Very Ample Line Bundles on $\overline{M}_{0,5}$ and on its Blowing ups

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Abstract. Let $X \subset \overline{M}_{0,5}$ be the blowing up of $\mathbb{P}^2$ at four linearly independent points. Fix $Q \in X$, an integer $y > 0$ and a very ample $L \in \text{Pic}(X)$. Let $u : Y \to X$ be the blowing up of $X$ at $Q$. Here we study when $h^1(X, I_{yQ} \otimes L) = 0$ and the line bundle $u^*(L)(-yu^{-1}(Q))$ is very ample.

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1. Introduction

Fix 4 distinct points $P_i \in \mathbb{P}^2$, $1 \leq i \leq 4$, such that no 3 of them are collinear. Any two 4-ples as above are projectively equivalent. Let $f : X \to \mathbb{P}^2$ be the blowing up of these 4 points. It is well-known that $X \cong \overline{M}_{0,5}$ (indeed, this is just the easiest case of both Keel’s and Kapranov’s celebrated constructions of $\overline{M}_{0,n}$ as a suitable sequence of blow-ups ([3])). The anticanonical bundle $\omega_X$ is very ample and it embeds $X$ into $\mathbb{P}^5$ as a degree 5 smooth Del Pezzo surface ([1]). In this embedding the image of $X$ contains exactly 10 lines, which may be described in the following way. Set $D_i := f^{-1}(P_i)$, $1 \leq i \leq 4$. For all $1 \leq i < j \leq 4$ let $D_{ij}$ denote the strict transform in $X$ of the line $\langle \{P_i, P_j\} \rangle$. These 10 divisors (each of them isomorphic to $\mathbb{P}^1$) are exactly the $-1$-curves of $X$, i.e. the integral curves $D$ such that $D^2 = D \cdot \omega_X = -1$, i.e. the curves of $X$ mapped to lines by the anticanonical embedding. These 10 integral curves are the boundary components of $\overline{M}_{0,5}$, i.e. the irreducible components of $\overline{M}_{0,5}\setminus M_{0,5}$. Here we will study some geometrical properties of embeddings along these 10 boundary components (see Propositions 3, 4, 5 and 6) and which blown ups of them are very ample (see Theorems 1 and

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Proposition 1. Fix any ordering $T_1, \ldots, T_{10}$ of the boundary components of $X$. Fix integers $a_i$, $1 \leq i \leq 10$. Set $b_4 := \min_{1 \leq i \leq 10} a_i$ and let $B_4$ any boundary component $T_i$ such that $b_4 = a_i$. Let $b_3$ be the minimal integer $a_i$ among the boundary components $T_i$ of $X$ disjoint from $B_4$ and call $B_3$ any such boundary component. Let $b_2$ be the minimal integer $a_i$ among the boundary components $T_i$ disjoint from $B_4 \cup B_3$ and call $B_2$ any such boundary component. There is a unique boundary component $B_1$, say $B_1 = T_j$, disjoint from $B_2, B_3, B_4$ and set $b_1 := a_j$. For all $1 \leq i < j \leq 4$ let $B_{ij}$ denote the unique boundary component intersecting both $B_i$ and $B_j$ and call $b_{ij}$ the integer $a_h$ if $B_{ij} = T_h$. There is $L \in \text{Pic}(X)$ such that $L \cdot T_i = a_i$ if and only if $b_{ij} + b_i + b_j = b_{hk} + b_h + b_k$ for all $1 \leq i < j \leq 4$ and all $1 \leq h < k \leq 4$. The following conditions are equivalent:

(i) $L$ is ample;
(ii) $L$ is very ample;
(iii) $a_i > 0$ for all $1 \leq i \leq 10$.

Assume $a_i > 0$ for all $1 \leq i \leq 10$. Then $h^1(X, L) = 0$, $h^0(X, L) = (b_1 + b_2 + b_{ij} + 2)(b_1 + b_2 + b_{ij} + 1)/2 - \sum_{i=1}^4 b_i(b_i + 1)/2$ and $L^2 = (b_1 + b_2 + b_{ij})^2 - \sum_{i=1}^4 b_i^2$.

Then we will prove the following results.

Theorem 1. Set $L := f^*(\mathcal{O}_{P^4}(t))(-m_1 D_1 - m_2 D_2 - m_3 D_3 - m_4 D_4)$ with $m_1 \geq m - 2 \geq m_2 \geq m_3 > 0$ and assume that $L$ is very ample. Let $y$ be a non-negative integer. Assume

$$t > y + m_1, 2t > m_1 + m_2 + m_3 + m_4 + y$$

Let $u : Y \to Q$ be the blowing up of $Q$. Set $D := u^{-1}(Q)$ and $R := u^*(L)(-yD)$. If $Q \notin \partial$, then $h^1(X, \mathcal{O}_Q \otimes L) = 0$ and $R$ is very ample. If $Q \notin \partial$, then the two inequalities in (1) are necessary and sufficient conditions for the very ampleness of $R$. If $Q \in \partial$, then the two inequalities in (1) are necessary conditions for the ampleness of $R$.

