International Mathematical Forum, 2, 2007, no. 48, 2347 - 2367

A Non-Uniform Bound on Normal Approximation of Randomized Orthogonal Array Sampling Designs

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Abstract

This paper gave a non-uniform bound on normal approximation of randomized orthogonal array sampling designs which is proposed by Owen and Tang in 1992 and 1993. Our method is Stein's method.

Mathematics Subject Classification: Primary 62E20, secondary 60F05

Keywords: non-uniform bound, normal approximation, computer experiment, orthogonal array sampling designs, Stein's method

1 Introduction

The problem of computing a value of integral over a high dimension is important in many scientific fields. Consider a deterministic function $Y = f(x) \in \mathbb{R}$ where $x \in [0,1]^d$ and f is known but hard to calculate. Then our aim is to estimate

$$\int_{[0,1]^d} f(x)dx,$$

that is the expectation of $f \circ X$, $\mu = E(f \circ X)$, where X is a random vector uniformly distributed on a unit hypercube $[0,1]^d$. The simplest way is to draw the samplings $X_1, X_2, ..., X_n$ independently and uniformly distributed from $[0,1]^d$ and use

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f \circ X_i \tag{1.1}$$

as an estimator of μ. This method is called the Monte Carlo method.(see[10],[14] and [16] for more details)

Beside the uniform sampling, there are various alternative ways to select the points X_i's for $\hat{\mu}$. For examples, lattice sampling([9]), Latin hypercube sampling([1],[13],[15],[18]), the orthogonal arrays([2],[6]) and scrambled net([3],[4]).

In 1996, Loh considered some classes of the orthogonal arrays and he gave a uniform bound on normal approximation. In this work, we shall establish a non-uniform bound by using the Stein's method. The organization of this paper is as follows. In section 2, we give some definitions, notations and state the main result. Some useful lemmas are proved in section 3 and we show the main result in section 4.

2 Preliminary Notes and Main Result

Definition 2.1. *An orthogonal array of strength t with index λ ($\lambda \geq 1$) is an $n \times d$ matrix with elements taken from the set $\{0, 1, ..., q-1\}$ such that for any $n \times t$ submatrix, each of the q^t possible rows appears the same number λ of times where d, n, q and t are positive integers with $t \leq d$ and $q \geq 2$. Of course, $n = \lambda q^t$.*

A class of this arrays is denoted by $OA(n, d, q, t)$(see [8] for more details).

In 1996, Loh considered the class $OA(n, 3, q, 2)$ when $n = q^2$ and constructed the sampling $X_1, X_2, ..., X_{q^2}$ on the unit cube $[0,1]^3$ as follows: Let
(a) π_1, π_2, π_3 be random permutations of $\{0, 1, ..., q-1\}$,
(b) $U_{i_1,i_2,i_3,j}$ be $[0,1]$ uniform random variables where $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$, $j \in \{1, 2, 3\}$, and
(c) $U_{i_1,i_2,i_3,j}$'s and π_k's be all stochastically independent.
An orthogonal array-based sample of size q^2, $\{X_1, X_2, ..., X_{q^2}\}$, is defined to be

$$\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \leq i \leq q^2\},$$

where, for each $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$ and $j \in \{1, 2, 3\}$,

$$X\big(i_1, i_2, i_3\big) = \big(X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3)\big),$$

$$X_j(i_1,i_2,i_3)=\frac{i_j+U_{i_1,i_2,i_3,j}}{q},$$

and $a_{i,j}$ is the $(i,j)^{th}$ element of some arbitary but fixed $A\in OA(q^2,3,q,2)$. So the estimator $\hat{\mu}$ of μ in (1.1) can be express in the form of

$$\hat{\mu}=\frac{1}{q^2}\sum_{i=1}^{q^2} f\circ X(\pi_1(a_{i,1}),\pi_2(a_{i,2}),\pi_3(a_{i,3})).$$

Owen([2]) gave an expression for the asymptotic variance $Var(\hat{\mu})$ of $\hat{\mu}$. In this paper, we assume that $Var(\hat{\mu})>0$ and define

$$W=\frac{1}{\sqrt{Var(\hat{\mu})}}(\hat{\mu}-\mu). \tag{2.1}$$

Loh([18]) gave a uniform bound on the normal approximation of W and Theorem 2.1 is his result.

Theorem 2.1. *Let Φ be the stantdard normal distribution function, i.e., $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$. Suppose that $E(f\circ X)^r<\infty$ for some even integer $r\geq 4$. Then*

$$sup\Big\{\big|P(W\leq w)-\Phi(w)\big|:-\infty<w<\infty\Big\}=O(q^{\frac{2-r}{2r-2}})\quad as\quad q\to\infty.$$

In 2006, Laipaporn and Neammanee([12]) improved the order in Theorem 2.1 to the rate $O(q^{-\frac{1}{2}})$ under the assumption that $E(f\circ X)^6<\infty$. In this work, we investigate a non-uniform bound. The following theorem is our main result.

Theorem 2.2. *Suppose that $E(f\circ X)^r<\infty$ for even number $r\geq 10$. Then, as $q\to\infty$, for $z\in\mathbb{R}$,*

$$\big|P(W\leq z)-\Phi(z)\big|\leq max\Big(\frac{1}{(1+|z|)^{1-\frac{2}{r}}}O(\frac{1}{q^{\frac{r-8}{2r}}}),\frac{1}{(1+|z|)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}})\Big).$$

3 Auxiliary Results

In this section, we will give some lemmas which used in proving the main result in section 4. In 1996, Loh defined a random function ρ_π be such that

$$\big(i_1,i_2,\rho_\pi(i_1,i_2)\big)=\big(\pi_1(a_{i,1}),\pi_2(a_{i,2}),\pi_3(a_{i,3})\big) \tag{3.1}$$

for some $i\in\{1,...,q^2\}$ and showed that W in (2.1) can be rewritten again as the form

$$W=\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}Y\big(i_1,i_2,\rho_\pi(i_1,i_2)\big) \tag{3.2}$$

where
$$Y(i_1,i_2,i_3)=\frac{1}{q^2\sqrt{\mathrm{Var}(\hat{\mu})}}\Big[f\circ X(i_1,i_2,i_3)-\mu-\sum_{j=1}^{3}\mu_j(i_j)-\sum_{1\le k<l\le 3}\mu_{k,l}(i_k,i_l)\Big],$$

$$\mu(i_1,i_2,i_3)=Ef\circ X(i_1,i_2,i_3),\ \ \mu_j(i_j)=\frac{1}{q^2}\sum_{\substack{i_k=0\\ k\neq j}}^{q-1}[\mu(i_1,i_2,i_3)-\mu],$$

and
$$\mu_{k,l}(i_k,i_l)=\frac{1}{q}\sum_{\substack{i_j=0\\ j\neq k,l}}^{q-1}[\mu(i_1,i_2,i_3)-\mu-\mu_k(i_k)-\mu_l(i_l)].$$

