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Some Polynomials of Flower Graphs

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Abstract. We define a class of graphs called flower and give some properties of these graphs. Then the explicit expressions of the chromatic polynomial and the flow polynomial is given. Further, we give classes of graphs with the same chromatic and flow polynomials.

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1. Introduction

There are several polynomials associated with a graph G. Polynomials play an important role in the study of graphs as it encode various information about a graph, such as the number of trees in G and other more striking information. Generalizing a graph polynomial for a specific class of graphs is an ongoing research.

The chromatic polynomial of a graph G, $\chi(G;\lambda)$, counts the number of ways of coloring the vertices of a graph properly with λ colors. The chromatic polynomial was introduced by Read [3] and have been widely studied. Explicit expressions of the chromatic polynomials of certain classes of graphs are known, we refer to [3, 6]. The flow polynomial, $F(G;k)$ counts the number of nowhere zero k-flows in G, we refer to [2] for further details.

In this paper we study a class of graphs called a complete flower. We begin by defining a graph called an $n\times m$- complete flower. Then we give an explicit expression of the chromatic polynomial of an $n\times m$- complete flower and certain subgraphs of an $n \times m$- complete flower. Furthermore, we give an explicit expression of the flow polynomial of certain subgraphs of an $n \times m$- complete flower. Finally, we give a general set of non-isomorphic graphs obtained from flower graphs, having the same chromatic and flow polynomials.

2. Flower graphs

In this section we give a definition and an example of a flower graph.

A graph G is called an $(n \times m)$-*flower graph* if it has the following set of vertices $V(G) = \{1, 2, \cdots, n, n+1, \cdots, n(m-1)\}$ and the edge set $E(G) = \{\{1,2\},\{2,3\},\cdots,\{n-1,n\},\{n,1\}\}\cup\{\{1,n+1\},\{n+1,n+2\},\{n+2,n+3\},$ $\{n+3,n+4\},\cdots\{n+m-3,n+m-2\},\{n+m-2,2\}\}\cup\{\{2,n+m-1\},$ $\{n+m-1,n+m\},\{n+m,n+m+1\},\{n+m+1,n+m+2\},\cdots,$ $\{n+2(m-2)-1,n+2(m-2)\},\{n+2(m-2),3\}\}\cup\cdots\cdots\cdots$ $\cup\{\{n,n+(n-1)(m-2)+1\},\{n+(n-1)(m-2)+1,n+(n-1)(m-2)+2\},\{n+(n-1)(m-2)+2,n+(n-1)(m-2)+3\},\{n+(n-1)(m-2)+3,n+(n-1)(m-2)+4\},\cdots,\{nm-1,nm\},\{nm,1\}\}$.

In other words, a graph G is called a $(n\times m)$-*flower graph* if it has n vertices which form an n-cycle and n sets of $m-2$ vertices which form m-cycles around the n cycle so that each m-cycle uniquely intersects with the n-cycle on a single edge. This graph will be denoted by $f_{n\times m}$. It is clear that $f_{n\times m}$ has $n(m-1)$

vertices and nm edges. The m-cycles are called the *petals* and the n-cycle is called the center of $f_{n\times m}$. The n vertices which form the center are all of degree 4 and all the other vertices have degree 2. The diagrams in Figure 1 are examples of a flower graphs.

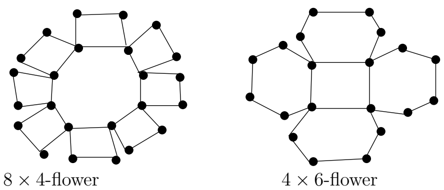


FIGURE 1

Removing a petal p of $f_{n\times m}$ is to take one m-cycle and delete all the vertices of degree 2 and their adjacent edges. We define an $(n \times m)$-*flower graph with i petals* to be an $(n \times m)$-flower graph with $n - i$ petals removed for $i \in \{1, 2, \cdots, n\}$. We denote an $(n \times m)$-flower graph with i petals by $f^i_{n\times m}$. It should be noted that $f_{n\times m} = f^n_{n\times m}$ and the positions of petals removed is irrelevant. Thus we have several non-isomorphic graphs represented by $f^i_{n\times m}$. The diagrams shown in Figure 2 are two non-isomorphic graphs represented by $f^3_{7\times 5}$.

