On Almost Subprojection Operators

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Abstract

In this paper, we introduce the class of almost subprojection operators on a Hilbert space H . We give some basic properties of such operators. We study the relation between almost subprojection operators and other operators. We show that the product of two commuting almost subprojection operators is almost subprojection. We show that the sum of two commuting almost subprojection operators is not necessarily almost subprojection. Finally we give some conditions under which the sum of two almost subprojections is almost subprojection.

1 Introduction

Let $L(H)$ be the algebra of all bounded linear operators acting on a Hilbert space H . Let T^* denotes the adjoint of any $T \in L(H)$. In [2] the author introduced the class of subprojection operators: $T \in L(H)$ is subprojection if $T^2 = T^*$. In this paper we introduce the class of almost subprojection: $T \in L(H)$ is called almost subprojection if T^2 is subprojection i.e. if $T^4 = T^{*2}$. In Section 2 of this paper we prove some general properties of almost subprojection operators. In Section 3 we study the relation between the class of almost subprojection operators and other classes of operators. In Section 4 we study the sum and the product of two almost subprojection operators.

2 Some general properties of almost subprojection

Proposition 2.1 *Let $T \in L(H)$ be almost subprojection then*

(i) T^* is almost subprojection.

(ii) If T^{-1} exists then T^{-1} is almost subprojection.

(iii) If $A \in L(H)$ and A and T are unitarily equivalent then A is almost subprojection.

(iv) If M is a closed subspace of H that reduces T then $T|M$ – the restriction of T to M –, is almost subprojection.

Proof. (i) Since T is almost subprojection then $T^4 = T^{*^2}$. Thus

$$(T^*)^4 = (T^4)^* = (T^{*^2})^* = ((T^*)^*)^2.$$

Thus T^* is almost subprojection.

(ii) Suppose T^{-1} exists then

$$(T^{-1})^4 = (T^4)^{-1} = (T^{*^2})^{-1} = (T^{-1})^{*^2}.$$

Thus T^{-1} is almost subprojection.

(iii) Let $A \in L(H)$ and let A and T unitarily equivalent, then there exists a unitary operator U in $L(H)$ such that $T = U^*AU$ which implies that $T^4 = U^*A^4U$ and $T^* = U^*A^*U$. Since T is almost subprojection, $T^4 = T^{*^2}$. Thus $U^*A^4U = U^*A^*UU^*A^*U = U^*A^{*^2}U$ which implies that $A^4 = A^{*^2}$. Thus A is almost subprojection.

(iv) Let M be a closed subspace of H that reduces T then $(T|M)^4 = T^4|M = T^{*^2}|M = (T|M)^{*^2}$. Thus $T|M$ is almost subprojection. ■

In the following example we show that unitarily equivalence in Proposition 2.1 can not be replaced by similarity.

Example 2.1 Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$ be operators acting on the two dimensional space $\mathbb{R}^2$ then direct computation shows that $S^4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = S^{*^2}$. Thus S is almost subprojection. Also direct calculation shows that $X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and $T = X^{-1}SX$ which means that T is similar to S . Now and again by direct computation we can show that $T^4 = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$ while $T^{*^2} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$. Thus $T^4 \neq T^{*^2}$ which means that T is not almost subprojection.

In the following example it is shown that we may have a nonzero almost subprojection operator in $L(H)$ whose square is zero.

Example 2.2 Consider the nonzero operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $\mathbb{R}^2$ then direct computation shows that $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Now $T^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $T^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ imply that $T^4 = T^{*^2}$. Thus T is almost subprojection.

Proposition 2.2 *The set $A(H)$ of all almost subprojection operators in $L(H)$ is strongly closed i.e. $\overline{A(H)}^{\text{SOT}}$ – the closure of $A(H)$ in the strong operator topology – equals $A(H)$.*

Proof. Let (S_n) be a sequence of operators in $A(H)$ such that (S_n) converges strongly to $S \in L(H)$. Since the product of operators is sequentially continuous in the strong operator topology ([1], p. 62), one concludes that (S_n^4) converges strongly to S^4 . Now since (S_n) converges strongly to S , we have $\|S_n x - Sx\| \rightarrow 0$ as $n \rightarrow \infty$ for each $x \in H$. Thus $\|S_n^* x - S^* x\| = \|(S_n^* - S^*)x\| = \|(S_n - S)^* x\| \leq \|(S_n - S)^*\| \|x\| = \|S_n - S\| \|x\| \rightarrow 0$ as $n \rightarrow \infty$. Thus (S_n^*) converges strongly to S^* which implies that (S_n^{*2}) converges strongly to S^{*2} . Thus (S_n^4) converges strongly to S^{*2} . Since the strong limit is unique, we have $S^4 = S^{*2}$ which means that $S \in A(H)$. Thus $A(H)$ is strongly closed. ■

In Section 3 we study the relation between the class of almost subprojection operators and other classes of operators.