Let I and K be uniformly distributed random variables on $\{0,1,...,q-1\}$, (I,K) uniformly distributed on $\big\{(i,k)|i,k=0,1,...,q-1,i\neq k\big\}$ and assume that they are indenpendent of all π_1,π_2,π_3 and $U_{i_1,i_2,i_3,j}$'s defined previously. Define

$$\widetilde{W}=W-S_1-S_2+S_3+S_4$$

where
$$S_1=\sum_{i_2=0}^{q-1}Y\big(I,i_2,\rho_\pi(I,i_2)\big),\qquad S_2=\sum_{i_2=0}^{q-1}Y\big(K,i_2,\rho_\pi(K,i_2)\big),$$
$$S_3=\sum_{i_2=0}^{q-1}Y\big(I,i_2,\rho_\pi(K,i_2)\big),\qquad S_4=\sum_{i_2=0}^{q-1}Y\big(K,i_2,\rho_\pi(I,i_2)\big).$$

Moreover, for each i_1,i_2 and $i_3\in\{0,1,2,...,q-1\}$, and $z\ge 0$, we also let

$$\tilde{\mu}(i_1,i_2,i_3)=EY(i_1,i_2,i_3)$$
$$Y_z(i_1,i_2,i_3)=Y(i_1,i_2,i_3)\mathbb{I}\big(\big|Y(i_1,i_2,i_3)\big|>1+z\big),$$
$$\widehat{Y}_z(i_1,i_2,i_3)=Y(i_1,i_2,i_3)\mathbb{I}\big(\big|Y(i_1,i_2,i_3)\big|\le 1+z\big),$$
$$\widehat{Y}=\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\widehat{Y}_z(i_1,i_2,\rho_\pi(i_1,i_2))$$

and
$$\widetilde{Y}=\widehat{Y}-\widehat{S}_{1,z}-\widehat{S}_{2,z}+\widehat{S}_{3,z}+\widehat{S}_{4,z}$$

where
$$\widehat{S}_{1,z}=\sum_{i_2=0}^{q-1}\widehat{Y}_z(I,i_2,\rho_\pi(I,i_2)),\quad \widehat{S}_{2,z}=\sum_{i_2=0}^{q-1}\widehat{Y}_z(K,i_2,\rho_\pi(K,i_2)),$$
$$\widehat{S}_{3,z}=\sum_{i_2=0}^{q-1}\widehat{Y}_z(I,i_2,\rho_\pi(K,i_2)),\quad \widehat{S}_{4,z}=\sum_{i_2=0}^{q-1}\widehat{Y}_z(K,i_2,\rho_\pi(I,i_2)),$$

and $\mathbb{I}$ is an indicator function. Loh([18]) showed that $\left(W, \widetilde{W}\right)$ is an exchangeable pair, S_1, S_2, S_3, S_4 are identically distributed and if $E(f \circ X)^r < \infty$ for a positive even integer r, then

$$ES_i^r = O(q^{-\frac{r}{2}}) \text{ for } i = 1,2,3,4, \tag{3.3}$$

$$\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} EY^r(i_1,i_2,i_3) = O(q^{3-r}), \tag{3.4}$$

and
$$\frac{1}{q}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} EY^2(i_1,i_2,i_3) = 1 + O(\frac{1}{q}). \tag{3.5}$$

By the fact that

$$E\left(\widetilde{W} - W\right)^r \leq O(q^{-\frac{r}{2}}) \tag{3.6}$$

for every positive even number r([12]) we have

$$\begin{aligned}\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} E|Y(i_1,i_2,i_3)^m Y_z(i_1,i_2,i_3)^n| &\leq \sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} \frac{E|Y(i_1,i_2,i_3)|^{m+n+t}}{(1+z)^t} \\ &\leq \frac{O(q^{3-m-n-t})}{(1+z)^t}\end{aligned} \tag{3.7}$$

for any integers m, n and t which $m \geq 0, n, t > 0$ and $m+n+t$ is an even number. The following lemmas are used in section 4 and in what follows, C denotes a generic absolute constant whose value may be different at each appearance.

Lemma 3.1.

1. If $E(f \circ X)^2 < \infty$, then $\left|E\widehat{Y}^2 - 1\right| \leq O(\frac{1}{q})$ as $q \to \infty$.

2. If $E(f \circ X)^r < \infty$, then, for any even number $r \geq 2$, $E|\widetilde{Y} - \widehat{Y}|^r \leq O(\frac{1}{q^{\frac{r}{2}}})$ as $q \to \infty$.

3. If $E(f \circ X)^{r+1} < \infty$, then, for any odd number $r \geq 3$, $E|\widetilde{Y} - \widehat{Y}|^r \leq O(\frac{1}{q^{\frac{r}{2}}})$ as $q \to \infty$.

Proof. see [11] on p.20. □

Lemma 3.2.

1. If $E(f \circ X)^2 < \infty$, then $E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y(i_1,i_2,i_3)\Big)^2 \leq O(q)$ as $q \to \infty$.

2. If $E(f \circ X)^4 < \infty$, then $E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y(i_1,i_2,i_3)\Big)^4 \leq O(q^2)$ as $q \to \infty$.