Theorem 2. Fix $Q \in X$ lying on exactly one boundary component, $T$. Let $u : Y \to X$ be the blowing up of $Q$. Set $D := u^{-1}(Q)$. Let $F$ denote the strict transform of $T$ in $X$. Fix an ample $R \in \text{Pic}(Y)$. Set $y := R \cdot D$ and $w := F \cdot R$. Set $L := u^*(R)(-yD)$. Let $B_3$ be a boundary component of $X$ such that $B_3 \cap T = \emptyset$ and with $b_3 := R \cdot B_3$ minimal among such components. Let $B_2$ be a boundary component of $X$ disjoint from $T \cup B_3$ and with $b_2 := R \cdot B_2$ minimal among such components. Let $B_1$ be the only boundary component of $X$ disjoint from $B_2 \cup B_3 \cup T$. Set $b_1 := B_1 \cdot R$. Let $B_{12}$ be the only boundary component of $X$ intersecting both $B_1$ and $B_2$. Set $b_{12} := B_{12} \cdot R$. Assume
Lemma 1. Set $L := f^*(\mathcal{O}_{\mathbb{P}^2}(t))(-m_1D_1 - m_2D_2 - m_3D_3 - m_4D_4)$ and assume $m_1 \geq m_2 \geq m_3 \geq m_4 > 0$ and $t \geq m_1 + m_2 + m_4$. Then $L$ is very ample.

Proof. As in the proof of [2], Lemma IV.4.12, we will use the fact that the tensor product of a spanned line bundle and a very ample line bundle is very ample. Set $L_0 := f^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $L_1 := f^*(\mathcal{O}_{\mathbb{P}^2}(1))(-D_1)$, $L_2 := f^*(\mathcal{O}_{\mathbb{P}^2}(2))(-D_1 - D_2)$, $L_3 := f^*(\mathcal{O}_{\mathbb{P}^2}(2))(-D_1 - D_2 - D_3)$ and $L_4 := f^*(\mathcal{O}_{\mathbb{P}^2}(3))(-D_1 - D_2 - D_3 - D_4) \cong \omega_X^1$. $L_1$, $L_2$ and $L_3$ are spanned (e.g. for $L_3$ use that the the set of all plane conics through $P_1, P_2, P_3$ have no further base point and that for any line $R$ through $P_1$ there is one such conic not tangent to $R$). $L_4$ is very ample (same proof or read [2], proof of IV.4.6). We have $L \cong L_0^{a_0} \otimes L_1^{a_1} \otimes L_2^{a_2} \otimes L_3^{a_3} \otimes L_4^{a_4}$ with $a_4 = m_4 > 0$, $a_3 = m_3 - m_4 \geq 0$, $a_2 = m_2 - m_3 \geq 0$, $a_1 = m_1 - m_2$ and $a_0 = t \geq 0$. Hence $L$ is very ample. \hfill $\Box$

Lemma 2. Set $L := f^*(\mathcal{O}_{\mathbb{P}^2}(t))(-m_1D_1 - m_2D_2 - m_3D_3 - m_4D_4)$. The following conditions are equivalent:

(i) $L$ is very ample;
(ii) $L$ is ample;
(iii) $L \cdot D_i > 0$ for all $1 \leq i \leq 4$, and $L \cdot D_{ij} > 0$ for all $1 \leq i < j \leq 4$;
(iv) $m_i > 0$ for all $1 \leq i \leq 4$, and $t > m_i + m_j$ for all $1 \leq i < j \leq 4$.

Proof. We copy the proof of [2], Th. IV.4.11, in which the assumption $L^2 > 0$ is never used, as implicitly remarked in [2], Cor. 4.13. Obviously, (iii) and (iv) are equivalent. Obviously, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv). Assume that (iv) is satisfied. Let $E_4$ be any boundary divisor such that $E_4 \cdot L$ is minimal. Let $E_3$ be any exceptional divisor such that $E_4 \cap E_3 = \emptyset$ and $E_3 \cdot L$ is minimal among the boundary divisors disjoint from $E_4$. Let $E_2$ be any boundary divisor such that $(E_4 \cup E_3) \cap E_2 = \emptyset$ and $E_2 \cdot L$ is minimal among the boundary divisors disjoint from $E_4 \cup E_3$. Let $E_1$ be any boundary divisor such that $(E_4 \cup E_3 \cup E_2) \cap E_1 = \emptyset$ and $E_1 \cdot L$ is minimal among the boundary divisors disjoint from $E_4 \cup E_3 \cup E_2$. Since the exceptional divisors $E_1, E_2, E_3, E_4$ are disjoint, they may be blown down simultaneously to obtain a morphism $u : X \to \mathbb{P}^2$ with them as exceptional divisors. Since (i), (ii), (iii) and (iv) do not depend from the choice of the blowing down $X \to \mathbb{P}^2$, without losing generality we may assume $E_i = D_i$ for all $i$. Hence $m_1 \geq m_2 \geq m_3 \geq m_4 > 0$. Since $D_{12}$ was an
allowable choice when we defined $E_3$, $t - m_1 - m_2 \geq m_3$. The very ampleness
of $L$ follows from Lemma 1.