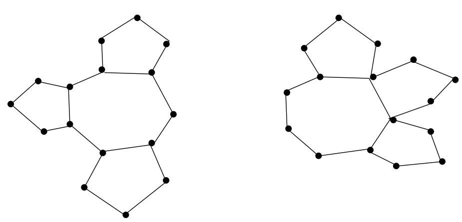


FIGURE 2. (7×5)-petal graph with 3 petals

3. CHROMATIC POLYNOMIALS

In this section we give an explicit expression of the chromatic polynomial of an $n \times m$-flower.

The following theorem is widely known in the literature, see [3].

Theorem 3.1. *If C_n is an n-cycle then*

$$\chi(C_n;\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$$

for $n \in \mathbb{N}$.

Proposition 3.2. *Let G be the graph of a complete $n \times m$-flower. Then the chromatic polynomial of G is*

$$\chi(G;\lambda) = [(\lambda-1)^{m-2} - (\lambda-1)^{m-3} \pm \cdots \pm (\lambda-1)]^n[(\lambda-1)^n + (-1)^n(\lambda-1)].$$

Proof. Without loss of generality, we use the deletion and contraction formula as shown in Figure 3. To ease notation each diagram of a graph in Figure 3 represents the chromatic polynomial of that graph. If we repeat the process

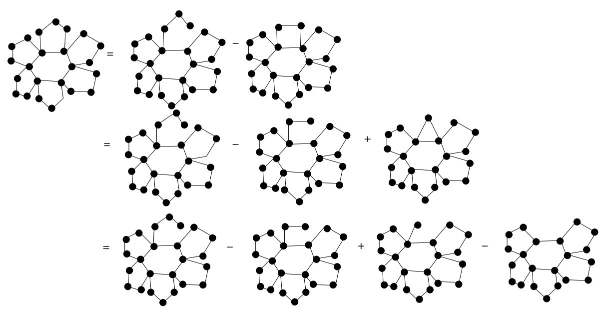


FIGURE 3

deleting and contracting all petals of $f_{n\times m}$ we get the result. □

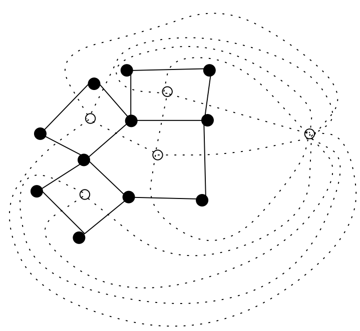
Corollary 3.3. *Let G be the graph of an $n \times m$-flower with i petals, for $i = \{1, 2, \cdots n\}$ Then*

$$\chi(G;\lambda) = [(\lambda-1)^{m-2} - (\lambda-1)^{m-3} \pm \cdots \pm (\lambda-1)]^i[(\lambda-1)^n + (-1)^n(\lambda-1)].$$

4. Flow polynomials

In this section we give an explicit expression of the flow polynomial of an $n \times m$-flower graph with i petals, for $i = \{1, 2, \cdots n-1\}$. We begin by constructing the dual graph of an $n\times m$-flower graph with i petals. Then we give an explicit expression of the chromatic polynomial of the dual graph of an $n \times m$-flower graph with i petals. Finally we convert the explicit expression of the chromatic polynomial of the dual graph into an explicit expression of the flow polynomial of an $n \times m$-flower graph with i petals.

Let G be a planar graph. To construct G^* the dual of G, we put a vertex in each face of G including the face out of G. Then we join two such vertices when the corresponding faces share an edge. The diagram in Figure 4 is an example of an $n \times m$-flower graph with i petals and its vertex join.