Proof.
1. By the fact that

$$\sum_{i_j=0}^{q-1} \tilde{\mu}(i_1, i_2, i_3) = 0 \quad \text{for} \quad j = 1, 2, 3, \tag{3.8}$$

(see [18], p.1212) and (3.5), we have

$$\begin{aligned}
&E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y(i_1, i_2, i_3)\Big)^2 \\
&= \sum_{i_1,i_2,i_3} EY^2(i_1, i_2, i_3) + \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}(i_1, i_2, i_3)\tilde{\mu}(j_1, j_2, j_3) \\
&= \sum_{i_1,i_2,i_3} EY^2(i_1, i_2, i_3) - \sum_{i_1,i_2,i_3} \tilde{\mu}^2(i_1, i_2, i_3) \\
&\leq O(q).
\end{aligned}$$

2. Note that

$$E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y(i_1, i_2, i_3)\Big)^4 = C_1M_1 + C_2M_2 + C_3M_3 + C_4M_4 + C_5M_5 \tag{3.9}$$

where

$$M_1 = \sum_{i_1,i_2,i_3} EY^4(i_1, i_2, i_3),$$

$$M_2 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY(i_1, i_2, i_3)Y^3(j_1, j_2, j_3),$$

$$M_3 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY^2(i_1, i_2, i_3)Y^2(j_1, j_2, j_3),$$

$$M_4 = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3) \\ (k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} EY^2(i_1, i_2, i_3)Y(j_1, j_2, j_3)Y(k_1, k_2, k_3),$$

$$\begin{aligned}
M_5 = &\sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{l_1,l_2,l_3 \\ (l_1,l_2,l_3)\neq(i_1,i_2,i_3) \\ (l_1,l_2,l_3)\neq(k_1,k_2,k_3)}} \\
&EY(i_1, i_2, i_3)Y(j_1, j_2, j_3)Y(k_1, k_2, k_3)Y(l_1, l_2, l_3)
\end{aligned}$$

and C_1, C_2, C_3, C_4, C_5 are constants.

From (3.4) and (3.8), we know that

$$
\begin{aligned}
|M_1| &= O(\frac{1}{q}), &(3.10)\\
\big|M_2\big| &= \Big|\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\tilde{\mu}(i_1,i_2,i_3)EY^3(j_1,j_2,j_3)\Big|\\
&= \Big|-\sum_{j_1,j_2,j_3}\tilde{\mu}(j_1,j_2,j_3)EY^3(j_1,j_2,j_3)\Big|\\
&\leq \sum_{j_1,j_2,j_3}EY^4(j_1,j_2,j_3)\\
&= O(\frac{1}{q}), &(3.11)\\
|M_3| &= \sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}EY^2(i_1,i_2,i_3)EY^2(j_1,j_2,j_3)\\
&\leq \big(\sum_{i_1,i_2,i_3}EY^2(i_1,i_2,i_3)\big)^2\\
&\leq O(q^2), &(3.12)\\
|M_4| &= \Big|\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\sum_{\substack{l_1,l_2,l_3\\(l_1,l_2,l_3)\neq(i_1,i_2,i_3)\\(l_1,l_2,l_3)\neq(j_1,j_2,j_3)}}EY^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(l_1,l_2,l_3)\Big|\\
& &(3.13)\\
&\leq \Big|\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}EY^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(i_1,i_2,i_3)\Big|\\
&\quad+\Big|\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}EY^2(i_1,i_2,i_3)\tilde{\mu}^2(j_1,j_2,j_3)\Big|\\
&\leq \sum_{i_1,i_2,i_3}EY^4(i_1,i_2,i_3)+\big(\sum_{i_1,i_2,i_3}EY^2(i_1,i_2,i_3)\big)^2\\
&\leq O(q^2), &(3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\big|M_5\big| = \Big|&\sum_{i_1,i_2,i_3}\sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\sum_{\substack{k_1,k_2,k_3\\(k_1,k_2,k_3)\neq(i_1,i_2,i_3)\\(k_1,k_2,k_3)\neq(j_1,j_2,j_3)}}\sum_{\substack{l_1,l_2,l_3\\(l_1,l_2,l_3)\neq(i_1,i_2,i_3)\\(l_1,l_2,l_3)\neq(j_1,j_2,j_3)\\(l_1,l_2,l_3)\neq(k_1,k_2,k_3)}}\\
&\tilde{\mu}(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(k_1,k_2,k_3)\tilde{\mu}(l_1,l_2,l_3)\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= 3\Big|\sum_{i_1,i_2,i_3}\ \sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}}\ \sum_{\substack{k_1,k_2,k_3\\(k_1,k_2,k_3)\neq(i_1,i_2,i_3)\\(k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} \tilde{\mu}^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(k_1,k_2,k_3)\Big|\\
&= 3\Big|\sum_{i_1,i_2,i_3}\ \sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(i_1,i_2,i_3)\\
&\quad+\sum_{i_1,i_2,i_3}\ \sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \tilde{\mu}^2(i_1,i_2,i_3)\tilde{\mu}(j_1,j_2,j_3)\tilde{\mu}(j_1,j_2,j_3)\Big|\\
&\le 3\Big(\sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) + \Big(\sum_{i_1,i_2,i_3} EY^2(i_1,i_2,i_3)\Big)^2\Big)\\
&\le O(q^2). \qquad (3.15)
\end{aligned}
$$

Hence from (3.9)-(3.15), we have 2. □

Lemma 3.3. *Suppose that $E(f\circ X)^4<\infty$. Then*

1. $E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\Big)^2 \le \frac{1}{(1+z)^2}O(\frac{1}{q})$ *as* $q\to\infty$.
2. $E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\Big)^4 \le \frac{1}{(1+z)^2}O(\frac{1}{q^2})$ *as* $q\to\infty$.

Proof. 1. It follows from (3.4) and (3.7) that

$$
\begin{aligned}
&E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y_z(i_1,i_2,i_3)\Big)^2\\
&\quad= \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3) + \sum_{i_1,i_2,i_3}\ \sum_{\substack{j_1,j_2,j_3\\(j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z(i_1,i_2,i_3)Y_z(j_1,j_2,j_3)\\
&\quad\le \frac{1}{(1+z)^2}\sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) + \sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3)\\
&\quad\le \frac{1}{(1+z)^2}O(\frac{1}{q}).
\end{aligned}
$$

2. By the same argument as (3.9) we see that

$$
E\Big(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1} Y_z(i_1,i_2,i_3)\Big)^4 = C_1M_{1,z}+C_2M_{2,z}+C_3M_{3,z}+C_4M_{4,z}+C_5M_{5,z} \qquad (3.16)
$$

where
$$M_{1,z} = \sum_{i_1,i_2,i_3} EY_z^4(i_1,i_2,i_3),$$
$$M_{2,z} = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z(i_1,i_2,i_3)Y_z^3(j_1,j_2,j_3),$$
$$M_{3,z} = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_z^2(i_1,i_2,i_3)Y_z^2(j_1,j_2,j_3),$$
$$M_{4,z} = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3) \\ (k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} EY_z^2(i_1,i_2,i_3)Y_z(j_1,j_2,j_3)Y_z(k_1,k_2,k_3),$$
$$M_{5,z} = \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} \sum_{\substack{k_1,k_2,k_3 \\ (k_1,k_2,k_3)\neq(i_1,i_2,i_3) \\ (k_1,k_2,k_3)\neq(j_1,j_2,j_3)}} \sum_{\substack{l_1,l_2,l_3 \\ (l_1,l_2,l_3)\neq(i_1,i_2,i_3) \\ (l_1,l_2,l_3)\neq(j_1,j_2,j_3) \\ (l_1,l_2,l_3)\neq(k_1,k_2,k_3)}} EY_z(i_1,i_2,i_3)Y_z(j_1,j_2,j_3)Y_z(k_1,k_2,k_3)Y_z(l_1,l_2,l_3)$$
and C_1, C_2, C_3, C_4, C_5 are some constants.