Lemma 3. Fix any ample $L \in \text{Pic}(X)$. Then $h^1(X, L) = 0$.

Proof. $L$ is very ample (Lemma 2) and hence there is $D \in |L|$ with $D$ smooth
and connected. Consider the exact sequence

$$0 \to \mathcal{O}_X \to L \to L|D| \to 0 \quad (2)$$

Since $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$, $h^1(X, L) = 0$ if and only if $h^1(D, L|D) = 0$. Since $\omega_X$ is ample $2p_a(D) - 2 = L^2 + \omega_X \cdot L < \deg(L|D)$. Hence $h^1(D, L|D) = 0$. □

Proof of Proposition 1. We claim that the first part is just a restatement of the proof of Proposition 2. Let $h : X \to \mathbb{P}^2$ be the blowing down of $B_1, B_2, B_3, B_4$. With the notation of the proof we have $z := b_1 + b_2 + b_{12}$ and we need to check that $z = b_i + b_j + b_{ij}$ for all $1 \leq i < j \leq 4$. To get the uniqueness it is sufficient to prove that every $M \in \text{Pic}(X)$ such that $M \cdot T_i = 0$ for all $i$ is trivial. This is true, because the boundary components generate $\text{Pic}(X)$ and on $X$ numerical equivalence and linear equivalence coincide. If $L$ is ample, then $h^1(X, L) = 0$ (Lemma 3) Working in the plane we see that $L^2 = z^2 - \sum_{i=1}^4 b_i^2$. We have $h^0(X, L) = \binom{z+2}{2} - \sum_{i=1}^4 b_i(b_i+1)/2$ (here we use $h^1(X, L) = h^2(X, L) = 0$).

The case $a_i = 1$ for all $i$ corresponds to the anticanonical embedding of $X$ as the Del Pezzo degree 5 surface of $\mathbb{P}^5$.

The proof of Lemma 2, i.e. the proof of [2], Th. 4.11, (but note that the condition $L^2 > 0$ is never used) gives the following result.

Lemma 4. Fix 5 distinct points $P_i \in \mathbb{P}^2$, $1 \leq i \leq 5$, such that no 3 of them are collinear. Let $u : Y \to \mathbb{P}^2$ be the blowing up of these 5 points. Set $D_i := u^{-1}(P_i)$, $1 \leq i \leq 5$. For all $1 \leq i < j \leq 5$ let $D_{ij}$ denote the strict transform in $Y$ of the line $\langle \{P_i, P_j\} \rangle$. Let $\Gamma$ denote the strict transform in $Y$ of the unique conic $\Gamma'$ containing the points $P_1, \ldots, P_5$. The images in $\mathbb{P}^4$ of the 16 curves $D_i$, $1 \leq i \leq 5$, $D_{ij}$, $1 \leq i < j \leq 5$, and $\Gamma$ are the 16 lines of the anticanonical model of $Y$. Fix integers $t$ and $m_i$, $1 \leq i \leq 5$ and set $L := u^*(\mathcal{O}_{\mathbb{P}^2}(1))(-\sum_{i=1}^5 m_i D_i)$. The following conditions are equivalent:

(i) $L$ is very ample;
(ii) $L$ is ample;
(iii) $L \cdot D_i > 0$ for all $1 \leq i \leq 4$, $L \cdot D_{ij} > 0$ for all $1 \leq i < j \leq 5$ and $L \cap \Gamma > 0$;
(iv) $m_i > 0$ for all $1 \leq i \leq 5$, $t > m_i + m_j$ for all $1 \leq i < j \leq 5$ and $2t > m_1 + m_2 + m_3 + m_4 + m_5$.

Remark 1. Take $Y$ and $L$ as in Remark 4 with $L$ very ample. The proof of Lemma 3 gives $h^1(X, L) = 0$. 
Remark 2. Let $A$ be a quasi-projective and smooth surface, $P \in A$ and $u : B \to A$ the blowing up of $P$. Set $D := u^{-1}(P)$. Hence for any integer $n \geq 0$ the effective Cartier divisor $(n+1)D \subset B$ is the infinitesimal neighborhood $D^{(n)}$ of order $n$ of $D$ in $B$. Fix a locally free sheaf $E$ on $B$. Since $D^2 = -1$, the conormal bundle $\mathcal{O}_D(-D)$ is the degree 1 line bundle on $D \cong \mathbb{P}^1$. Thus for every integer $n > 0$ we have an exact sequence
\[ 0 \to (E|D)(n+1) \to E|D^{(n)} \to E|D^{(n-1)} \to 0 \tag{3} \]
Now assume that $E|D$ is trivial. Using (3) and induction on $n$ we get that $E|D^{(n)}$ is trivial for all $n$, i.e. that the restriction $E|\hat{D}$ of $E$ to the infinitesimal neighborhood $\hat{D}$ is trivial. The formal function theorem ([2], III.11.1) gives that $u_*(E)$ is locally free and that the natural map $u^*(u_*(E)) \to E$ is an isomorphism. Now assume that $A$ is projective. Since $R^i u_*(\mathcal{O}_B) = 0$ for all $i > 0$ and $u_*(\mathcal{O}_B) = \mathcal{O}_A$, we get $h^i(B, E) = h^i(A, F)$ for all $i \geq 0$. Conversely, if there is a vector bundle $F$ on $A$ such that $u^*(F) \cong A$, then $E|D$ is trivial.

