Asymptotic Estimates of the Eigenvalues of Some Integral Operators

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Abstract

In this paper, we give asymptotic estimates of the eigenvalues of certain positive Fredholm operators

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1 Introduction

Throughout let J be a fixed closed subinterval of the real line $\mathbb{R}$. Suppose that D is a simply-connected domain containing the real closed interval J and φ is any function, which maps D conformally onto Δ , where Δ is the open unit disk of complex plain $\mathbb{C}$. Then let us define a function K_D on $D \times D$ by

$$K_D(\zeta, z) = \frac{\varphi'(\zeta)^{\frac{1}{2}} \overline{\varphi'(z)^{\frac{1}{2}}}}{1 - \varphi(\zeta) \overline{\varphi(z)}} \quad \text{for all } \zeta, z \in D,$$

for either of the branches of $\varphi'^{\frac{1}{2}}$. This function is independent of the choice of mapping function φ , see [3,p.410]. For example, if $D = \Delta$, then

$$K_D(z, w) = \frac{1}{1 - z\overline{w}} \quad (\text{take } \varphi(z) = z)$$

and if D is the right half plane, then

$$K_D(z, w) = \frac{1}{z + \bar{w}} \quad (\text{take } \varphi(z) = \frac{z-1}{z+1}).$$

By restricting the function K_D to the square $J \times J$ we obtain a compact symmetric operator T_D on $L^2(J)$ defined by

$$T_D f(s) = \int_J K_D(s, t) f(t) dt \quad (f \in L^2(J), s \in J).$$

This operator is always positive in the sense of operator theory (i.e. $(Tf, f) \geq 0$ for all $f \in L^2(J)$), see [3]. We shall use $\lambda_n(T_D)$ to denote the eigenvalues of T_D .

The main aim of this work is to prove in detail the following theorem.

Theorem 1 *If D_1, D_2 are two discs and if $D = D_1 \cap D_2$ contains the real closed interval J , then*

$$\lambda_n(K_D) \simeq \lambda_n(K_{D_1} + K_{D_2})$$

where $a_n \simeq b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$.

This is a special case of a theorem in [3, Theorem 1] but we give a different proof.

2 Preliminaries And Factorization Through E^2

To prove above Theorem 1 we need the following lemma but we recall some definitions first.

The space $H^\infty(\Delta)$ is just the set of all bounded analytic function on Δ with the uniform norm. For $1 \leq p < \infty$, $H^p(\Delta)$ is the set of all functions f analytic on Δ such that

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty. \quad (1)$$

The p -th root of the left hand side of (1) here defines a complete norm on $H^p(\Delta)$. For more information on this spaces see [1, 2 and 4]. In the case of $p = 2$, H^2 be the familiar Hardy space of all functions analytic on Δ with square-summable Maclaurin coefficients.

Let D be a simply connected domain in $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ and let φ be a Riemann mapping function for D , that is, a conformal map of D onto Δ . An analytic

function g on D is said to be of (Smirnov) class $E^2(D)$ if there exists a function $f \in H^2(\Delta)$ such that

$$g(z) = f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (z \in D)$$

where $\varphi'^{\frac{1}{2}}$ is a branch of the square root of φ' . We define $\|g\|_{E^2(D)} = \|f\|_{H^2(\Delta)}$. Thus, by construction, $E^2(D)$ is a Hilbert space with

$$\langle g_1, g_2 \rangle_{E^2(D)} = \langle f_1, f_2 \rangle_{H^2(\Delta)}$$

where $g_i(z) = f_i(\varphi(z))\varphi'(z)^{\frac{1}{2}}$, ($i = 1, 2$) and the map $U_\varphi : H^2(\Delta) \rightarrow E^2(D)$ given by

$$U_\varphi f(z) = f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (f \in H^2(\Delta), z \in D)$$

is an isometric bijection. For more information on this spaces see [1 and 6]. If ∂D is a rectifiable Jordan curve then the same formula

$$V_\varphi f(z) = f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (f \in L^2(\partial\Delta), z \in \partial D)$$

defines an isometric bijection V_φ of $L^2(\partial\Delta)$ onto $L^2(\partial D)$, the L^2 space of normalized arc length measure on ∂D where ∂D and $\partial\Delta$ denote the boundary of D and Δ respectively. The inverse

$$V_\psi = V_\varphi^{-1} : L^2(\partial D) \rightarrow L^2(\partial\Delta)$$

of V_φ is given by

$$V_\psi g(w) = g(\psi(w))\psi'(w)^{\frac{1}{2}} \quad (g \in L^2(\partial D), w \in \partial\Delta, \psi = \varphi^{-1}).$$