From (3.7),
$$|M_{1,z}| \leq \frac{1}{(1+z)^2 O(q^3)}, \tag{3.17}$$
$$\big|M_{2,z}\big| \leq \big\{\sum_{i_1,i_2,i_3} E|Y_z(i_1,i_2,i_3)|\big\}\big\{\sum_{j_1,j_2,j_3} E|Y_z^3(j_1,j_2,j_3)|\big\}$$
$$= \frac{1}{(1+z)^4} O(\frac{1}{q^2}), \tag{3.18}$$
$$\big|M_{3,z}\big| \leq \big\{\sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3)\big\}^2$$
$$= \frac{1}{(1+z)^4} O(\frac{1}{q^2}), \tag{3.19}$$
$$\big|M_{4,z}\big| \leq \big\{\sum_{i_1,i_2,i_3} EY_z^2(i_1,i_2,i_3)\big\}\big\{\sum_{j_1,j_2,j_3} E|Y_z(j_1,j_2,j_3)|\big\}\big\{\sum_{l_1,l_2,l_3} E|Y_z(l_1,l_2,l_3)|\big\}$$
$$= \frac{1}{(1+z)^5} O(\frac{1}{q^3}), \tag{3.20}$$
and
$$|M_{5,z}| \leq \big(\sum_{i_1,i_2,i_3} E|Y_z(i_1,i_2,i_3)|\big)^4 \leq \big\{\frac{1}{(1+z)^3} O(\frac{1}{q^4})\big\}^4. \tag{3.21}$$

Hence from (3.16)-(3.21), we have 2.

□

Let g_z be the solution of the Stein's equation

$$g_z'(\omega) - \omega g_z(\omega) = h(\omega) - \Phi(z)$$

where $h(x) = \mathbb{I}(x \leq z)$. It is well known that

$$g_z(\omega) = \begin{cases} \sqrt{2\pi}e^{\omega^2/2}\Phi(\omega)[1-\Phi(z)] & \text{if} \quad \omega < z, \\ \sqrt{2\pi}e^{\omega^2/2}\Phi(z)[1-\Phi(\omega)] & \text{if} \quad \omega \geq z. \end{cases}$$

Moreover, $\omega g_z(\omega)$ is an increasing function of ω, and for all real ω, u, v, t and s,

$$0 < g_z(w) \leq \min(\frac{\sqrt{2\pi}}{4}, \frac{1}{|z|}), \quad \text{for} \quad z \neq 0 \tag{3.22}$$

$$|g_z'(w)| \leq 1, \tag{3.23}$$

$$|g_z'(w) - g_z'(v)| \leq 1 \tag{3.24}$$

and

$$|g_z'(w+s) - g_z'(w+t) - \int_t^s h(w+u)du\big| \leq \mathbb{I}\big(z - max(s,t) < w < z - min(s,t)\big) \tag{3.25}$$

([5], p.9-10).

Lemma 3.4. *For each $z > 0$, let $h : \mathbb{R} \to \mathbb{R}$ be defined by*

$$h(w) = (wg_z(w))' \tag{3.26}$$

If $E(f \circ X)^r < \infty$ for some positive even number r, then

$$E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)dudt \leq \frac{1}{(1+z)^{1-\frac{2}{r}}}O(\frac{1}{\sqrt{q}}) \quad as\ q \to \infty,$$

where $K(t) = \frac{q-1}{4}\Big(\widetilde{Y} - \widehat{Y}\Big)\Big(\mathbb{I}\big(0 \leq t \leq \widetilde{Y} - \widehat{Y}\big) - \mathbb{I}\big(\widetilde{Y} - \widehat{Y} \leq t < 0\big)\Big)$.

Proof. By the fact that

$$0 \leq h(w) \leq \begin{cases} 4(1+z^2)e^{\frac{z^2}{8}}(1-\Phi(z)) & \text{if} \quad w \leq \frac{z}{2} \\ 4(1+z^2)e^{\frac{z^2}{2}}(1-\Phi(z)) & \text{if} \quad \frac{z}{2} < w \leq z \text{ or } \omega > z \end{cases} \tag{3.27}$$

([5], p.48) and

$$1-\Phi(z)\le \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}z}\quad \text{for}\ \ z>0 \tag{3.28}$$

([7], p.23), we have

$$h(w)\le \begin{cases} C(1+z) & \text{if}\ \ \frac{z}{2}\le w\le z,\\ \frac{C}{(1+z)^2} & \text{if}\ \ w<\frac{z}{2}\ \ \text{or}\ \ w>z \end{cases} \tag{3.29}$$

where C is a constant. From (3.29) and the fact that

$$E\int_{-\infty}^{\infty}\int_t^0 K(t)dudt \le E\int_{-\infty}^{\infty}|t|K(t)dt\le qE|\widetilde{Y}-\widehat{Y}|^3\le O(\frac{1}{\sqrt{q}}), \tag{3.30}$$

we have

$$\begin{aligned} &E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)dudt\\ &\quad = E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)\mathbb{I}(\widehat{Y}+u<\frac{z}{2}\ \ \text{or}\ \ \widehat{Y}+u>z)dudt\\ &\quad\quad + E\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y}+u)K(t)\mathbb{I}(\frac{z}{2}\le \widehat{Y}+u\le z)dudt\\ &\quad \le \frac{C}{(1+z)^2}O(\frac{1}{\sqrt{q}})+C(1+z)E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+u\ge\frac{z}{2})dudt. \end{aligned} \tag{3.31}$$