In [3] the author introduces the class of 2-normal operators. $T \in L(H)$ is 2-normal if and only if $T^2 T^* = T^* T^2$.

Proposition 3.1 *If $T \in L(H)$ is almost subprojection then T is 2-normal.*

Proof. Since T is almost subprojection then

$$T^4 = T^{*2} \quad (*)$$

Multiplying $(*)$ above on right by T^2 we get

$$T^6 = T^{*2} T^2. \quad (A)$$

Multiplying $(*)$ above on the left by T^2 we get

$$T^6 = T^2 T^{*2}. \quad (B)$$

From (A) and (B) above we conclude that $T^{*2} T^2 = T^2 T^{*2}$ which implies that T^2 is normal, thus, by ([3], Proposition 1.6, p. 192) T is 2-normal. ■

The converse of Proposition 3.1 is not in general true as seen by the following example:

Example 3.1 Consider the operator $T = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ acting on the two-dimensional complex plane $\mathbb{C}^2$ then by direct computation we have $T^4 = \begin{pmatrix} 13 & 21i \\ -21i & 34 \end{pmatrix}$ while $T^{*2} = \begin{pmatrix} 2 & 3i \\ -3i & 5 \end{pmatrix}$ which means that T is not almost subprojection. Also by direct computation we can show that $TT^* = \begin{pmatrix} 2 & 3i \\ -3i & 5 \end{pmatrix} = T^* T$. Thus T is normal which implies, by ([3], Proposition 2.1, p. 192) that T is 2-normal.

Proposition 3.2 *If $T \in L(H)$ is subprojection then it is almost subprojection.*

Proof. Since T is subprojection then

$$T^2 = T^*. \quad (*)$$

Multiplying $(*)$ on the left by T^2 we get

$$T^4 = T^2 T^*. \quad (A)$$

Multiplying $(*)$ on the right by T^* we get

$$T^2 T^* = T^{*^2}. \quad (B)$$

From (A) and (B) above we conclude that $T^4 = T^{*^2}$. Thus T is almost subprojection. ■

The converse of Proposition 3.2 is not in general true as the following example shows:

Example 3.2 Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $\mathbb{R}^2$, then direct calculation shows that $T^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = T^{*^2}$. Thus T is almost subprojection.

However and again by direct calculation we can show that $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = T^*$. Thus T is not subprojection.

In [4] the author introduces the class of 3-normal operators. $T \in L(H)$ is 3-normal operator if $T^3 T^* = T^* T^3$.

Proposition 3.3 *If $T \in L(H)$ is both 2-normal and 3-normal then T is almost subprojection.*

Proof. Since T is 3-normal, then

$$T^3 T^* = T^* T^3. \quad (*)$$

Multiplying $(*)$ above on the right by T and on the left by T^* we get

$$\begin{aligned} T^4 T^{*^2} &= T T^* T^3 T^* \\ &= T T^* T^* T^3 \\ &= T T^{*^2} T^3 \\ &= T^{*^2} T T^3 \quad (\text{since } T \text{ is 2-normal}) \\ &= T^{*^2} T^4. \end{aligned}$$

Thus T is almost subprojection. ■

Example 3.3 Consider the operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ acting on $\mathbb{R}^2$, then direct computation shows that $T^2T^* = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ while $T^*T^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus T is not 2-normal which implies – by Proposition 3.1 – that T cannot be almost subprojection. Also by direct computation one can show that $T^3T^* = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = T^*T^3$. Thus T is a 3-normal operator which is not almost subprojection.

Example 3.4 Consider the operator $T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ acting on $\mathbb{R}^2$, then direct calculation shows that $T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T^{*^2}$. Thus T is almost subprojection. However and by direct calculation we can show that $T^3T^* = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$ while $T^*T^3 = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ which means that T is not 3-normal.

