Base Locus of Linear Systems Containing a Fixed Finite Set

E. Ballico¹

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Let S be a finite subset of a projective variety. Here we give some condition on a linear system V on X which imply that S is the scheme-theoretic base locus of V .

Mathematics Subject Classification: 14N05

Keywords: base locus

Let S be a finite subset of a projective variety. Here we give some condition on a linear system V on X which imply that S is the scheme-theoretic base locus of V . For a very refined study of this problem when $X = \mathbf{P}^2$, see [2]. We will first prove the following result.

Proposition 1. *Fix integers n, d, s such that $n \geq 2$, $d \geq 2$, $d^n \geq 25$, and $\binom{n+s-1}{\leq} s \leq \binom{n+s}{n} - n - 1$. Let $Z \subset \mathbf{P}^n$ a general complete intersection of n degree d hypersurfaces. Let $S \subset Z$ be any subset such that $\sharp(S) = s$. Then $h^0(\mathbf{P}^n, \mathcal{I}_S(d-1)) = 0$, $h^1(\mathbf{P}^n, \mathcal{I}_S(t)) = 0$ for all $t \geq d$, $h^0(\mathbf{P}^n, \mathcal{I}_S(d)) = \binom{n+d}{n} - s$ and S is the scheme-theoretic base locus of the linear system $|\mathcal{I}_S(d)|$.*

The assumption $d^n \geq 25$ is satisfied, unless either $n = 2$ and $d = 2, 3, 4$ or $d = 2$ and $n = 3, 4$.

An immediate consequence of Proposition 1 is the following result.

Corollary 1. *Fix integers n, d, s such that $n \geq 2$, $d \geq 2$, $d^n \geq 25$, and $\binom{n+s-1}{\leq} s \leq \binom{n+s}{n} - n - 1$. Let $S \subset \mathbf{P}^n$ be a general subset such that $\sharp(S) = s$. Then $h^0(\mathbf{P}^n, \mathcal{I}_S(d-1)) = 0$, $h^1(\mathbf{P}^n, \mathcal{I}_S(t)) = 0$ for all $t \geq d$, $h^0(\mathbf{P}^n, \mathcal{I}_S(d)) = \binom{n+d}{n} - s$ and S is the scheme-theoretic base locus of the linear system $|\mathcal{I}_S(d)|$.*

¹The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

Proposition 2. Fix integers n, d, s, e such that $n \geq 2$, $d \geq 2$, $e > 0$, and $\binom{n+s-1}{-}n \leq s \leq \binom{n+s}{n} - 1$. Fix a general $\binom{n+s}{n} - s$ linear subspace M of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$ and a general $(n - \binom{n+s}{n} + s)$ -dimensional linear subspace of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d+e))$. Let Z_1 the scheme-theoretic intersection of the zero-loci of the homogeneous forms in M . Let Z be the scheme-theoretic intersection of Z_1 and of the the zero-loci of the homogeneous forms in M' . Z_1 is a smooth complete intersection of dimension $n - \binom{n+s}{n} + s$ and of degree $d^{n - \binom{n+s}{n} + s}$. Z is a finite set of $\gamma := d^{n - \binom{n+s}{n} + s} \cdot (d+e)^{\binom{n+s}{n} - s}$ points. Assume $\gamma \geq 25$. Let $S \subset Z$ be any subset such that $\sharp(S) = s$. S is the scheme-theoretic intersection of Z_1 and the base locus of the linear system $|\mathcal{I}_S(d+e)|$.

The assumption $\gamma \geq 25$ is satisfied if $d^{(n-1)} \cdot (d+e) \geq 25$. Now we fix S and consider the stratification of $\mathbf{P}^n \setminus S$ by the base locus of $\mathcal{I}_{S \cup \{P\}}(d)$, $P \in \mathbf{P}^n \setminus S$.

