A Numerical Solution of the One-Dimensional Inverse Parabolic Problem Using Chebyshev T Polynomials

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Abstract

In this paper, a numerical solution of an inverse non-dimensional heat conduction problem will be considered. By using a sensor located at a point inside the body and measuring the temperature at a point $x = x_1$, $0 < x_1 < 1$, and applying the Chebyshev polynomials base function to the inverse heat conduction problem, we determine a stable numerical solution to this problem.

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1 Introduction

The numerical solution of the inverse heat conduction problem (IHCP) requires to determine an unknown boundary condition from an additional information. Inverse heat conduction problems have many applications in various branches of science and engineering. Mechanical and chemical engineers, mathematicians and specialists in many other sciences branches are interested in inverse problems, each with different application in mind [1]-[6].

In this work, by using the ChebyshevT-set a stable solution for an inverse heat conduction problem is presented.

2 Problem statement

In this section, let us consider the following IHCP

$$u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t < T_0, \quad (1)$$

$$u(x, 0) = f(x), \quad 0 < x_1 < x < 1, \quad (2)$$

$$u(x_1, t) = g(t), \quad 0 < t < T_0, \quad (3)$$

$$u_x(1, t) = h(t), \quad 0 < t < T_0, \quad (4)$$

where T_0 is a given positive constant, and the sensor is located at x_1 , $0 < x_1 < 1$, for measuring temperature, at this point, $f(x)$ is piecewise-continuous known function, $g(t)$ and $h(t)$ are infinitely differentiable functions, while the temperature $u(x, t)$ and heat flux $u_x(0, t)$ are unknown which remain to be determined. The problem (1)-(4) may be divided into two separate problems. The first problem is

$$u_t = u_{xx}, \quad x_1 < x < 1, \quad 0 < t < T_0, \quad (5)$$

$$u(x, 0) = f(x), \quad x_1 < x < 1, \quad (6)$$

$$u(x_1, t) = g(t), \quad 0 < t < T_0, \quad (7)$$

$$u_x(1, t) = h(t), \quad 0 < t < T_0. \quad (8)$$

This problem may be analyzed as a direct problem, for the portion of the body from $x = x_1$ to $x = 1$ with known boundary conditions. There is a unique stable solution to the direct problem (5)-(8)[5]. The second problem is the

following IHCP

$$u_t = u_{xx}, \quad 0 < x < x_1, \quad 0 < t < T_0, \quad (9)$$

$$u(x_1, t) = g(t), \quad 0 < t < T_0. \quad (10)$$

In the next section, the above IHCP will be considered, the heat in the body and heat flux at the boundary $x = 0$ will be obtained by solving this problem numerically.

Theorem 2.1 (*Existence*)

For piecewise-continuous functions $f(x)$, $g(t)$, and $h(t)$ on their domains, the problem (5)-(8) has a unique solution [7].

3 Numerical solution of the IHCP (9)-(10)

In order to solve the problem (9)-(10) we need an extra condition, and we choose the condition

$$u_x(x_1, t) = k(t), \quad 0 \leq t \leq T_0, \quad (11)$$

where $k(t)$ can be obtained from the solution of the direct problem (5)-(8). The problem (9)-(11) is a Cauchy problem. The solution of this problem, exists and is unique but not always stable [7]. In this work, stable solution will be given to the Cauchy problem (9)-(11). Now using the transformations

$$\tau = -1 + \frac{2}{T_0}t, \quad (12)$$

and

$$T(x, \tau) = u(x, \frac{(1 + \tau)T_0}{2}), \quad (13)$$

(9)-(11) becomes

$$T_\tau = \frac{T_0}{2}T_{xx}, \quad 0 < x < x_1, \quad -1 < \tau < 1, \quad (14)$$

$$T(x_1, \tau) = G(\tau), \quad -1 \leq \tau \leq 1, \quad (15)$$

$$T_x(x_1, \tau) = K(\tau), \quad -1 \leq \tau \leq 1. \quad (16)$$

This problem is called a Cauchy problem. Now, we seek a solution for this problem in the form

$$T(x, \tau) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{ij} T_j(\tau) (x - x_1)^i, \quad (17)$$

where $T_j(\tau)$ is a ChebyshevT polynomials of degree j , and $C_{i,j}$ are constant numbers which must be determined. By putting (17) in (14) and equating the coefficient of each power of $(x - x_1)^i$ to zero, we find

$$C_{i+2,j} = \frac{2T'_{j+1}(\tau)}{T_0(i+1)(i+2)T_j(\tau)} C_{i,j+1}. \quad (18)$$