From Examples 3.3 and 3.4 we conclude that the classes of almost subprojection operators and 3-normal operators are independent. Example 3.4 shows that the converse of Proposition 3.3 is not – in general – true.

We show by the following two examples that the class of normal operators and the class of almost subprojections are independent.

Example 3.5 Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ be an operator acting on $\mathbb{R}^2$, then by direct calculation we have $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = T^*T$. Thus T is normal. Also by direct calculation we can show that $T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix}$ while $T^{*^2} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. Thus T is not almost subprojection.

Example 3.6 Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be an operator acting on $\mathbb{R}^2$, then by direct calculation one can show that $T^4 = 0 = T^{*^2}$. Thus T is almost subprojection. Also by direct calculation we can show that $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T^*T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus T is not normal.

In the following proposition we give a condition under which an almost subprojection in $L(H)$ becomes normal.

Proposition 3.4 *If $S \in L(H)$ is almost subprojection and if $I - S$ is almost subprojection then S is normal.*

Proof. Since $I - S$ is almost subprojection then $(I - S)^4 = (I - S)^{*2}$. Thus by direct calculation we get

$$I - 4S + 6S^2 - 4S^3 + S^4 = I - 2S^* + S^{*2}. \quad (\text{A})$$

Using the fact that $S^4 = S^{*2}$, (A) becomes

$$2S^3 - 3S^2 + 2S = S^*. \quad (\text{B})$$

Multiplying (B) on the left by S we get

$$SS^* = 2S^4 - 3S^3 + 2S^2. \quad (\text{C})$$

Multiplying (B) on the right by S we get

$$S^*S = 2S^4 - 3S^3 + 2S^2. \quad (\text{D})$$

From (C) and (D) we obtain that $SS^* = S^*S$ which implies that S is normal. ■

Proposition 3.5 *If S is an almost subprojection operator and if S is similar to an idempotent T then S is a subprojection.*

Proof. Since S and T are similar, there is an invertible operator $N \in L(H)$ such that $T = N^{-1}SN$ which implies that $T^2 = N^{-1}S^2N$. Since $T^2 = T$, $N^{-1}SN = N^{-1}S^2N$ which implies that $S^2 = S$. Since S is almost subprojection, $S^4 = S^{*2}$ which implies that $S^2 = S^4 = S^{*2} = (S^2)^* = S^*$. Thus S is subprojection. ■

Proposition 3.6 *If $T \in L(H)$ is almost subprojection and T^{-1} exists then T^2 is unitary.*

Proof. Since T is almost subprojection, T^2 is subprojection. Thus by ([2], Proposition 2.6, p. 232) $T^2 = T^2(T^*)^2T^2$. Multiplying the last equation on the right by $(T^2)^{-1}$ we get $T^2(T^2)^{-1} = T^2(T^*)^2T^2(T^2)^{-1}$ which implies that $I = T^2(T^*)^2$. Since T is almost subprojection, T is 2-normal. Thus $T^2(T^*)^2 = T^2T^*T^* = T^*T^2T^* = (T^*)^2T^2$. Thus we have $(T^*)^2T^2 = I = T^2(T^*)^2$. Hence T^2 is unitary. ■

Theorem 3.1 *An invertible operator $T \in L(H)$ is a scalar multiple of a unitary operator if and only if $\|T\| \|T^{-1}\| = 1$.*

Proof. ([5], p. 181). ■

Proposition 3.7 *If $T \in L(H)$ is almost subprojection and contraction then T is a scalar multiple of a unitary.*

Proof. Since T is almost subprojection then—by Proposition 3.6,— T^2 is unitary. Hence we have $\|T^{-1}\| = \|TT^{-1}T^{-1}\| = \|T(T^{-1})^2\| = \|T(T^2)^{-1}\| \leq \|T\| \|(T^2)^{-1}\| = \|T\| \|(T^*)^2\| = \|T\| \leq 1$. Thus $\|T\| \|T^{-1}\| \leq 1$. Also we have $1 = \|I\| = \|TT^{-1}\| \leq \|T\| \|T^{-1}\|$. Thus $\|T\| \|T^{-1}\| = 1$. Hence—by Proposition T is a scalar multiple of a unitary. ■

4

In this fourth and last section we study the sum and the product of two almost subprojection operators in $L(H)$.