Lemma 2 Suppose that D is a disc or a codisc or a half-plane and $\gamma' \subseteq \overline{D}$ be a circular arc (or a straight line) then for every $g \in E^2(D)$,

$$\frac{1}{2\pi} \int_{\gamma'} |g(z)|^2 |dz| \leq \|g\|_{E^2(D)}^2 = \frac{1}{2\pi} \int_{\partial D} |g(z)|^2 |dz|.$$

Proof. This is Corollary 1.3 to Lemma 1.2 in [5].

Suppose now that D contains our fixed interval J . By restricting φ to J we obtain a linear operator $S_D : E^2(D) \rightarrow L^2(J)$ defined by

$$S_D f(s) = f(s) \quad (f \in E^2(D), s \in J).$$

Then S_D is compact operator and $T_D = S_D S_D^*$ is the compact, positive integral operator on J with kernel K_D :

$$K_D(s, t) = \frac{\varphi'(s)^{\frac{1}{2}} \overline{\varphi'(t)^{\frac{1}{2}}}}{1 - \varphi(s)\overline{\varphi(t)}}$$

for all $s, t \in J$. This is proved in [3].

Definition 1 Let H and H' be Hilbert spaces and suppose that T is a compact, positive operator on H . If $S : H' \rightarrow H$ is a compact operator such that $T = SS^*$, then S is called a quasi square-root of T . We call H' the domain space of S .

Remark 1 Suppose that D_1, D_2 are simply-connected domains containing J and let T_{D_1}, T_{D_2} be continuous positive operators on a Hilbert space $L^2(J)$ and suppose that for each i , S_{D_i} is a quasi square-root of T_{D_i} with domain space $E^2(D_i)$. If $T_+ = T(K_{D_1} + K_{D_2}) = T(\sum_{i=1}^2 K_{D_i})$ so that

$$T_+f(s) = \int_J (K_{D_1}(s, t) + K_{D_2}(s, t))f(t)dt \quad (f \in L^2(J), s \in J),$$

then $T_+ : L^2(J) \rightarrow L^2(J)$ is compact, positive integral operator and T_+ has the quasi square-root $S_+ : E^2(D_1) + E^2(D_2) \rightarrow L^2(J)$,

$$S_+(f_1, f_2) = S_{D_1}f_1 + S_{D_2}f_2$$

so that

$$\begin{aligned} S_+(f_1, f_2)(s) &= S_{D_1}f_1(s) + S_{D_2}f_2(s) \\ &= f_1(s) + f_2(s) \quad (f \in L^2(J), s \in J). \end{aligned}$$

Lemma 3 Let T_1, T_2 be compact operators on a Hilbert space H and suppose that S_1, S_2 are quasi square-root of T_1, T_2 with domain H_1, H_2 respectively.

- i) If there exists a continuous operator $V : H_2 \rightarrow H_1$ such that $S_2 = VS_1$ then $(T_2f, f) \leq k(T_1f, f)$ for some $k > 0$ and so $\lambda_n(T_2) \simeq O(\lambda_n(T_1))$ ($n \geq 0$).
- ii) If there exists continuous operators $V : H_2 \rightarrow H_1$ and $W : H \rightarrow H$ such that $S_2 = WS_1V$, then $\lambda_n(T_2) \simeq O(\lambda_n(T_1))$.

Proof. See [3, page 407].