Theorem 1. Let X be an integral n -dimensional projective variety and a very ample linear system (not necessarily complete) on X , say associated to the line bundle L . For any finite $A \subset X_{\text{reg}}$ let $\pi_A : X_A \rightarrow X$ denote the blowing up of A and $E_A := \cup_{P \in A} \pi_A^{-1}(P)$ (the union of all exceptional divisors). For every integer x such that $1 \leq x \leq 6$ fix a general $A_x \subset X$ with $\sharp(A_x) = x$ and let V_{A_x} , $1 \leq x \leq 6$, denote the linear system $\pi_{A_x}^*(V) \cap H^0(X_{A_x}, \pi_{A_x}^*(L)(-E_{A_x}))$. Assume that all linear systems V_{A_x} , $1 \leq x \leq 6$, are very ample. Fix an integer s such that $1 \leq s \leq \dim(V) - n - 1$, and a general $A \subset X$ such that $\sharp(A) = s$. Then the linear system V_A on X_A is base point free.

Remark 1. Let C be an integral projective curve and A a base point free linear system on C , say associated to the linear subspace $V \subseteq H^0(C, L)$, $L \in \text{Pic}(C)$. For a general $P \in C$ the linear system $A(-P)$ associated to $V \cap H^0(C, \mathcal{I}_P \otimes L)$ is base point free if and only if the morphism $h_A : C \rightarrow \mathbf{P}^n$, $n := \dim(V) - 1$, is birational onto its image. Assume that this is the case and fix an integer a such that $1 \leq a \leq n - 3$. Fix a general $(P_1, \dots, P_a) \in (\mathbf{P}^n)^a$. In characteristic zero the linear projection of $h_A(C)$ from the linear subspace $\langle \{h_A(P_1), \dots, h_A(P_a)\} \rangle$ is birational onto its image by the linear independence of a general hyperplane section ([1]). Hence the linear system $A(-P_1 - \dots - P_a)$ is base point free. In positive characteristic the existence of very strange projective curves shows that this is not always true.

Remark 2. Let $Z \subset \mathbf{P}^n$ a zero-dimensional scheme and t a positive integer such that $h^1(\mathbf{P}^n, \mathcal{I}_Z(t)) = 0$. By Castelnuovo-Mumford's lemma the homogeneous ideal of Z is generated by forms of degree at most $t + 1$. Hence Z is the scheme-theoretic base locus of the linear system $|\mathcal{I}_Z(t+1)|$.

Proof of Proposition 1. Let $\Gamma_{n,d}$ denote the set of all n -dimensional linear subspaces $V \subset H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d))$ such that the scheme-theoretic base locus B_V of V is a disjoint union of d^n points. Set $I := \{(V, P) \in \Gamma_{n,d} \times \mathbf{P}^n : P \in B_V\}$ (the incidence correspondence) and let $\pi : I \rightarrow \Gamma_{n,d}$ be the map

induced by the projection on the first factor. π is proper, étale and of degree d^n . Let G be the Galois group of the finite map π . G is well-defined, because $\Gamma_{n,d}$ is irreducible. Fixing any $W \in \Gamma_{n,d}$ and any ordering of the d^n points in B_W we may see G as a subgroup of the symmetric group S_{d^n} . We assume that W is general in $\Gamma_{n,d}$.

Claim: G contains the alternating group A_{d^n} .

