : [0, a_1] \times [0, a_2] \rightarrow E \) in which \( u \) has continuous partial derivatives and \( u(t_1, t_2) \in \bigcap_{i=1}^2 D(H_i) \) and \( u \) satisfies in 2-RACP.

If either \( H_1 \) and \( H_2 \) are densely defined or \( E \) is an ideal in \( E'' \) then for every \( u_0 \in \bigcap_{i=1}^2 D(H_i^2) \) there exits a unique solution of the Cauchy problem 2-RACP. Furthermore, denote by \( \{ S(t, s) \}_{t, s \geq 0} \) the two-parameter integrated semigroup generated by \((H_1, H_2)\) then \( u(t_1, t_2) = \frac{\partial}{\partial t_1} S(t_1, 0) \frac{\partial}{\partial t_2} S(0, t_2) u_0 \) for all \( t_1, t_2 \geq 0 \).

Moreover, if \( u_0 \geq 0 \), then \( u(t_1, t_2) \geq 0 \) for all \( t_1, t_2 \geq 0 \).

**Proof.** The proof of the existence and uniqueness of solutions is completely similar to the proof of Theorem 4.1 of [8] with the difference that \( u_0 \in \bigcap_{i=1}^2 D(H_i^2) \subseteq \bigcap_{i=1}^2 D(\tilde{H}_i) \). Also by Proposition 1.16 one can see that if \( u_0 \geq 0 \), then \( u(t_1, t_2) \geq 0 \) for all \( t_1, t_2 \geq 0 \). \( \square \)

**References:**


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