By Lemma 3.1(1,2) and the fact that $K(t)=0$ for $|t|>|\widehat{Y}-\widetilde{Y}|$,

$$\begin{aligned} &E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+u\ge\frac{z}{2})dudt\\ &\le E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+|\widetilde{Y}-\widehat{Y}|\ge\frac{z}{2})dudt\\ &\le qE|\widetilde{Y}-\widehat{Y}|^3\mathbb{I}(\widehat{Y}+|\widetilde{Y}-\widehat{Y}|\ge\frac{z}{2})\\ &\le q\{E|\widetilde{Y}-\widehat{Y}|^{3r}\}^{\frac{1}{r}}\{P(\widehat{Y}+|\widetilde{Y}-\widehat{Y}|\ge\frac{z}{2})\}^{\frac{r-1}{r}}\\ &\le O(\frac{1}{\sqrt{q}})\{\frac{E\widehat{Y}^2+|\widetilde{Y}-\widehat{Y}|^2}{z^2}\}^{\frac{r-1}{r}}\\ &\le \frac{1}{(1+z)^{2-\frac{2}{r}}}O(\frac{1}{\sqrt{q}}). \end{aligned}$$

Hence the lemma follows from this fact and (3.31). $\square$

Lemma 3.5. *Let $\mathcal{B}$ be the $\sigma-$algebra generated by π_1, π_2, π_3 and $U_{i_1,i_2,i_3,j}$'s. Then*
1. $E\big|E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty} K(t)dt\}\big| \leq O(\frac{1}{\sqrt{q}})$ *as* $q \to \infty$.
2. $E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty} K(t)dt\}\big)^2 = (1+z)^{\frac{1}{6}} O(\frac{1}{\sqrt[3]{q}})$ *as* $q \to \infty$
where $E^{\mathcal{B}}X$ is the conditional expectation of X given $\sigma-$ algebra $\mathcal{B}$.

Proof. 1. First, we note that

$$E\big|E^{\mathcal{B}}\{qS_k^2 - 1\}\big| = O(\frac{1}{\sqrt{q}}), \tag{3.32}$$

$$qE\big|E^{\mathcal{B}} S_j S_k\big| = O(\frac{1}{q}) \quad \text{as } q \to \infty \tag{3.33}$$

for any $j,k = 1,2,3,4$ ([18], p.1218-1221) and for each $r,s \in \mathbb{N}$ such that $r+s$ is an even number, we have

$$E|S_{1,z}|^r \leq \frac{1}{(1+z)^s} O(\frac{1}{q^s}), \tag{3.34}$$

$$ES_{1,z}^2 = ES_{2,z}^2 = ES_{3,z}^2 = ES_{4,z}^2 \leq \frac{1}{(1+z)^4} O(\frac{1}{q^4}) \tag{3.35}$$

and
$$E|S_i S_{j,z}| \leq \frac{1}{(1+z)^2} O(\frac{1}{q^2\sqrt{q}}) \tag{3.36}$$

for $i,j = 1,2,3,4$ where

$$S_{1,z} = \sum_{i_2=0}^{q-1} Y_z(I, i_2, \rho_\pi(I, i_2)), \qquad S_{2,z} = \sum_{i_2=0}^{q-1} Y_z(K, i_2, \rho_\pi(K, i_2))$$

$$S_{3,z} = \sum_{i_2=0}^{q-1} Y_z(I, i_2, \rho_\pi(K, i_2)), \qquad S_{4,z} = \sum_{i_2=0}^{q-1} Y_z(K, i_2, \rho_\pi(I, i_2))$$

([11], p.21-22). Note from (3.35) and (3.36) that,

$$E|E^{\mathcal{B}}(S_{j,z}S_k)| \leq EE^{\mathcal{B}}|S_{j,z}S_k| = E|S_{j,z}S_k| \leq \frac{1}{(1+z)^2} O(\frac{1}{q^2\sqrt{q}}), \tag{3.37}$$

and
$$E|E^{\mathcal{B}}(S_{j,z}S_{k,z})| \leq E|S_{j,z}S_{k,z}| \leq \{ES_{j,z}^2\}^{\frac{1}{2}} \{ES_{k,z}^2\}^{\frac{1}{2}} \leq \frac{1}{(1+z)^4} O(\frac{1}{q^4}) \tag{3.38}$$

for any $j,k = 1,2,3,4$. From (3.32), (3.33), (3.37) and (3.38), we have

$$
\begin{aligned}
&E\big|E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big| \\
&= E\big|E^{\mathcal{B}}\{1-\frac{(q-1)}{4}(\widehat{S}_{1,z}+\widehat{S}_{2,z}-\widehat{S}_{3,z}-\widehat{S}_{4,z})^2\}\big| \\
&\le \sum_{k=1}^{4} E\big|E^{\mathcal{B}}((q-1)\widehat{S}_{k,z}^2-1)\big| + q\sum_{1\le j<k\le 4} E\big|E^{\mathcal{B}}(\widehat{S}_{j,z}\widehat{S}_{k,z})\big| \\
&= \sum_{k=1}^{4} E\big|E^{\mathcal{B}}\{(q-1)(S_k-S_{k,z})^2-1\}\big| + q\sum_{1\le j<k\le 4} E\big|E^{\mathcal{B}}(S_j-S_{j,z})(S_k-S_{k,z})\big| \\
&\le \sum_{k=1}^{4} E\big|E^{\mathcal{B}}((q-1)S_k^2-1)\big| + q\sum_{k=1}^{4} E\big|E^{\mathcal{B}}S_{k,z}S_k\big| + q\sum_{k=1}^{4} E\big|E^{\mathcal{B}}S_{k,z}^2\big| \\
&\quad + q\sum_{1\le j<k\le 4} E\big|E^{\mathcal{B}}(S_jS_k-S_{j,z}S_k-S_jS_{k,z}+S_{j,z}S_{k,z})\big| \\
&\le O(\frac{1}{\sqrt{q}}).
\end{aligned} \tag{3.39}
$$

2. It follows from (3.3) and (3.34) that

$$
\begin{aligned}
&E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^4 \\
&= E\Big\{E^{\mathcal{B}}\{1-\frac{(q-1)}{4}(\widehat{S}_{1,z}+\widehat{S}_{2,z}-\widehat{S}_{3,z}-\widehat{S}_{4,z})^2\}\Big\}^4 \\
&\le \frac{1}{q(q-1)}\sum_{i=0}^{q-1}\sum_{\substack{k=0\\k\ne i}}^{q-1} E\Big\{E^{\mathcal{B}}\big[1-\frac{(q-1)}{4}(\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j)) \\
&\quad + \sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(i,j)))^2\big]\Big\}^4 \\
&= \frac{1}{q(q-1)}\sum_{i=0}^{q-1}\sum_{\substack{k=0\\k\ne i}}^{q-1} E\Big\{1-\frac{(q-1)}{4}[\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j)) + \sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(k,j)) \\
&\quad - \sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j)) - \sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(i,j))]^2\Big\}^4 \\
&\le \frac{C}{q(q-1)}\sum_{i=0}^{q-1}\sum_{\substack{k=0\\k\ne i}}^{q-1}\Big\{1+q^4E(\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j)))^8 + q^4E(\sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(k,j)))^8 \\
&\quad + q^4E(\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j)))^8 + q^4E(\sum_{j=0}^{q-1}\widehat{Y}_z(k,j,\rho_\pi(i,j)))^8\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= C + Cq^3\big(\sum_{i=0}^{q} E(\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j)))^8\big) + Cq^2\big(\sum_{i=0}^{q}\sum_{\substack{k=0\\k\neq i}}^{q-1} E(\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(k,j)))^8\big)\\
&= C + Cq^4 E\widehat{S}_{1,z}^{\,8}\\
&= O(1).
\end{aligned}
$$