Proposition 2. For any integer $s$ such that $1 \leq s \leq 4$ let $f_s : X_s \to \mathbb{P}^2$ be the blowing up of $P_1, \ldots, P_s$. Set $\mathcal{O}_{X_s}(t; a_1, \ldots, a_s) := f_s^*(\mathcal{O}_{\mathbb{P}^2}(t))(-\sum_{i=1}^s D_i)$ for all integers $t, a_1, \ldots, a_s$. Fix $R \in \text{Pic}(X_s)$, say $R := \mathcal{O}_{X_s}(t; a_1, \ldots, a_s)$. $R$ is spanned and only if the following conditions are satisfied:

(a) If $s = 1$, then assume $R^2 \geq 0$ and $R \cdot D_1 \geq 0$, i.e. assume $t \geq a_1 \geq 0$.
(b) If $2 \leq s \leq 4$ assume $R \cdot D_i \geq 0$ for all $1 \leq i \leq s$ and $R \cdot D_{ij}$ for all $1 \leq i < j \leq s$, i.e. assume $a_i \geq 0$ for all $1 \leq i \leq s$ and $t \geq a_i + a_j$ for all $1 \leq i < j \leq s$.

If these conditions are satisfied, then $h^1(X_s, R) = 0$.

Proof. The “only if” part is trivial. The “i.e.” part in (a) and (b) is easily checked. Assume $s = 1$ and that the conditions in (a) is satified. $X_1$ is the Hirzebruch surface $F_1$ and its ruling $v : F_1 \to \mathbb{P}^1$ is associated to the linear system $\mathcal{O}_{X_1}(1; 1)$. The condition $R \cdot D_1 \geq 0$ is equivalent to $a_1 \geq 0$. Since $t \geq 0$, we immediately got that $R$ is spanned and $h^1(X_1, R) = 0$. Now assume $s \geq 2$ and that the result is true for the positive integers $s' < s$. The curves $D_i$, $1 \leq i \leq s$, and $D_{ij}$, $1 \leq i < j \leq s$, will be called the boundary curves of $X_s$. Let $T$ be a boundary curve for which $R \cdot T$ is minimal. If $R \cdot T = 0$, then $R$ comes from a blowing down $v : X_s \to A$ (Remark 2). Since $A \cong X_{s-1}$, we may use the inductive assumption, if in the case $s = 2$ we check that $R^2 > 0$ (see below. Now assume $R \cdot T > 0$. The minimality of the integer $R \cdot T$ implies $a_i > 0$ for all $1 \leq i \leq s$ and $t > a_i + a_j$ for all $1 \leq i < j \leq s$. Set $E_s := T$. Let $E_{s-1}$ be a boundary curve such that $E_{s-1} \cdot E_s = 0$ and with $E_{s-1} \cdot R$ minimal among the boundary curves disjoint from $E_s$. And so on. Write $R = \mathcal{O}_{X_s}(z; b_1, \ldots, b_s)$ with respect to the morphism $X_s \to \mathbb{P}^2$ obtained blowing down the curves $E_1, \ldots, E_s$. By construction $b_1 \geq \cdots \geq b_s$. The first part of the proof gives $b_s > 0$ and $z > b_1 + b_2$. First assume $s = 2$. We have $(R(-b_2 D_2) \cdot D_2 = 0$. Hence $R(-b_2 D_2) = v^*(L)$ with $L \in X_1$. We have $L \cdot D_1 = R \cdot D_1 = b_1 > 0$ and $L \cdot h = R \cdot D_{12} - b_2 = b_1 - b_2 > 0$. Hence $L$ is spanned and this is
true even if we only assumed \( z \geq b_1 + b_2 \) and \( b_1 \geq b_2 \geq 0 \). Hence \( R \) is spanned. Now assume \( s \geq 3 \). At the time we chose \( E_3 \) we could have chosen exactly 3 divisors disjoint from it: \( E_1, E_2 \) and another one, \( E_{12} \), intersecting both \( E_1 \) and \( E_2 \). The minimality condition for the curves \( E_1, \ldots, E_s \) gives \( z - b_1 - b_2 \geq b_3 \). As in Lemma 1 we get that \( R \) is very ample. As in Lemma 3 we get \( h^1(X_s, R) = 0 \).

**Proposition 3.** Fix a very ample \( L \in \text{Pic}(X) \) and a boundary component \( T \) of \( X \). Let \( h_L : X \rightarrow \mathbb{P}^n \), \( n := h^0(X, L) - 1 \), denote the embedding associated to \( L \). Then the curve \( h_L(T) \) is a degree \( L \cdot T \) rational normal curve in its linear span.

