r -Difference Differential Operator D_r and Solution of ODEs

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Abstract

Let $f : [a, b] \rightarrow R$ be a real -valued function, we introduce the linear operator D_r as follow:

$$D_r f(x) = f'(x) - r f(x) = (D - r) f(x). \quad (1)$$

In the sequel, we investigate properties of D_r and find a shift exponential formula and use it in solving of nonhomogeneous differential equations with constant coefficients that can be written as follow:

$$\left(\prod_{r=1}^m D_{r_i}\right)y(x) = f(x). \quad (2)$$

Keywords: differential operator D , r -difference differential operator D_r , Shift exponential formula, Particular solution

1 Introduction

1.1. The linear operator differential method is used in solving of linear ODE and linear PDE with constant coefficients. Suppose that r is constant, then the solution of equation $y' - ry = (D - r) y = f(x)$ is written as follow:

$$y = \frac{1}{D - r} f(x) = e^{rx} \int e^{-rx} f(x) dx. \quad (3)$$

(3) is an important toll in solving of ODEs and PDEs with constant coefficients.

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2 Main Results

Definition1. Suppose that $f : I \rightarrow R$ is differentiable, we define the r -difference differential operator D_r as follow:

$$\forall X \in I = [a, b], \quad D_r f(x) = f'(x) - r f(x) = (D - r) f(x), \quad (4)$$

then two operators $D - r$ and D_r are equivalent, so $D_0 \equiv D$. From (3), we have

$$D_r(e^{rx} f(x)) = e^{rx} f'(x), \quad (5)$$

and in general ,

$$D_r(e^{ax} f(x)) = e^{ax} (D + a - r) f(x). \quad (6)$$

Now, Replacing $f(x)$ by $\int f(x) dx$, we obtain

$$D_r(e^{rx} \int f(x) dx) = e^{rx} \int f(x) dx. \quad (7)$$

The inverse of (7), may be written in the following form:

$$\frac{1}{D_r} (e^{rx} f(x)) = e^{rx} \int f(x) dx. \quad (8)$$

Equality (8) may be written as follow

$$\frac{1}{D_r} (f(x)) = e^{rx} \int e^{-rx} f(x) dx. \quad (9)$$

(9) is used in solving of ODEs and PDEs.

Lemma 1. For each of the following equalities, prove the accuracy

$$D_r(e^{rx} \int e^{-rx} f(x) dx) = f(x). \quad (10)$$

$$\frac{1}{D_r} (f(x)) = e^{rx} \int e^{-rx} f(x) dx. \quad (11)$$

Proof. The equality (11) is inverse of (10).

Lemma 2. Suppose that $f \in C^k [a, b]$, then prove that

$$(i) \quad D_r^k(e^{rx} f(x)) = e^{rx} D^k f(x). \quad (12)$$

$$(ii) \frac{1}{D_r^k} (e^{rx} f(x)) = e^{rx} \frac{1}{D_r^k} f(x). \quad (13)$$

In general

$$D_r^k (e^{ax} f(x)) = e^{ax} D_{r-a}^k f(x). \quad (14)$$

$$\frac{1}{D_r^k} (e^{ax} f(x)) = e^{ax} \frac{1}{D_{r-a}^k} f(x). \quad (15)$$

Proof. Is easy, proof is possible by definition 1 and mathematics induction.

Theorem (shift exponential formula) suppose that P is a polynomial, then

$$P(D_r) (e^{rx} f(x)) = e^{rx} P(D) f(x). \quad (16)$$

$$\frac{1}{P(D_r)} (e^{rx} f(x)) = e^{rx} \frac{1}{P(D)} f(x). \quad (17)$$

Generally

$$P(D_r) (e^{ax} f(x)) = e^{ax} P(D_{r-a}) f(x). \quad (18)$$

$$\frac{1}{P(D_r)} (e^{ax} f(x)) = e^{ax} \frac{1}{P(D_{r-a})} f(x). \quad (19)$$

Proof. Since D_r is a linear operator then D_r is invertable and D_r^{-1} is linear, D_r^k (k is integer) and $P(D_r)$ are also linear, we can write

$$\begin{aligned} P(D_r) (e^{rx} f(x)) &= (\sum_{j=0}^n a_j D_r^j) (e^{rx} f(x)) \\ &= \sum_{j=0}^n e^{rx} a_j D_r^j f(x) = e^{rx} P(D) f(x). \end{aligned}$$