$$
\begin{aligned}
&\text{Hence } E\big(E^{\mathcal{B}}\{1-\textstyle\int_{-\infty}^{\infty}K(t)dt\}\big)^2\\
&= E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^2\mathbb{I}\big(|E^{\mathcal{B}}(1-\int_{-\infty}^{\infty}K(t)dt)| < q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\big)\\
&\quad + E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^2\mathbb{I}\big(|E^{\mathcal{B}}(1-\int_{-\infty}^{\infty}K(t)dt)| \geq q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\big)\\
&\leq E\big|E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big|q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\\
&\quad + \big\{E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^4\big\}^{\frac{1}{2}}\big\{P\big(|E^{\mathcal{B}}(1-\int_{-\infty}^{\infty}K(t)dt)| \geq q^{\frac{1}{6}}(1+z)^{\frac{1}{6}}\big)\big\}^{\frac{1}{2}}\\
&\leq (1+z)^{\frac{1}{6}}O(\frac{1}{\sqrt[3]{q}}) + \frac{E\big(E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big)^4}{\big(q^{\frac{2}{3}}(1+z)^{\frac{2}{3}}\big)^{\frac{1}{2}}}\\
&\leq (1+z)^{\frac{1}{6}}O(\frac{1}{\sqrt[3]{q}}).
\end{aligned}
$$

□

4 Proof of the main result(Theorem 2.2)

To bound $\big|P(W \leq z) - \Phi(z)\big|$, it suffices to consider $z > 0$ because we use the fact that $\Phi(z) = 1 - \Phi(-z)$ and apply the result to $-W$ when $z \leq 0$. So, from now on, we assume $z > 0$.

Let $\gamma = \frac{q}{4}E|\widehat{Y} - \widetilde{Y}|^3$. We note from Lemma 3.1(3) that, $\gamma = O(\frac{1}{\sqrt{q}})$.

By [12] we know that $\sup\big\{\big|P(W \leq z) - \Phi(z)\big| \ : -\infty < z < \infty\big\} = O(\frac{1}{\sqrt{q}})$. Hence $\big|P(W \leq z) - \Phi(z)\big| \leq \frac{1}{(1+z)^{1-\frac{2}{r}}}O(\frac{1}{\sqrt{q}})$ for $0 < z \leq 1$. Assume that $z > 1$. Note that

$$
|P(W \leq z) - \Phi(z)| \leq P(W \neq \widehat{Y}) + |P(\widehat{Y} \leq z) - \Phi(z)|, \tag{4.1}
$$

and
$$
P(W \neq \widehat{Y}) = P\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z) \geq 1\Big)
$$

$$
\begin{aligned}
&\le E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\mathbb{I}(|Y(i,j,\rho_\pi(i,j))|>1+z)\Big)\\
&=\frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}P(|Y(i,j,k)|>1+z)\\
&\le\frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\frac{E|Y(i,j,k)|^4}{(1+z)^4}\\
&=\frac{1}{(1+z)^4}O(\frac{1}{q^2}).
\end{aligned}
$$

Thus we need to bound only the second term on the right hand side of (4.1).

Case 1 $(1+z)\gamma\ge 1$.
By Lemma 3.1(1) and the fact that $\gamma\ge\dfrac{1}{1+z}$, we have

$$P(\widehat{Y}>z)=P(\widehat{Y}+1>1+z)\le\frac{E|\widehat{Y}+1|^2}{(1+z)^2}\le\frac{C}{(1+z)^2}\le\frac{C\gamma}{1+z}.$$

Hence
$$
\begin{aligned}
|P(\widehat{Y}\le z)-\Phi(z)|&=|1-P(\widehat{Y}>z)-\Phi(z)|\\
&=P(\widehat{Y}>z)+(1-\Phi(z))\\
&\le\frac{1}{1+z}O(\frac{1}{\sqrt{q}})
\end{aligned}
$$

where we have used the fact that $1-\Phi(z)\le\dfrac{C}{(1+z)^2}\le\dfrac{C\gamma}{1+z}$ in the last inequality.
Case 2 $(1+z)\gamma<1$.
In view of Theorem 1.2 of [12], we can show that

$$\big|P(\widehat{Y}\le z)-\Phi(z)\big|=\big|Eg_z'(\widehat{Y})-E\widehat{Y}g_z(\widehat{Y})\big|\le T_1+T_2+T_3+T_4 \tag{4.2}$$

where
$$
\begin{aligned}
T_1&=\big|E\int_{-\infty}^{\infty}\{g_z'(\widehat{Y})-g_z'(\widehat{Y}+t)\}K(t)dt\big|,\\
T_2&=\big|Eg_z'(\widehat{Y})E\int_{-\infty}^{\infty}K(t)dt-Eg_z'(\widehat{Y})\int_{-\infty}^{\infty}K(t)dt\big|,\\
T_3&=\Big|Eg_z'(\widehat{Y})-Eg_z'(\widehat{Y})E\int_{-\infty}^{\infty}K(t)dt\Big|,\\
T_4&=|\widetilde{\Delta}g_z(\widehat{Y})\big|,
\end{aligned}
$$

and
$$\widetilde{\Delta}f(\widehat{Y})=\frac{1}{q}Ef(\widehat{Y})\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\widehat{Y}_z(i_1,i_2,i_3)\text{ for any function }f.$$

From Lemma 3.2(1) and Lemma 3.3(1) we have

$$E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3))^2+E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y_z(i_1,i_2,i_3))^2\le O(q).$$

And, by (3.7) and (3.8),

$$\begin{aligned}&E(\sum_{i_1,i_2,i_3}\sum_{j_1,j_2,j_3}Y(i_1,i_2,i_3)Y_z(j_1,j_2,j_3))\\&=\sum_{i_1,i_2,i_3}EY(i_1,i_2,i_3)Y_z(i_1,i_2,i_3)-\sum_{i_1,i_2,i_3}\tilde{\mu}(i_1,i_2,i_3)EY_z(i_1,i_2,i_3)\\&=O(q).\end{aligned}$$

