When $M$-convexity of a Function Implies its $N$-convexity for Some Means $M$ and $N$

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Devoted to Professor Boris Paneah on the occasion of his 70th birthday

Abstract. Let $M$ be an arbitrary strict mean in an interval $J$ and $M_p^{[\varphi]}$ be a weighted quasi-arithmetic mean of the weight $p \in (0; 1)$ and a generator $\varphi : J \to \mathbb{R}$. We prove that, for all intervals $I \subset J$ and for all continuous functions $f : I \to J$, the condition $f$ is $M_p^{[\varphi]}$-affine implies $f$ is $M$-convex is satisfied iff $M$ is a quasi-arithmetic mean. Some variants of this result are proved and an open problem is posed.

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1. Introduction

Let $J \subset \mathbb{R}$ be an interval. A function $M : J^2 \to \mathbb{R}$ is called a mean in $J$ if
\[ \min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in J. \]

If, for all $x, y \in J, x \neq y$, these inequalities are strict, $M$ is called strict; and symmetric, if $M(x, y) = M(y, x)$. Every mean $M$ in $J$ is reflexive, i.e.
\[ M(x, x) = x, \quad x \in J; \]
consequently, $M(I^2) = I$ for every subinterval $I \subseteq J$ and $M$ is a mean in $I$. Obviously, every reflexive function $M : J^2 \to \mathbb{R}$ which is increasing in each variable is a mean.
Let $I \subset J$ be an interval. A function $f : I \to J$ is said to be $M$-convex, $M$-concave, $M$-affine if, respectively, (cf. J. Aczél [1], G. Aumann [3], and J. Matkowski & J. Rätz [8]),

\begin{align*}
 f(M(x, y)) &\leq M(f(x), f(y)), \quad x, y \in I; \\
 f(M(x, y)) &\geq M(f(x), f(y)), \quad x, y \in I; \\
 f(M(x, y)) &= M(f(x), f(y)), \quad x, y \in I.
\end{align*}

If $M = A$ where $A(x, y) := \frac{x + y}{2}$, then $A$-convexity coincides with the Jensen-convexity.

Let us mention that the convexity of a function with respect to a non arithmetic mean appears in some characterization of $L^p$-norm [7] and the Gamma function [5].

Let $N$ and $M$ be some means in an interval $J$. In the present paper we show that:

if $N$ is a weighted quasi-arithmetic mean and for any subinterval $I \subset J$ and any continuous function $f : I \to J$,

\begin{equation*}
 f \text{ is } N\text{-affine implies that } f \text{ is } M\text{-convex},
\end{equation*}

then $M$ must be quasi-arithmetic too. If moreover both means are symmetric, then $M = N$.

We also show that assuming either some special condition on the generator of quasi-arithmetic mean $N$ or the continuity of $M$, one can substantially weaken the assumptions in this result.

A simple example shows that the quasi-arithmeticity of the means is meaningful. An open problem is posed.

2. Auxiliary results

Recall that for every continuous and strictly monotonic function $\varphi : J \to \mathbb{R}$ and $p \in [0, 1]$, the function $M^\varphi_p : J^2 \to J$,

\begin{equation*}
 M^\varphi_p(x, y) := \varphi^{-1}(p\varphi(x) + (1 - p)\varphi(y)), \quad x, y \in J,
\end{equation*}

is a mean. $M^\varphi_p$ is called a weighted quasi-arithmetic mean; the function $\varphi$ is referred to as its generator, and the numbers $p$ and $1 - p$ its weights. For $p = \frac{1}{2}$ this mean is denoted by $M^{[\varphi]}$ and is called quasi-arithmetic.

**Remark 1.** The mean $M^\varphi_p$ is strict iff $p \in (0, 1)$. Moreover we have $M^\varphi_0(x, y) = y$ and $M^\varphi_1(x, y) = x$ for all $x, y \in I$, i.e. $M^\varphi_0$ and $M^\varphi_1$ are the projective means.

**Remark 2.** Suppose that $\varphi, \phi : J \to \mathbb{R}$ are continuous and strictly monotonic, and $p, q \in (0, 1)$. Then $M^\varphi_p = M^{[\phi]}_q$ if, and only if, $q = p$ and there are $a, b \in \mathbb{R}$, $a \neq 0$, such that

\begin{equation*}
 \varphi(x) = a\phi(x) + b, \quad x \in J,
\end{equation*}
Let \( \varphi \) of the mean \( M_p^{[\varphi]} \) is increasing and, if it is convenient, that \( 0 \in \varphi(J) \).

**Remark 3.** Let \( M \) be a mean in an interval \( J \). Considering \( M \)-convex (\( M \)-affine) functions we can assume, without any loss of generality, that \( 0 \in \text{int} \ J \) (or \( 0 \in J \)).