Proposition 4.1 *If S, T are two commuting almost subprojection then ST is almost subprojection.*

Proof. Since $ST = TS$ then we have

$$\begin{aligned}
 (ST)^4 &= S^4 T^4 \\
 &= S^{*^2} T^{*^2} \\
 &= S^* S^* T^* T^* \\
 &= S^* T^* S^* T^* \\
 &= (TS)^* (TS)^* \\
 &= ((TS)^*)^2 \\
 &= ((ST)^*)^2.
 \end{aligned}$$

Thus ST is almost subprojection. ■

Corollary 4.1 *If $S \in L(H)$ is almost subprojection then so is S^n for any positive integer n .*

Proposition 4.2 *If S, T are two commuting almost subprojection operators in $L(H)$ then $S + T$ is not necessarily almost subprojection.*

Proof. Consider the almost subprojection operator T then $(T+T)^4 = (2T)^4 = 16T^4$ while $((T+T)^*)^2 = (T^* + T^*)^2 = (2T^*)^2 = 4T^{*^2}$. Thus $(T+T)^4 \neq ((T+T)^*)^2$ which implies that $T+T$ is not almost subprojection. ■

In the following we give a condition under which the sum of two almost subprojection operators becomes almost subprojection.

Proposition 4.3 *If S, T are two almost subprojection operators in $L(H)$ such that $ST = TS = 0$ then $S + T$ is almost subprojection.*

Proof.

$$\begin{aligned}
 (S+T)^4 &= ((S+T)^2)^2 \\
 &= (S^2 + ST + TS + T^2)^2 \\
 &= (S^2 + T^2)^2 \quad (\text{since } ST = TS = 0) \\
 &= S^4 + S^2 T^2 + T^2 S^2 + T^4 \\
 &= S^{*^2} + T^{*^2} \quad (\text{since } S^2 T^2 = 0 = T^2 S^2) \\
 (S+T)^{*^2} &= (S^* + T^*)^2 \\
 &= S^{*^2} + S^* T^* + T^* S^* + T^{*^2} \\
 &= S^{*^2} + T^{*^2} \quad (\text{since } S^* T^* = 0 = T^* S^*).
 \end{aligned}$$

Thus $(S + T)^4 = (S + T)^{*^2}$ which implies that $S + T$ is almost subprojection.

■

Proposition 4.4 *The direct sum and the tensor product of two almost subprojection operators in $L(H)$ are almost subprojection operators.*

Proof. Let $x = x_1 \oplus x_2 \in H_1 \oplus H_2$, and let $T, S \in L(H)$ be almost subprojection operators then

$$\begin{aligned}(T \oplus S)^4(x) &= (T \oplus S)^4(x_1 \oplus x_2) \\ &= T^4(x_1) \oplus S^4(x_2) \\ &= T^{*^2}(x_1) \oplus S^{*^2}(x_2) \\ &= (T^{*^2} \oplus S^{*^2})(x_1 \oplus x_2) \\ &= (T \oplus S)^{*^2}(x).\end{aligned}$$

Thus $(T \oplus S)^4 = (T \oplus S)^{*^2}$ which means that $T \oplus S$ is almost subprojection. Also

$$\begin{aligned}(T \otimes S)^4(x) &= (T \otimes S)^4(x_1 \otimes x_2) \\ &= T^4(x_1) \otimes S^4(x_2) \\ &= T^{*^2}(x_1) \otimes S^{*^2}(x_2) \\ &= (T^{*^2} \otimes S^{*^2})(x_1 \otimes x_2) \\ &= (T \otimes S)^{*^2}(x).\end{aligned}$$

Thus $(T \otimes S)^4 = (T \otimes S)^{*^2}$ which implies that $T \otimes S$ is almost subprojection.

■

References

- [1] Halmos, P.R. (1982). A Hilbert Space Problem Book, Springer-Verlag, New York.
- [2] Jibril, A.A.S. (2003). On subprojection operators, International Mathematical Journal, Vol. 4, no. 3, 229–238.
- [3] Jibril, A.A.S. (1996). On 2-normal operators, Dirasat, Natural and Engineering Sciences, Vol. 23, No. 2, 190–195.
- [4] Jibril, A.A.S. On power normal operators. Submitted
- [5] Shah, N.C. and Sheth, I.H. (1975). The maximal vector subspace of an operator. Math. Japon., 20:181.

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