3 Main Result

Now we are ready to prove the theorem, which mentioned in the introduction. We prove that if D_1, D_2 are two discs containing the closed real interval J , then

$$\text{i) } \lambda_n(K_{D_1 \cap D_2}) = O(\lambda_n(K_{D_1} + K_{D_2}))$$

$$\text{ii) } \lambda_n(K_{D_1} + K_{D_2}) = O(K_{D_1 \cap D_2}).$$

What follows are the main steps in the proof of this theorem. Suppose that D is a simply connected and bounded domain. Let φ be a Riemann mapping function for D and suppose that $\psi = \varphi^{-1}$ is the inverse function of φ . An analytic function f on D is said to be of class $H^\infty(D)$ if it is bounded on D .

Proposition 4 *If $\psi' \in H^1$, then $H^\infty(D) \subseteq E^2(D)$.*

Proof. Suppose that $f \in H^\infty(D)$. For $z \in D$, define $g(z) = f(\psi(z))\psi'(z)^{1/2}$. Then $f\psi \in H^\infty(D)$ and $\psi'^{1/2} \in H^2$. It follows that $(f\psi)\psi'^{1/2} \in H^2$. Hence $f \in E^2(D)$.

Proposition 5 *Suppose that ∂D is a rectifiable Jordan curve, then*

i) $\psi' \in H'$

ii) *Each function $f \in E^2(D)$ has a non-tangential limit $\tilde{f} \in L^2(\partial D)$. The map $f \rightarrow \tilde{f}$ is an isometric isomorphism and $\|f\|_{E^2(D)}^2 = \frac{1}{2\pi} \int_{\partial D} |\tilde{f}(z)|^2 |dz|$.*

iii) *If D is a Smirnov domain, then polynomials (thus H^∞) are dense in $E^2(D)$.*

iv) *If D is a convex region, it is a Smirnov domain.*

v) *$E^2(D)$ coincides with the $L^2(\partial D)$ closure of the polynomials if and only if D is a Smirnov domain*

Proof. See [1, pages 44, 170 and 173]. The definition of smirnov domain is given in [1, page 173]

Remark 2 *If $f \in L^2(\partial\Delta)$, then the formula*

$$P_0 f(z) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_{\partial\Delta} f(\zeta) \frac{1}{1 - \bar{\zeta}z} |d\zeta| \quad (|z| < 1)$$

defines a continuous linear operator $P_0 : L^2(\partial\Delta) \rightarrow H^2(\Delta) = E^2(\Delta)$ and $\|P_0\| = 1$. In fact, P_0 is essentially the orthogonal projection of $L^2(\partial\Delta)$ onto $H^2(\Delta)$. Note that if $f(e^{ni\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{ni\theta}$ then $P_0 f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($a_n \in l^2(\mathbb{Z})$).

Remark 3 *There are similar results for any discs in [3, page 423]. That is, if D is a disc, or codisc or half-plane, then the formula*

$$P f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

defines a continuous linear operator $P : L^2(\partial D) \rightarrow E^2(D)$ with $\|P\| = 1$. Throughout suppose now that D_1, D_2 are plane discs and let $D = D_1 \cap D_2$ contains the real closed interval J . For $k = 1, 2$, let $\gamma_k = \partial D_k \cap (D_1 \cap D_2)$. We shall exhibit continuous operators

$$N : E^2(D) \rightarrow E^2(D_1) \oplus E^2(D_2) \quad \text{and} \quad M : E^2(D_1) \oplus E^2(D_2) \rightarrow E^2(D).$$

To define N suppose first that $G = \{f : f \text{ is a polynomial in } E^2(D)\}$. Note that then $\overline{G} = E^2(D)$ because D is convex.

If $f \in G$, then for all $z \in D$, Cauchy's Integral Formula gives

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} dw + \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z} dw.$$

For $f \in G$ and $1 \leq k \leq 2$, define a function f_k on D_k by

$$f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw \quad (z \in D_k),$$

and define a function $\tilde{f}_k$ on ∂D_k by

$$\tilde{f}_k(z) = \begin{cases} f(\zeta) & \text{if } \zeta \in \gamma_k \\ 0 & \text{if } \zeta \in \partial D_k - \gamma_k \end{cases} \quad (z \in \partial D_k).$$

Lemma 6 Suppose that $f \in G$ and $1 \leq k \leq 2$ then

i) $\tilde{f}_k \in L^2(\partial D_k)$ and $f_k \in E^2(D_k)$.

ii) The formula

$$V_1 f = (f_1, f_2)$$

defines a continuous linear operator $V_1 : G \rightarrow E^2(D_1) \oplus E^2(D_2)$ and so that V_1 has an extension N by continuity to $E^2(D)$.