**Proof.** The curve \( h_L(T) \) is a degree \( L \cdot T \) curve. Since \( h^1(X, L) = 0 \) (Lemma 2), \( h_L(T) \) is a rational normal curve in its linear span if and only if \( h^1(X, L(-T)) = 0 \), i.e. if and only if \( h^1(X, (L \otimes \omega_X^*(-T) \otimes \omega_X) = 0 \). Set \( A := L \otimes \omega_X^*(-T) \). Let \( E \) be a boundary component of \( X \). If \( E = T \), then \( A \cdot E = L \cdot E - \omega_X \cdot E - E^2 > 2 \). If \( E \cap T = \emptyset \), then \( A \cdot E = L \cdot E - \omega_X \cdot E \geq 2 \). Now assume \( E \neq T \) and \( E \cap T \neq \emptyset \). Since \( E \cdot T = 1 \), we get \( A \cdot E > 0 \). The same proof gives \( L(-T) \cdot > 0 \) for all boundary components \( E \) of \( X \). Since \( \omega_X^* \) is ample, we get \( A \cdot E > 0 \) for every boundary component \( E \) of \( X \). Hence \( A \) is very ample (Lemma 2). Since \( h^1(X, \mathcal{O}_X) = 0 \), Kodaira vanishing is true and easy in arbitrary characteristic. Hence \( h^1(X, A \otimes \omega_X) = 0 \), concluding the proof.

**Proposition 4.** Fix a very ample \( L \in \text{Pic}(X) \) and two boundary components \( T, T' \) of \( X \) such that \( T \neq T' \) and \( T \cap T' \neq \emptyset \). Let \( h_L : X \rightarrow \mathbb{P}^n \), \( n := h^0(X, L) - 1 \), denote the embedding associated to \( L \). Then the curve \( h_L(T) \) (resp. \( h_L(T') \)) a degree \( L \cdot T \) (resp. \( L \cdot T' \)) rational normal curve in its linear span and \( \langle h_L(T) \rangle \cap \langle h_L(T') \rangle \) is the unique point \( T \cap T' \).

**Proof.** Since \( h^1(X, L) = 0 \) (Lemma 2), Proposition 3 applied to \( T \) and \( T' \) shows that Lemma 4 is true if and only if \( h^1(X, (L \otimes \omega_X^*(-T - T') \otimes \omega_X) = 0 \). Set \( A := L \otimes \omega_X^*(-T - T') \). Let \( E \) be a boundary component of \( X \). If \( E = T \), then \( A \cdot E = L \cdot E - \omega_X \cdot E - T \cdot T' \geq 2 \). The same is true if \( E = T' \). If \( E \cap (T \cup T') = \emptyset \), then \( A \cdot E = L \cdot E - \omega_X \cdot E \geq 2 \). Now assume \( E \neq T \), \( E \neq T' \) and \( E \cap (T \cup T') \neq \emptyset \). Notice that \( E \cdot (T \cup T') = 1 \) by the configuration of the boundary components of \( X \). Hence \( A \cdot E > 0 \). The same proof gives \( L(-T - T') \cdot E \geq 0 \) for all boundary components \( E \) of \( X \). Since every effective divisor if \( X \) is linearly equivalent to a sum with non-negative coefficients of boundary divisors, \( L(-T) \) \( \omega_X^* \) is very ample, \( A \cdot E > 0 \) for every boundary component \( E \). Hence \( A \) is very ample (Lemma 2). Since \( h^1(X, \mathcal{O}_X) = 0 \), Kodaira vanishing is true and easy in arbitrary characteristic. Hence \( h^1(X, A \otimes \omega_X) = 0 \), concluding the proof.

**Proposition 5.** Fix a very ample \( L \in \text{Pic}(X) \), an integer \( m \) such that \( 2 \leq m \leq 4 \), and \( m \) mutually disjoint boundary components \( T_i, 1 \leq i \leq i, \) of \( X \). Let \( h_L : X \rightarrow \mathbb{P}^n \), \( n := h^0(X, L) - 1 \), denote the embedding associated to \( L \). Assume \( L \cdot E \geq 2 \) for every boundary component \( E \) of \( X \). Then the curve \( h_L(T_1 \cup \cdots \cup T_m) \) spans a linear space
of dimension $m - 1 + \sum_{i=1}^{m} L \cdot T_i$, i.e. the linear spans $\langle h_L(T_i) \rangle$, $1 \leq i \leq m$, of the rational normal curves $h_L(T_i)$, $1 \leq i \leq m$, are linearly independent in $\mathbf{P}^n$.

Proof. Set $B := L(-\sum_{i=1}^{m} T_i)$ and $A := B \otimes \omega_X$. As in the proof of Proposition 3 and 4 it is sufficient to prove that $B \cdot E \geq 2$ for every boundary component $E$. If $E \cap T_i = \emptyset$ for all $i$, then $B \cdot E > 0$. If $E = T_i$ for some $i$, then $B \cdot E = L \cdot T_i - T_i^2 \geq 2$. If $E \neq T_i$ for any $i$, but $E \cap T_j \neq \emptyset$ for exactly one index $j$, say for exactly $s$ indices $j$, then we get $B \cap E = L \cap E - s$. Since $s \leq 2$, we just use our assumption. □

Remark 3. The case $L = \omega_X$ shows that for $3 \leq m \leq 4$ we cannot drop the assumption “$L \cdot E \geq 2$ for every boundary component $E$ of $X$ intersecting two of the curves $T_1, \ldots, T_m$” in the statement of Proposition 4. We do not know how much we may weak it and we do not know if we always may drop it for $m = 2$.