Since $P(D_r)$ is linear, then

$$\frac{1}{P(D_r)} (e^{rx} f(x)) = e^{rx} \frac{1}{P(D)} f(x).$$

Solution of ODEs with constant coefficients

Suppose that $P_n(D) = D_{r_1} D_{r_2} \dots D_{r_n}$, Such that $r_i \quad i = 1, 2, \dots, n$ can be real or complex numbers, in this case the former ODE will be written as follow:

$$\left(\prod_{j=1}^n D_{r_j} \right) y = f(x). \quad (20)$$

Solution of equation (20) is written in the following form

$$y = \frac{1}{\prod_{j=1}^n D_{r_j}} f(x). \quad (21)$$

Suppose that $D_{r_j} \quad j = 1, 2, \dots, n$ are distinct, namely $D_{r_i} \neq D_{r_j}$, (21) is written in one of the following forms

$$(i) \quad y = \left(\sum_{j=1}^n \frac{A_j}{D_{r_j}} \right) f(x) = A_j \frac{1}{D_{r_j}} f(x), \quad (22)$$

where

$$A_{r_j} = \lim_{D \rightarrow r_j} (D - r_j) \prod_{i=1}^n D_{r_i} = \frac{1}{\prod_{j \neq i} (r_j - r_i)}$$

$$(ii) \quad y = \frac{1}{D_{r_1}} \left(\frac{1}{D_{r_2}} \left(\dots \frac{1}{D_{r_n}} f(x) \right) \right). \quad (23)$$

Remark. Let $r_1 = r_2 = \dots = r_n = r$, $k \leq n$.

In this case, using equality (23) is better.

Example 1. Solve the following ODE:

$$y' - 3y = e^{3x}(3x^2 + 10x + 2).$$

Solution. Write $(D - 3)y = e^{3x}(3x^2 + 10x + 2)$ as follow

$$y = \frac{1}{D - 3} (e^{3x}(3x^2 + 10x + 2)) = e^{3x} \int (3x^2 + 10x + 2) dx = e^{3x}(x^3 + 5x^2 + 2x + c).$$

Example 2. Find the solution of the following ODE:

$$(D^2 - 5D + 6)y = e^{2x}.$$

Solution.

$$\begin{aligned} y &= \frac{1}{D^2 - 5D + 6} e^{2x} = \left(\frac{1}{D^3} - \frac{1}{D^2} \right) e^{2x} = e^{3x} \int e^{-x} dx - e^{2x} \int dx \\ &= c_1 e^{2x} + c_2 e^{3x} - e^{2x} - x e^{2x} = k_1 e^{2x} + k_2 e^{3x} - x e^{2x}. \end{aligned}$$

Example 3. Find the particular solution of the following ODE:

$$y''' - 3y' + 2y = (D - 1)^2(D - 2)y = D_1^2 D_{-2} y = e^x(24x^2 + 18x + 4).$$

Solution. Write

$$\begin{aligned} y &= \frac{1}{D_1^2 D_{-2}} [e^x(24x^2 + 18x + 4)] = e^x \frac{1}{D^2 D_{-3}} (24x^2 + 18x + 4) \\ &= e^x \frac{1}{D_{-3}} (2x^4 + 3x^3 + 2x^2) = e^x \left(\frac{2}{3}x^4 + \frac{1}{9}x^3 + \frac{5}{9}x^2 \right). \end{aligned}$$

Example 4. Solve the following linear ODE:

$$y''' - 7y' - 6y = 5e^{-x} - 3e^{-2x} + 9e^{3x}.$$

Solution. Write

$$y''' - 7y' - 6y = D_{-1} D_{-2} D_{-3} y = 5e^{-x} - 3e^{-2x} + 9e^{3x},$$

then

$$\begin{aligned} \frac{1}{D_{-1} D_{-2} D_{-3}} (5e^{-x} - 3e^{-2x} + 9e^{3x}) &= \left(\frac{-1}{D_{-1}} + \frac{\frac{1}{2}}{D_{-2}} + \frac{\frac{1}{2}}{D_3} \right) (5e^{-x} - 3e^{-2x} + 9e^{3x}) \\ &= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{5}{4} x e^{-x} - \frac{3}{5} x e^{-2x} + \frac{9}{20} x e^{3x}. \end{aligned}$$

3 Discussion and results

The r - difference differential operator D_r method is a new method which can solve all of the linear non- homogeneous equations with constant coefficients.

4 References

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