Proof i) Let φ_k be a Riemann mapping function for D_k and suppose V_{φ_k} and U_{φ_k} are as in Section 2.

Consider the Figure 1

where the map $P_k : L^2(\partial D_k) \rightarrow E^2(D_k)$ given by

$$P_k f(z) = \frac{1}{2\pi i} \int_{\partial D_k} \frac{f(\zeta)}{\zeta - z} d\zeta$$

is a continuous linear operator with $\|P_k\| = 1$ (from Remark 3).

Since $f_k = P_k \tilde{f}_k$, it follows that $f_k \in E^2(D_k)$. So $(f_1, f_2) \in E^2(D_1) \oplus E^2(D_2)$. Since

$$\begin{aligned} \|f_k\|_{E^2(D_k)}^2 &= \|P_k \tilde{f}_k\|_{E^2(D_k)}^2 \leq \|\tilde{f}_k\|_{L^2(\partial D_k)}^2 = \frac{1}{2\pi} \int_{\gamma_k} |f(\zeta)|^2 |d\zeta| \\ &\leq \|f\|_{E^2(D)}^2 \end{aligned}$$

$$\begin{array}{ccc}
L^2(\partial D_k) & \xleftarrow{V_{\varphi_k}} & L^2(\partial \Delta) \\
\downarrow P_k & & \downarrow P_0 \\
E^2(D_k) & \xleftarrow{U_{\varphi_k}} & H^2(\Delta) = E^2(\Delta)
\end{array}$$

Figure 1:

and

$$\begin{aligned}
\|V_1 f\|_{E^2(D_1) \oplus E^2(D_2)}^2 &= \|(f_1, f_2)\|_{E^2(D_1) \oplus E^2(D_2)}^2 = \|f_1\|_{E^2(D_1)}^2 + \|f_2\|_{E^2(D_2)}^2 \\
&\leq 2\|f\|_{E^2(D)}^2.
\end{aligned}$$

It follows that the map $f \rightarrow (f_1, f_2)$ is a continuous linear operator $G \rightarrow E^2(D_1) \oplus E^2(D_2)$. Now suppose that N is an extension by continuity to $E^2(D)$. Note that then $\|N\|^2 \leq 2$.

If we denote $F = H^\infty(D_1) \oplus H^\infty(D_2)$ then $\overline{F} = E^2(D_1) \oplus E^2(D_2)$.

Lemma 7 *The map $V_2 : F \rightarrow E^2(D)$, is given by*

$$V_2(f_1, f_2)(z) = f_1(z) + f_2(z), \quad ((f_1, f_2) \in F, \quad z \in D),$$

is a continuous operator so that V_2 has an extension M by continuity to $E^2(D_1) \oplus E^2(D_2)$.

Proof If $(f_1, f_2) \in F$ then by Propositions 4 and 5

$$f_i \in H^\infty(D) \subseteq E^2(D) \quad (1 \leq i \leq 2)$$

and

$$V_2(f_1, f_2) \in H^\infty(D) \subseteq E^2(D).$$

So we have

$$\begin{aligned}
\|V_2(f_1, f_2)\|_{E^2(D)}^2 &= \|f_1 + f_2\|_{E^2(D)}^2 \\
&= \|f_1\|_{E^2(D)}^2 + \|f_2\|_{E^2(D)}^2 + 2 \operatorname{Re} \langle f_1, f_2 \rangle_{E^2(D)} \\
&\leq \|f_1\|_{E^2(D)}^2 + \|f_2\|_{E^2(D)}^2 + (\|f_1\|_{E^2(D)}^2 + \|f_2\|_{E^2(D)}^2) \\
&= 2(\|f_1\|_{E^2(D)}^2 + \|f_2\|_{E^2(D)}^2) \\
&\quad \text{(by Corollary 2)} \\
&\leq 2(2\|f_1\|_{E^2(D_1)}^2 + 2\|f_2\|_{E^2(D_2)}^2) \\
&= 4\|(f_1, f_2)\|_{E^2(D_1) \oplus E^2(D_2)}^2.
\end{aligned}$$

Hence $\|V_2\|^2 \leq 4$ and V_2 is a continuous linear operator. Let now M be extension by continuity to $E^2(D_1) \oplus E^2(D_2)$. Note that then $\|M\|^2 \leq 4$.