From

$$T(x_1, \tau) = \sum_{j=0}^{\infty} C_{0,j} T_j(\tau) = G(\tau),$$

we conclude that

$$C_{0,0} = \frac{1}{\pi} \int_{-1}^1 \frac{G(\tau)}{\sqrt{1-\tau^2}} d\tau, \quad (19)$$

and

$$C_{0,j} = \frac{2}{\pi} \int_{-1}^1 \frac{G(\tau) T_j(\tau)}{\sqrt{1-\tau^2}} d\tau. \quad (20)$$

Similarly from $T_x(x_1, \tau) = K(\tau)$ we obtain

$$C_{1,0} = \frac{1}{\pi} \int_{-1}^1 \frac{K(\tau)}{\sqrt{1-\tau^2}} d\tau, \quad (21)$$

and

$$C_{1,j} = \frac{2}{\pi} \int_{-1}^1 \frac{K(\tau) T_j(\tau)}{\sqrt{1-\tau^2}} d\tau. \quad (22)$$

Substituting the above results in (17), we find

$$T(x, \tau) = \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{2}{T_0}\right)^s \left(\prod_{k=1}^s \frac{T'_{j+k}(\tau)}{T_{j+k-1}(\tau)}\right) \left(\frac{C_{0,j+s}}{(2s)!} T_j(\tau) (x - x_1)^{2s} + \frac{C_{1,j+s}}{(2s+1)!} T_j(\tau) (x - x_1)^{2s+1}\right) \quad (23)$$

Now, if $G(\tau)$ and $K(\tau)$ belong to Holmgren class $H(x_1, 1, C_1, 0)$ [5] then (23) is a solution to the problem (14)-(16) [5]. Therefore, if we approximate $G(\tau)$,

$K(\tau)$ by polynomials of degree n , then by the orthogonality of ChebyshevT polynomials with respect to the weight function $(1 - \tau^2)^{-1/2}$, and the following properties

$$\int_{-1}^1 \frac{G(\tau)T_{j+s}(\tau)}{\sqrt{1-\tau^2}} d\tau = 0, \quad j+s > n. \quad (24)$$

and

$$\int_{-1}^1 \frac{K(\tau)T_{j+s}(\tau)}{\sqrt{1-\tau^2}} d\tau = 0, \quad j+s > n. \quad (25)$$

we will obtain an approximate solution in the form

$$T(x, \tau) \cong \sum_{s=0}^n \sum_{j=0}^n \left(\frac{2}{T_0}\right)^s \left(\prod_{k=1}^s \frac{T'_{j+k}(\tau)}{T_{j+k-1}(\tau)}\right) \left(\frac{C_{0,j+s}}{(2s)!} T_j(\tau)(x-x_1)^{2s} + \frac{C_{1,j+s}}{(2s+1)!} T_j(\tau)(x-x_1)^{2s+1}\right) \quad (26)$$

for the problem (14)-(16). The unknown heat flux at $x = 0$ will be obtained by differentiation of (23) with respect to x and putting $x = 0$.

Now, by giving the following theorem the stability of solution (23) will be proved.