Remark 4. Fix a very ample $L \in \text{Pic}(X)$. Let $h_L : X \to \mathbf{P}^n$, $n := h^0(X, L) - 1$, denote the embedding associated to $L$. Let $\Delta$ denote the set of all the boundary components of $X$. It is easy to check that $\Delta \in |\omega_X^2|$. Since $h^1(X, L) = 0$, $h_L$ gives a “linearly independent” embedding of $\Delta$ (i.e. the rational normal curves images of the boundary components are as skew as possible) if and only if $h^1(X, L \otimes \omega_X^2) = 0$.

The proof of Proposition 4 gives the following result.

Proposition 6. Fix a very ample $L \in \text{Pic}(X)$, an integer $z \geq 2$ and a boundary component $T$ of $X$. Let $h_L : X \to \mathbf{P}^n$, $n := h^0(X, L) - 1$, denote the embedding associated to $L$. Assume $L \cdot E \geq z^2$ for every boundary component $E$ of $X$ such that $E \cap T = \emptyset$. Then $h^1(X, L(-zT)) = 0$ and $h_L$ gives a linearly normal embedding of the scheme $zT$.

Proof of Theorem 1. Lemma 2 gives $m_i > 0$ for all $1 \leq i \leq 4$, and $t > m_1 + m_2$. Hence there is at least one such integer $y \geq 0$. Since if $y = 0$ there is nothing to prove, we may assume $y > 0$. First assume $Q \notin \partial$. By assumption we have $t > m_i + m_j$ for all $i \neq j$ and $t > m_i + y$ for all $i$. By Lemma 4 and Remark 1 $R$ is very ample and $h^1(Y, R) = 0$. The latter equality is equivalent to $h^1(X, \mathcal{O}_Q \otimes L) = 0$. Lemmas 4, Remark 1 and the openness of very ampleness among line bundles $R$ with $h^1(Y, R) = 0$ also give the last assertion of Theorem 1. □

Remark 5. Fix $L$, $Q$ and $y$ as in the statement of Theorem 1 and let $h_L : X \to \mathbf{P}^n$, $N := h^0(X, L) - 1$, denote the embedding of $X$ induced by the complete linear system $|L|$. The statement of Theorem 2 is equivalent to say that the osculating sequence of $h_L(X)$ at $h_L(Q)$ is the expected one.

Remark 6. Fix $L$, $Q$, $y$, $Y$, and $R$ as in the statement of Theorem 1. First assume $Q \in D_i$. Let $D_i \subset Y$ denote the strict transform of $D_i$ in $Y$. Since
deg(L|D_i) = m_i, if y ≥ m_i + 2, then h^1(Y, R) > 0, if y = m_i + 1, then \(\tilde{D}_i\) is in the base locus of |R|, while if y = m_i, then R|D_i is not very ample. More precisely, if y = m_i, then either the base locus of |R| intersects \(\tilde{D}_i\) or |R| contracts \(\tilde{D}_i\) to a point. Now assume \(Q \in D_{ij}\). Let \(D_{ij} \subset Y\) denote the strict transform of \(D_{ij}\) in \(Y\). Since \(\text{deg}(L|D_{ij}) = t - m_i - m_j\), if \(t \leq m_i + m_j + y - 2\), then \(h^1(Y, R) > 0\), if \(t = m_i + m_j + y - 1\), then \(\tilde{D}_{ij}\) is in the base locus of |R|, while if \(t = m_i + m_j + y\), then R|\(D_{ij}\) is not very ample. More precisely, if \(t = m_i + m_j + y\), then either the base locus of |R| intersects \(\tilde{D}_{ij}\) or |R| contracts \(\tilde{D}_{ij}\) to a point.

**Remark 7.** Fix L, Q, y, Y, and R as in the statement of Theorem 1. Since \(h^1(X, L) = h^2(X, L) = 0\), Riemann-Roch gives \(h^0(X, L) = (L^2 - \omega_X \cdot L) + \chi(\mathcal{O}_X) = (t^2 - m_1^2 - m_2^2 - m_3^2 - m_4^2 + 3t + m_1 + m_2 + m_3 + m_4)/2 + 1\). Notice that \(h^1(Y, R) = h^1(X, \mathcal{I}_y \otimes L)\) and that if \(h^1(X, \mathcal{I}_y \otimes L) = 0\), then \(h^0(X, L) \geq \text{length}(yQ) = (y + 1)y/2\).