Hence

$$\begin{aligned}T_4&\le\frac{1}{qz}E|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\widehat{Y}_z(i_1,i_2,i_3)|\\&\le\frac{1}{qz}\{E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\widehat{Y}_z(i_1,i_2,i_3))^2\}^{\frac{1}{2}}\\&=\frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}).\end{aligned}\tag{4.3}$$

From Lemma 3.1(1), $E|g'_z(\widehat{Y})|\le\frac{C}{(1+z)^2}$ ([17], p.248) and

$$E\int_{-\infty}^{\infty}K(t)dt=E(\widehat{Y}^2)-\widetilde{\Delta}\widehat{Y}\quad\text{where}\quad|\widetilde{\Delta}\widehat{Y}|\le O(\frac{1}{q})$$

([11], p.16), we have

$$T_3=\big|Eg'_z(\widehat{Y})\big|\Big|1-E\int_{-\infty}^{\infty}K(t)dt\Big|\le\frac{C}{(1+z)^2}\big|1-E\widehat{Y}^2+\Delta\widehat{Y}\big|\le\frac{1}{(1+z)^2}O(\frac{1}{q}).\tag{4.4}$$

and

$$\begin{aligned}T_2&=\big|Eg'_z(\widehat{Y})E^{\mathcal{B}}\Big\{E\int_{-\infty}^{\infty}K(t)dt-\int_{-\infty}^{\infty}K(t)dt\Big\}\big|\\&\le\frac{C}{(1+z)^2}\big|E\int_{-\infty}^{\infty}K(t)dt-1\big|+\big|Eg'_z(\widehat{Y})E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big|\\&\le\frac{C}{(1+z)^2}\big|1-E\widehat{Y}^2+\widetilde{\Delta}\widehat{Y}\big|+\big|Eg'_z(\widehat{Y})\mathbb{I}(\widehat{Y}\le\frac{z}{2})E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big|\\&\quad+\big|Eg'_z(\widehat{Y})\mathbb{I}(\widehat{Y}<\frac{z}{2})E^{\mathcal{B}}\{1-\int_{-\infty}^{\infty}K(t)dt\}\big|\\&=\frac{1}{(1+z)^2}O(\frac{1}{q})+T_{21}+T_{22}\end{aligned}\tag{4.5}$$

where $$T_{21} = \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} \le \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big|$$

and $$T_{22} = \big|Eg_z'(\widehat{Y})\mathbb{I}(\widehat{Y} > \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big|.$$

Note that $g_z'(\omega) = [\sqrt{2\pi}\omega e^{\omega^2/2}\Phi(\omega) + 1][1 - \Phi(z)]$ for $\omega \le z$. Hence from this fact, Lemma 3.5(1),(3.27) and the fact that $e^{-\frac{3z^2}{8}} \le \dfrac{C}{1+z}$, we have

$$\begin{aligned}
T_{21} &\le \big|E\{\sqrt{2\pi}(\frac{z}{2})e^{\frac{z^2}{8}}\Phi(\frac{z}{2}) + 1\}\{1 - \Phi(z)\}\mathbb{I}(\widehat{Y} \le \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\le \{\sqrt{2\pi}(\frac{z}{2})e^{\frac{z^2}{8}} + 1\}\{\frac{e^{-\frac{z^2}{2}}}{z}\}E\big|E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\le \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \qquad (4.6)
\end{aligned}$$

From Lemma 3.1(1) and Lemma 3.5(2),

$$\begin{aligned}
T_{22} &\le \big|E\mathbb{I}(\widehat{Y} > \frac{z}{2})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big| \\
&\le \{P(\widehat{Y} > \frac{z}{2})\}^{\frac{1}{2}}\{E\big|E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty} K(t)dt\}\big|^2\}^{\frac{1}{2}} \\
&\le \frac{1}{(1+z)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}})E\widehat{Y}^2 \\
&= \frac{1}{(1+z)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}}). \qquad (4.7)
\end{aligned}$$

Hence, by (4.5)-(4.7), we have

$$T_2 \le \frac{1}{(1+z)^{\frac{11}{12}}}O(\frac{1}{q^{\frac{1}{6}}}). \qquad (4.8)$$

To finish the proof of Theorem 2.2, it remains to bound T_1.
By (3.25) we have

$$T_1 \le T_{11} + T_{12} + T_{13} \qquad (4.9)$$

where
$$\begin{aligned}
T_{11} &= E\mathbb{I}(|\widetilde{Y} - \widehat{Y}| < \frac{1+z}{2})\int_{-\infty}^{\infty} \mathbb{I}(z - \max(0,t) < \widehat{Y} < z - \min(0,t))K(t)dt, \\
T_{12} &= E\mathbb{I}(|\widetilde{Y} - \widehat{Y}| < \frac{1+z}{2})\int_{-\infty}^{\infty}\int_t^0 h(\widehat{Y} + u)K(t)dudt, \\
T_{13} &= E\mathbb{I}(|\widetilde{Y} - \widehat{Y}| \ge \frac{1+z}{2})\int_{-\infty}^{\infty} \{g_z'(\widehat{Y}) - g_z'(\widehat{Y} + t)\}K(t)dt
\end{aligned}$$

and h be defined as in Lemma 3.4.
First, we consider T_{11}. For $\delta > 0$, let $f_\delta : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_\delta(t) = \begin{cases} 0 & \text{if} \quad t < z - 2\delta, \\ (1+t+\delta)(t-z+2\delta) & \text{if} \quad z-2\delta \le t \le z+2\delta, \\ 4\delta(1+t+\delta) & \text{if} \quad t > z+2\delta. \end{cases} \tag{4.10}$$