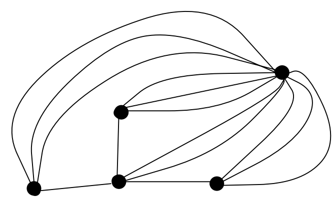


FIGURE 4. (5×4)-petal graph with 3 petals and its vertex join

Proposition 4.1. *Let G be an $n \times m$-flower graph with i petals, and G^* its dual, Then $G^* \cong \hat{t}_{i+1}$ up to parallel class, where $\hat{t}_{i+1}$ is a vertex join of a tree of $i+1$ vertices and $i = \{1, 2, \cdots n-1\}$.*

Proof. Let $G = f^i_{n\times m}$. Thus G has i petals and $i+2$ faces. We start by considering petal faces and the center face. The center is the only faces that shares an edge with each petal. Thus if we put a vertex in each face and join vertices whose corresponding faces share an edge, we have a tree on $i+1$ vertices, t_{i+1}. It is clear from definition that no two faces of petals share an edge but each petal shares $m-1$ edges with the outer face. Now if we consider the vertex in the outface, then each vertex in the petal face joins to it $m-1$

times. Furthermore, since the are $(n-i)>0$ petals missing, the center face shares $n-i$ edges with the outer face. Thus we can join the vertex in the center face and outer face. But the vertex in the outer face is joined to all the other vertices in the other faces. Thus forming a vertex join of the graph formed by the petals and center, which is a tree in this case. □

The following theorems are well known in the literature, see [6].

Theorem 4.2. *Let G be a graph and let $\hat{G}$ be its vertex join. Then*

$$\chi(\hat{G};\lambda)=\lambda\chi(G;\lambda-1).$$

Theorem 4.3. *Let G and H be graphs and let $G\cong H$ up to parallel class. Then*

$$\chi(G;\lambda)=\chi(H;\lambda).$$

Theorem 4.4. *If t_n is a tree of n vertices, then*

$$\chi(t_n;\lambda)=\lambda(\lambda-1)^{n-1}$$

The following theorem relates the Flow polynomial of G and the chromatic polynomial of its dual G^*, see [7].

Theorem 4.5. *If G is planar, the flow polynomial of G is the chromatic polynomial of G^*, such that*

$$F(G;\lambda)=\frac{1}{\lambda^{k(G)}}\chi(G^*;\lambda)$$

where $k(G)$ is the number of connected subgraphs of G.

Proposition 4.6. *let G be the graph of $n\times m$-flower with i petals. Then the flow polynomial of G,*

$$F(G;\lambda)=(\lambda-1)(\lambda-2)^i$$

for $i=\{1,2,\cdots n-1\}$.

Proof. By Proposition 4.1, we know that G^* is the vertex join of a tree of i vertices up to parallel class. Thus by applying Theorem 4.3, we get the chromatic polynomial of G^*, $\chi(G^*;\lambda) = \chi(\hat{t}_{i+1};\lambda)$ where $\hat{t}_{i+1}$ is a vertex join of a tree of $i+1$ vertices. Furthermore, by Theorem 4.2, $\chi(\hat{t}_{i+1};\lambda) = \lambda\chi(t_{i+1};\lambda-1)$. Thus $\chi(G^*;\lambda) = \chi(\hat{t}_{i+1};\lambda) = \lambda\chi(t_{i+1};\lambda-1)$. Hence by applying Theorem 4.4, we get $\chi(G^*;\lambda) = \lambda(\lambda-1)(\lambda-2)^i$. It is clear that a petal graph has one component. Thus by applying Theorem 4.5, we get $F(G;\lambda) = (\lambda-1)(\lambda-2)^i$. □

It should be noted that the explicit expression of the flow polynomial given, is only for flower graphs which are not complete.

It was indicated earlier that $f^i_{n\times m}$ for $i = \{1, 2, \cdots n-1\}$ represents a flower graph with i petals, and the positions of petals missing is irrelevant. Thus we have non-isomorphic graphs represented by $f^i_{n\times m}$. It is clear that, all non-isomorphic graphs represented by $f^i_{n\times m}$ have the same chromatic and flow polynomials.

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