**Proof of Theorem 2.** Y is a weak del Pezzo surface ([1]). Hence \(\omega_Y^*\) is spanned, \(h^0(Y, \omega_Y^*) = 5\), \(h^1(Y, \omega_Y^*) = 0\), the anticanonical morphism \(h_{\omega_Y^*} : X \to \mathbb{P}^4\) is an embedding outside \(F\) and it contracts \(F\) to a rational double point of the normal surface \(h_{\omega_Y^*}(Y)\) ([1], pp. 64–66). The surface \(h_{\omega_Y^*}(Y)\) is the complete intersection of two quadric hypersurfaces. The total transform in \(Y\) of a boundary component of \(X\) will be called a block of \(Y\). Hence \(Y\) has 10 blocks, 9 of them being exceptional curves of the first kind mapped to lines in the anticanonical model and the other one being the divisor \(F + D\). Let \(B_4\) be the unique reducible block. Notice that \(b_4 := R \cdot B_4 = y + w\). Among all blocks disjoint form \(B_4\) fix one of them, \(B_3\), such that \(b_3 := R \cdot B_3\) is minimal. Among all blocks disjoint form \(B_4 \cup B_3\) fix one of them, \(B_2\), such that \(b_2 := R \cdot B_2\) is minimal. Among all blocks disjoint form \(B_4 \cup B_3 \cup B_2\) fix one of them, \(B_1\), such that \(b_1 := R \cdot B_1\) is minimal. Hence \(b_1 \geq b_2 \geq b_3 > 0\). These 4 disjoint blocks may be blowing down simultaneously. Their blowing down gives a morphism \(h : Y \to \mathbb{P}^2\). Set \(Q_i := h(B_i), 1 \leq i \leq 4\). Since \(D^2 = -1, F^2 = -2\) and \(D \cdot F = 1\), there is an integer \(z\) such that \(R \cong h^*(\mathcal{O}_{\mathbb{P}^2}(z))(-\sum_{i=1}^{3} b_iB_i - (y + w)F - (2y + w)D)\). We have \(\omega_Y^* \cong h^*(\mathcal{O}_{\mathbb{P}^2}(3))(-\sum_{i=1}^{3} B_i - F - 2D)\). We know that the anticanonical morphism contracts \(F\) and it is an embedding outside \(F\). Let \(B_{ij}, 1 \leq i < j \leq 4\), be the strict transform in \(Y\) of the line \(\langle Q_i, Q_j \rangle\). The curves \(B_{ij}, 1 \leq i < j \leq 4\), are the other 6 blocks of \(Y\). Since \(R \cdot B_{ij} = z - b_i - b_j\) for all \(1 \leq i < j \leq 4\), we have \(z > b_i + b_j\) and \(z > b_i + y + w\). Since when we defined \(B_3\) we could have used \(B_{12}\), we have \(z - b_i - b_j > b_3\). Let \(A_0 := h^*(\mathcal{O}_{\mathbb{P}^2}(1)), A_1 := h^*(\mathcal{O}_{\mathbb{P}^2}(3))(-\sum_{i=1}^{4} B_i), A_2 := h^*(\mathcal{O}_{\mathbb{P}^2}(2))(-B_1 - B_2 - B_3), A_3 := h^*(\mathcal{O}_{\mathbb{P}^2}(2))(-B_1 - B_2)\) and \(A_4 := h^*(\mathcal{O}_{\mathbb{P}^2}(1))(-B_1)\). Notice that \(A_1 \cong u^*(\omega_X^*)\). \(A_1\) is spanned, \(|A_1|\) contracts \(D\) and it induces the anticanonical embedding of \(X\). Obviously, \(A_0\) is spanned. We claim that \(A_2, A_3, A_4\) are spanned. The linear system \(|A_4|\) is induced by the set of all lines through \(Q_1\)
and hence $A_4$ is spanned outside $B_1$. Since the block $B_1$ is irreducible, $A_4$ is spanned, because this set of lines has no tangent vector of $P^2$ at $Q_1$ in its base locus. Since $B_4 = F + D$, $A_3 \cong u^*(\mathcal{O}_{P^2}(2))(-D_1 - D_2 - D_3)$. Hence $A_3$ is spanned. Since $A_2 \cong u^*(\mathcal{O}_{P^2}(2))(-D_1 - D_2 - D_3)$, it is spanned. There are integers $\alpha, \beta$ such that $R \cong A_1^{(y+w)} \otimes A_2^{(b_3 - y - w)} \otimes A_3^{(b_2 - b_3)} \otimes A_4^{(b_1 - b_2)} \otimes A_0^{(z - b_1 - b_2 - b_3)}(\alpha F + \beta D)$. Since $F^2 = -2$, $F \cdot D = 1$, $D^2 = -1$, $R \cdot F = w$ and $R \cdot D = y$, and $B_1 \cap B_4 = \emptyset$ for $1 \leq i \leq 3$, we get $\alpha = 0$ and $\beta = -y$. Set $A_5 := f^*(\mathcal{O}_{P^2}(2))(-F - 2D)$. It is easy to check that $A_5$ is spanned, that $|A_5|$ embeds $Y \setminus F$ and that it contracts $F$ to a point. Thus if $z \geq b_1 + b_2 + w + 3y$, then $A_0^{(z - b_1 - b_2 - b_3)}(-yD)$ is spanned and its associated morphism contracts $F$ and it is an embedding outside $F$. Thus if $z \geq b_1 + b_2 + w + 3y$, then $R$ is very ample.