Note that f_δ is a nondecreasing function,

$$f'_\delta(t) \ge \begin{cases} 1+z-\delta & \text{if} \quad z-2\delta \le t \le z+2\delta, \\ 0 & \text{otherwise} , \end{cases} \tag{4.11}$$

$$\begin{aligned}
&E\big|\widehat{Y} f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\big| \\
&\le 4E|\widehat{Y}|(1+|\widehat{Y}|+|\widetilde{Y}-\widehat{Y}|)|\widetilde{Y}-\widehat{Y}| \\
&\le 4E|\widehat{Y}||\widetilde{Y}-\widehat{Y}| + 4E\widehat{Y}^2|\widetilde{Y}-\widehat{Y}| + 4E|\widehat{Y}||\widetilde{Y}-\widehat{Y}|^2 \\
&\le 4\{E\widehat{Y}^2\}^{\frac{1}{2}}\{E|\widetilde{Y}-\widehat{Y}|^2\}^{\frac{1}{2}} + 4\{E(\widehat{Y}^2)^{\frac{r}{r-1}}\}^{\frac{r-1}{r}}\{E|\widetilde{Y}-\widehat{Y}|^r\}^{\frac{1}{r}} \\
&\quad + 4\{E\widehat{Y}^2\}^{\frac{1}{2}}\{E|\widetilde{Y}-\widehat{Y}|^4\}^{\frac{1}{2}} \\
&\le O(\frac{1}{\sqrt{q}}) + O(\frac{1}{\sqrt{q}})\big\{E\widehat{Y}^2|\widehat{Y}|^{\frac{2}{r-1}}\big\}^{\frac{r-1}{r}} \\
&\le O(\frac{1}{\sqrt{q}}) + O(\frac{1}{\sqrt{q}})\big\{(q^2(1+z))^{\frac{2}{r-1}}E\widehat{Y}^2\big\}^{\frac{r-1}{r}} \\
&\le O(\frac{1}{\sqrt{q}}) + (1+z)^{\frac{2}{r}}O(\frac{1}{q^{\frac{r-8}{2r}}}),
\end{aligned} \tag{4.12}$$

and
$$\begin{aligned}
\widetilde{\Delta} f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y}) &\le \frac{1}{q}|Ef_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\widehat{Y}_z(i_1,i_2,i_3)| \\
&\le \frac{4}{q}\Big|E(1+\widehat{Y}+|\widetilde{Y}-\widehat{Y}|)|\widetilde{Y}-\widehat{Y}|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}\widehat{Y}_z(i_1,i_2,i_3)\Big| \\
&\le \frac{4}{q}\Big|E(1+\widehat{Y})|\widetilde{Y}-\widehat{Y}|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3)\Big| \\
&\quad + \frac{4}{q}\Big|E(1+\widehat{Y})|\widetilde{Y}-\widehat{Y}|\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y_z(i_1,i_2,i_3)\Big| \\
&\quad + \frac{4}{q}\Big|E|\widetilde{Y}-\widehat{Y}|^2\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3)\Big| \\
&\quad + \frac{4}{q}\Big|E|\widetilde{Y}-\widehat{Y}|^2\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y_z(i_1,i_2,i_3)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{4}{q}\big\{E(1+\widehat{Y})^2\big\}^{\frac{1}{2}}\big\{E|\widetilde{Y}-\widehat{Y}|^2(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3))^2\big\}^{\frac{1}{2}}\\
&\quad + \frac{4}{q}\big\{E(1+\widehat{Y})^2\big\}^{\frac{1}{2}}\big\{E|\widetilde{Y}-\widehat{Y}|^2(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y_z(i_1,i_2,i_3))^2\big\}^{\frac{1}{2}}\\
&\quad + \frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{2}}\big\{E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3))^2\big\}^{\frac{1}{2}}\\
&\quad + \frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{2}}\big\{E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y_z(i_1,i_2,i_3))^2\big\}^{\frac{1}{2}}\\
&\le \frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{4}}\big\{E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y(i_1,i_2,i_3))^4\big\}^{\frac{1}{4}}\\
&\quad + \frac{4}{q}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{4}}\big\{E(\sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1}\sum_{i_3=0}^{q-1}Y_z(i_1,i_2,i_3))^4\big\}^{\frac{1}{4}} + O(\frac{1}{q})\\
&\le O(\frac{1}{q}).
\end{aligned}
\tag{4.13}
$$

Using the same argument as in Lemma 2.3 of [11],we can show that

$$
E\int_{-\infty}^{\infty} f_\delta'(\widehat{Y}+t)K(t)dt = E\widehat{Y}f_\delta(\widehat{Y}) - \widetilde{\Delta}f_\delta(\widehat{Y}). \tag{4.14}
$$

Thus, from (4.11)-(4.14), we have

$$
\begin{aligned}
T_{11} &= E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})\mathbb{I}(z-\max(0,t)<\widehat{Y}<z-\min(0,t))K(t)dt\\
&\le E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})\mathbb{I}(z-2|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+2|\widetilde{Y}-\widehat{Y}|)K(t)dt\\
&\le \frac{2}{1+z}E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|}\mathbb{I}(|\widetilde{Y}-\widehat{Y}|<\frac{1+z}{2})(1+z-|\widetilde{Y}-\widehat{Y}|)\\
&\quad \mathbb{I}(z-2|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+2|\widetilde{Y}-\widehat{Y}|)K(t)dt\\
&\le \frac{2}{1+z}E\int_{|t|\le|\widetilde{Y}-\widehat{Y}|} f'_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y}+t)\mathbb{I}(z-2|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+2|\widetilde{Y}-\widehat{Y}|)K(t)dt\\
&\le E\Big|\widehat{Y}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\Big| + \Big|\widetilde{\Delta}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})\Big|\\
&\le \frac{1}{(1+z)^{1-\frac{2}{r}}}O(\frac{1}{q^{\frac{r-8}{2r}}}).
\end{aligned}
\tag{4.15}
$$

By Lemma 3.4,

$$
T_{12} \le \frac{1}{(1+z)^{1-\frac{2}{r}}}O(\frac{1}{\sqrt{q}}). \tag{4.16}
$$

By (3.24) and Lemma 3.1(2)

$$\begin{aligned}
T_{13} &\leq E\mathbb{I}(|\widetilde{Y}-\widehat{Y}| \geq \frac{1+z}{2}) \int_{-\infty}^{\infty} K(t)dt \\
&\leq qE\mathbb{I}(|\widetilde{Y}-\widehat{Y}| \geq \frac{1+z}{2})|\widetilde{Y}-\widehat{Y}|^2 \\
&\leq q\big\{P(|\widetilde{Y}-\widehat{Y}| \geq \frac{1+z}{2})\big\}^{\frac{1}{2}}\big\{E|\widetilde{Y}-\widehat{Y}|^4\big\}^{\frac{1}{2}} \\
&\leq C\big\{\frac{E|\widetilde{Y}-\widehat{Y}|^2}{(1+z)^2}\big\}^{\frac{1}{2}} \\
&= \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \qquad (4.17)
\end{aligned}$$

From (4.9), (4.15)-(4.17), we have the main theorem.

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Received: January 29, 2007