Proof of the Claim: By assumption $d^n \geq 25$. By [1], Th. 2.4, it is sufficient to show that G is at least 5-transitive. Let $\pi_2 : I \rightarrow \mathbf{P}^n$ be the map induced by the second projection. π_2 is surjective and each fiber of π_2 is isomorphic to a vector space of dimension $\binom{n+d}{n} - 1$. Hence I is irreducible. The 1-transitivity of G is equivalent to the irreducibility of I . Fix an integer x such that $2 \leq x \leq 6$ and assume that G is $(s-1)$ -transitive. We will show that G is x transitive. Fix $P_1, \dots, P_{x-1} \in B_W$ with $P_i \neq P_j$ for all $i \neq j$. Set $G' := \{g \in G : g(P_i) = P_i \text{ for } 1 \leq i \leq x-1\}$. Set $A := H^0(\mathbf{P}^n, \mathcal{I}_{\{P_1, \dots, P_{x-1}\}}(d))$. A induces a permutation group G'' on $B_W \setminus \{P_1, \dots, P_{x-1}\}$. Since G is $(x-1)$ -transitive to show that G is x -transitive it is sufficient to show that G'' is transitive. Since W is general and $x \leq \binom{n+d}{n}$, we may see $P_1, \dots, P_{x-1}$ as $x-1$ general points of $\mathbf{P}^n$. Set $D := \langle \{P_s, P_{s+1}\} \rangle$. Hence $W \in A$, $\dim(A) = \binom{n+d}{n} - s + 1$ and there is an irreducible family $\{V_t\}_{t \in T}$, T a non-empty open subset of the line D , $V_t \in A$ for all $t \in T$, such that $V_{P_s} = W$, $t \in B_{V_t}$ for all $t \in T$, and $h^1(\mathbf{P}^n, \mathcal{I}_{\{P_1, \dots, P_{x-1}, t\}}(d)) = 0$ for all $t \in T$. This family shows that G'' is transitive, proving the Claim.

Recall that W is general in $\Gamma_{n,d}$. By the Claim it is sufficient to show the existence of $S \subset B_W$ such that $\sharp(S) = s$. Then $h^0(\mathbf{P}^n, \mathcal{I}_S(d-1)) = 0$, $h^1(\mathbf{P}^n, \mathcal{I}_S(t)) = 0$ for all $t \geq d$, $h^0(\mathbf{P}^n, \mathcal{I}_S(d)) = \binom{n+d}{n} - s$ and S is the scheme-theoretic base locus of the linear system $|\mathcal{I}_S(d)|$. Since $h^0(\mathbf{P}^n, \mathcal{I}_{B_W}(d)) = n$, it is obvious the existence of $S \subset B_W$ such that $\sharp(S) = s$. Then $h^0(\mathbf{P}^n, \mathcal{I}_S(d-1)) = 0$, $h^1(\mathbf{P}^n, \mathcal{I}_S(t)) = 0$ for all $t \geq d$, $h^0(\mathbf{P}^n, \mathcal{I}_S(d)) = \binom{n+d}{n} - s$. For any such S the scheme-theoretic base locus Δ of the linear system $|\mathcal{I}_S(d)|$ is strictly contained in B_W . A weak form of the Cayley-Bacharach property gives $\sharp(B_W \setminus \Delta) \geq 2$. Fix $P \in B_W \setminus S$. Since G is $(d^n - 2)$ -transitive and $\sharp(B_W \setminus \Delta)$, there is $g \in G$ such that $g(S) = S$ and $g(P) \in \Delta$. Hence $P \in \Delta$, contradiction. $\square$

Proof of Proposition 2. We use the set-up of the Proof of Proposition 1 with trivial modifications. For instance, we intersect every base locus with Z_1 and instead of $\Gamma_{n,d}$ we take an open subset of the Grassmannian of all $(n - \binom{n+s}{n} + s)$ -dimensional linear subspaces of $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d+e))$. The assumption $\gamma \geq 25$ allows us to use the Claim of the proof of Proposition 1, without doing then by hands the few exceptional cases. $\square$

Proof of Theorem 1. The statement of Theorem 1 is just an axiomatization of the proof of Proposition 1. $\square$

REFERENCES

- [1] J. Rathmann, The uniform position principle for curves in characteristic p , Math. Ann. 276 (1987), no. 4, 565–579.
- [2] Z. C. Teitler, On the intersection of curves through a set of points in $\mathbb{P}^2$, J. Pure Appl. Algebra 209 (2007), no. 2, 571–581.

Received: May 23, 2007