3. The Twisted Cotangent Bundle

**Remark 8.** Let $D \subset X$ be an effective divisor. Consider the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

(4)

in which the injective map $j$ is obtained by the multiplication by a local equation of $D$. Hence $j|D \equiv 0$. Thus $\text{Tor}_1^\mathcal{O}_X(\mathcal{O}_D, \mathcal{O}_D) \cong \mathcal{O}_D(-D)$. Notice that $\mathcal{O}_D(-D)$ is a line bundle of degree $-D^2$ on $D$.

**Remark 9.** For any integer $a$ let $\mathcal{O}_{D_i}(a)$ denote the degree $a$ line bundle on $D_i \cong P^1$.

**Claim:** We have an exact sequence on $X$:

$$0 \to f^*(\Omega_{P^2}^1) \to \Omega_X^1 \to \bigoplus_{i=1}^4 \mathcal{O}_{D_i}(-1) \to 0$$

(5)

**Proof of the Claim:** The functoriality property of the cotangent sheaves gives the existence of the injective map $j : f^*(\Omega_{P^2}^1) \to \Omega_X^1$. Obviously, $\text{Coker}(j)$ is supported by $D_1 \cup D_2 \cup D_3 \cup D_4$. To check that $\text{Coker}(j)$ is as claimed, it is sufficient to do it around each $D_i$, i.e. it is sufficient to prove that an exact sequence like (5) holds for a blowing up $u : Y \to W$ of $P \in W$, $W$ a smooth surface. In this local situation it is obvious that the restriction to $D := u^{-1}(P)$ is a rank one line bundle on $D \cong P^1$. The only problem is to compute its degree $x$. Recall that $D_i^2 = -1$ for all $i$ and $\Omega_{D_i}^1 \cong \mathcal{O}_{D_i}(-2)$, we have an exact sequence Since $c_2^1 + c_2 = 12X(\mathcal{O}_X) = 12$ and $\omega_X^2 = 5$, $c_2(TX) = c_2(\Omega_X^1) = 7$. Similarly, $c_2(F_{F_1}) = 4$, where the Hirzebruch surface $F_1$ is the blowing-up $\tilde{f} : F_1 \to P^2$ of a point $Q \in P^2$. Set $Q$ as the intersection of two lines $L_1, L_2 \subset P^2$. Look at the minimal free resolution of the sheaf $\mathcal{O}_Q$ seen as an $\mathcal{O}_{P^2}$-sheaf:

$$0 \to \mathcal{O}_{P^2}(-2) \to \mathcal{O}_{P^2}(-1)^{\oplus 2} \to \mathcal{O}_Q \to 0$$
We get $c_1(\mathcal{O}_Q) = 0$ and $c_2(\mathcal{O}_Q) = -1$. This is true for any point of a smooth surface. We use the exact sequence

$$0 \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_D(a) \rightarrow \mathcal{O}_Z \rightarrow 0$$

(case $a > 0$) or the exact sequence

$$0 \rightarrow \mathcal{O}_D(a) \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_Z \rightarrow 0$$

in which $Z$ is the union of $|a|$ points. We get $c_1(\mathcal{O}_D(a)) = c_1(\mathcal{O}_D)$ and $c_2(\mathcal{O}_D(a)) = c_2(\mathcal{O}_D) - a$. From the exact sequence

$$\mathcal{O}_Y(-D) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_D \rightarrow 0$$

we get $c_2(\mathcal{O}_D) = 0$ and hence $c_2(\mathcal{O}_D(a)) = -a$. Look again at the exact sequence (5) with $\mathcal{O}_{D_i}(x)$ instead of $\mathcal{O}_{D_i}(-1)$ and take the corresponding exact sequence associated to the map $\tilde{f} : F_1 \rightarrow \mathbb{P}^2$. We get $c_2(\mathcal{O}_{F_1}) = c_2(\tilde{f}^*(\mathcal{O}_{\mathbb{P}^2})) + c_2(\mathcal{O}(x)) + c_1(\tilde{f}^*(\mathcal{O}_{\mathbb{P}^2})) \cdot c_1(\mathcal{O}(x))$, i.e. $4 = 3 - x + 0$. Thus $x = -1$

**Remark 10.** Notice that $\mathcal{O}_{\mathbb{P}_z} \cong TP^2(-3)$. The Euler’s sequence of $TP^2$ gives $h^1(P^2, \mathcal{O}_{\mathbb{P}_z}(z)) = 0$ for all $z \neq 0$ and that $\mathcal{O}_{\mathbb{P}_z}(2)$ is spanned.

From (5), Remark 10 and Lemma 2 we immediately get the following result.

**Proposition 7.** Fix integers $z, b_i, 1 \leq i \leq 4$, such that $b_i > 0$ for all $i$ and $z \geq 3 + b_i + b_j$ for all $1 \leq i < j \leq 4$. Set $L := f^*(\mathcal{O}_{P^2}(z))(-\sum_{i=1}^{4} b_i D_i)$ and $E := \mathcal{O}_X \otimes L$. Then $h^1(X, E) = 0$ and $E$ is spanned. If $b_i \geq 2$ for all $i$, then the morphism from $X$ into a Grassmannian induced by the pair $(E, H^0(X, E))$ is an embedding and the line bundle $\mathcal{O}_{P(E)}(1)$ on $P(E)$ is very ample.

**References**


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