Theorem 3.1 *If $G(\tau)$ and $K(\tau)$, belong to Holmgren class $H(x_1, 1, C_1, 0)$ and $G(\tau)$ and $K(\tau)$, are approximated by two polynomials of degree n , then the approximate solution (23) is stable.*

Proof. *Let us consider T as a solution in the form (23) and T^* as a perturb solution of IHCP (9)-(11). Then in order to prove the stability of solution (23), we have to show that for any given α if*

$$|G(\tau) - G^*(\tau)| < \alpha, \quad (27)$$

$$|K(\tau) - K^*(\tau)| < \alpha, \quad (28)$$

then there exist positive numbers β_1 and β_2 such that

$$|T(0, \tau) - T^*(0, \tau)| < \beta_1, \quad (29)$$

$$|T_x(0, \tau) - T_x^*(0, \tau)| < \beta_2. \quad (30)$$

Using the equation (23), we obtain

$$|T(0, \tau) - T^*(0, \tau)| \leq \sum_{s=0}^{\infty} \sum_{j=0}^{\infty} |T_j(\tau)| \left(\frac{2}{T_0}\right)^s \left| \prod_{k=1}^s \frac{T'_{j+k}(\tau)}{T_{j+k-1}(\tau)} \right|$$

$$\left[\frac{1}{(2s)!} \int_{-1}^1 \frac{|G(\tau) - G^*(\tau)| |T_{j+s}(\tau)|}{\sqrt{1-\tau^2}} d\tau + \frac{1}{(2s+1)!} \int_{-1}^1 \frac{|K(\tau) - K^*(\tau)| |T_{j+s}(\tau)|}{\sqrt{1-\tau^2}} d\tau \right],$$

from which, we find

$$|T(0, \tau) - T^*(0, \tau)| \leq \sum_{s=0}^n \sum_{j=0}^n \left(\frac{3}{\pi}\right) \left(\frac{2}{T_0}\right)^s M M_2 \left(\frac{\alpha}{(2s)!} + \frac{\alpha}{(2s+1)!}\right),$$

where

$$\left| \int_{-1}^1 \frac{1}{\sqrt{1-\tau^2}} d\tau \right| = M_2.$$

Finally

$$|T(0, \tau) - T^*(0, \tau)| \leq \alpha M_3.$$

This completes the proof of stability of solution(23).

4 A test problem

the purpose of this section is to illustrate the applicability of the method. This may be done by given some examples.

Example 4.1 In this example we solve the problem (9)-(11) with given data

$$u(x_1, t) = x_1^2 + 2t, \quad 0 \leq t \leq T_0,$$

$$u_x(x_1, t) = 2x_1, \quad 0 \leq t \leq T_0.$$

Table 1. The Temperature and heat flux on the boundary $x = 0$ for $T_0 = 2$ when $x_1 = .25$.

t	Numerical solution		Exact solution	
	$u(0, t)$	$u_x(0, t)$	$u(0, t)$	$u_x(0, t)$
0.025	0.05	0.0	0.05	0.0
0.225	0.45	0.0	0.45	0.0
0.425	0.85	0.0	0.85	0.0
0.625	1.25	0.0	1.25	0.0
0.825	1.65	0.0	1.65	0.0
1.025	2.05	0.0	2.05	0.0

Example 4.2 In this example, we consider the problem (9)-(11) with given data

$$\begin{aligned} u(x_1, t) &= \exp(-t) \sin x_1, & 0 \leq t \leq T_0, \\ u_x(x_1, t) &= \exp(-t) \cos x_1, & 0 \leq t \leq T_0. \end{aligned}$$

Table 2. The Temperature and heat flux on the boundary $x = 0$ for $T_0 = 2$ when $x_1 = .25$.

t	Numerical solution		Exact solution	
	$u(0, t)$	$u_x(0, t)$	$u(0, t)$	$u_x(0, t)$
0.025	1.40×10^{-5}	0.975091	0.0	0.97531
0.225	1.19×10^{-5}	0.798218	0.0	0.798516
0.425	1.14×10^{-5}	0.653941	0.0	0.65377
0.625	1.77×10^{-6}	0.535288	0.0	0.535261
0.825	1.15×10^{-6}	0.438252	0.0	0.438235
1.025	1.64×10^{-7}	0.358794	0.0	0.358796

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