PROOF OF THEOREM 1

i) Consider the Figure 2 where V_1 is as in Lemma 6

Note that here $T_+ = S_+ S_+^*$ and $T_D = S_D S_D^*$. By definition of V_1 , we have $S_D f = S_+ V_1 f$ for every $f \in G$. Thus, by continuity of V_1 , $S_D f = S_+ N f$ for every $f \in E^2(D)$ and so $S_D = S_+ N$. So for $g \in L^2(J)$,

$$\begin{aligned} \langle S_D S_D^* g, g \rangle &= \|S_D^* g\|^2 = \|N^* S_+^* g\|^2 \\ &\leq \|N^*\|^2 \|S_+^* g\|^2 = \|N\|^2 \langle S_+ S_+^* g, g \rangle \\ &\leq 2 \langle S_+ S_+^* g, g \rangle \end{aligned}$$

That is,

$$S_D S_D^* \leq 2 S_+ S_+^*.$$

Hence by Lemma 3

$$\lambda_n(S_D S_D^*) \leq 2 \lambda_n(S_+ S_+^*)$$

as required.

ii) Consider the Figure 3 where V_2 is as in Lemma 7. By definition of V_2 , it follows that $S_+(f_1, f_2) = S_D V_2(f_1, f_2)$ for every $(f_1, f_2) \in H^\infty(D_1) \oplus H^\infty(D_2)$. Thus, by continuity of V_2 ,

$$S_+(f_1, f_2) = S_D M(f_1, f_2) \quad \text{for every} \quad (f_1, f_2) \in E^2(D_1) \oplus E^2(D_2)$$

$$\begin{array}{ccccc}
 E^2(D_1) \oplus E^2(D_2) & \xrightarrow{V_2} & E^2(D) & & \\
 (f_1, f_2) & \xrightarrow{\quad} & f_1|_D + f_2|_D & \xrightarrow{\quad} & f \\
 & \searrow S_+ & & \downarrow S_D & \downarrow \\
 & & f_1|_J + f_2|_J & & f|_J \\
 & & & \searrow & \\
 & & & & L^2(J)
 \end{array}$$

Figure 3:

and so $S_+ = S_D M$. So for $g \in L^2(J)$, we have

$$\begin{aligned}
 \langle S_+ S_+^* g, g \rangle &\leq \|M\|^2 \langle S_D S_D^* g, g \rangle \\
 &\leq 4 \langle S_D S_D^* g, g \rangle
 \end{aligned}$$

i.e.,

$$S_+ S_+^* \leq 4 S_D S_D^*.$$

Consequently, from Lemma 3,

$$\lambda_n(S_+ S_+^*) \leq 4 \lambda_n(S_D S_D^*).$$

References

- [1] Duren, P. L., Theory of Hp Spaces Academic Press, New York, 1970.
- [2] Koosis, P., Introduction to Hp Spaces Cambridge University Press, Cambridge, Second Edition, 1998
- [3] Little, G., Equivalences of positive integrals operators with rational kernels., Proc. London. Math. Soc. (3) 62 (1991), 403-426.
- [4] Pommerenke, C., Boundary Behaviour of Conformal Maps, Springer-Verlag, Berlin, Heidelberg, 1992.
- [5] Soykan Y., An Inequality of Fejer-Riesz Type, Cankaya University, Journal of Arts and Sciences, Issue: 5, May 2006, 51-60

- [6] Soykan, Y., On equivalent characterisation of elements of Hardy and Smirnov spaces., Int. Math. Forum. 2, 2007, no. 24, 1185 - 1191.
- [7] Soykan, Y., Boundary behaviour of analytic curves, Int. Journal of Math. Analysis, Vol. 1, 2007, no. 3, 133 - 157.

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