To show this take an arbitrary \( x_0 \in J \), put \( J - x_0 := \{ x - x_0 : x \in J \} \), and define \( N : (J - x_0)^2 \to \mathbb{R} \) by

\[
N(u,v) := M(u + x_0, v + x_0) - x_0, \quad u,v \in J - x_0.
\]

It is easy to verify that \( N \) is a mean in \( J - x_0 \), and if \( M = M_p^{[\varphi]} \) then \( N = M_p^{[\varphi]} \) where \( \phi : (J - x_0) \to \mathbb{R} \) is given by \( \phi(u) := \varphi(u + x_0) \). Moreover, if \( f : I \to J \) is \( M \)-convex, then \( g : (I - x_0) \to J - x_0 \) defined by

\[
g(u) := f(u + x_0) - x_0, \quad u \in J - x_0,
\]

is \( N \)-convex. Indeed, for all \( u,v \in J - x_0 \) we have

\[
g(N(u,v)) = f(N(u,v) + x_0) - x_0 = f(M(u + x_0, v + x_0)) - x_0
\]

\[
\leq M(f(u + x_0), f(v + x_0)) - x_0
\]

\[
= M([f(u + x_0) - x_0] + x_0, [f(v + x_0) - x_0] + x_0) - x_0
\]

\[
= M(g(u) + x_0, g(v) + x_0) - x_0
\]

\[
= N(g(u), g(v)).
\]

If \( f : I \to J \) is \( M \)-affine, then a similar reasoning shows that \( g \) is \( N \)-affine.

Let \( J \subset \mathbb{R} \) be an interval and \( p \in (0,1) \). In the sequel we say that a function \( g : J \to \mathbb{R} \) is \( p \)-convex (resp., \( p \)-concave, \( p \)-affine) if \( g \) is convex (resp. concave, affine) with respect to the weighted arithmetic mean \( A_p(x,y) := px + (1-p)y \). In particular, \( \frac{1}{2} \)-convexity (\( \frac{1}{2} \)-concavity, \( \frac{1}{2} \)-affinity) coincides with Jensen convexity (Jensen cocavity, Jensen affinity, respectively).

**Remark 4.** It follows from the Daróczy-Páles identity

\[
\frac{x + y}{2} = p \left( (1-p)x + \frac{x + y}{2} \right) + (1-p) \left( (1-p)\frac{x + y}{2} + py \right)
\]

that every \( p \)-convex (\( p \)-concave, \( p \)-affine) function is Jensen convex (resp., Jensen concave, Jensen affine).

**Lemma 1.** Let \( J \subset \mathbb{R} \) be an interval, \( \varphi : J \to \mathbb{R} \) continuous and strictly increasing and \( p \in (0,1) \). Then

1. \( f : I \to J \) is \( M_p^{[\varphi]} \)-convex (\( M_p^{[\varphi]} \)-concave, \( M_p^{[\varphi]} \)-affine) iff the function \( \varphi \circ f \circ \varphi^{-1} \) is \( p \)-convex (\( p \)-concave, \( p \)-affine) in \( \varphi(I) \);

2. if \( f : I \to J \) is \( M_p^{[\varphi]} \)-affine and continuous at least at one point, then there are \( a,b \in \mathbb{R} \) such that \( \varphi \circ f \circ \varphi^{-1}(u) = au + b \) for all \( u \in \varphi(I) \).
Proof. Suppose that \( f \) is \( M_p^\varphi \)-convex in convex in \( I \). Then, for all \( x, y \in I \).

\[
f(\varphi^{-1}(p\varphi(x) + (1-p)\varphi(y))) \leq \varphi^{-1}(\varphi^{-1}(p\varphi(f(x)) + (1-p)\varphi(f(y)))).
\]

For arbitrary \( u, v \in \varphi(I) \), taking here \( x := \varphi^{-1}(u), y := \varphi^{-1}(v) \) and making use of the increasing monotonicity of \( \varphi \), we hence get

\[
\varphi \circ f \circ \varphi^{-1}(pu + (1-p)v) \leq p\varphi \circ f \circ \varphi^{-1}(u) + (1-p)\varphi \circ f \circ \varphi^{-1}(u),
\]

which proves that \( \varphi \circ f \circ \varphi^{-1} \) is \( p \)-convex. In the same way we can show the remaining assertions of part 1.

Now the second part of the lemma is a consequence of the Daróczy-Páles identity lemma and the classical theory of Jensen convex (or affine) functions (cf. M. Kuczma p. [6]).

\[\square\]

3. Some results

Clearly, every \( M \)-affine function is \( M \)-convex. We begin with the following

**Theorem 1.** Let \( \varphi : J \to \mathbb{R} \) be continuous and strictly monotonic in an open interval \( J \subset \mathbb{R} \) such that \( \varphi(J) = \mathbb{R} \), and \( p \in (0,1) \) a fixed number. Suppose that \( M : J^2 \to J \) is a mean. If, for all continuous functions \( f : J \to J \),

\[
f \text{ is } M_p^\varphi \text{-affine } \implies f \text{ is } M \text{-convex},
\]

then \( M = M_p^{\varphi^q} \) for some \( q \in [0,1] \). If, moreover, \( M \) is a strict mean, then \( q \in (0,1) \). Furthermore, if \( p = \frac{1}{2} \) and \( M \) is strict and symmetric, then \( M = M^\varphi \).

**Proof.** By Lemma 1, taking into account that \( \varphi(J) = \mathbb{R} \), we infer that, for all \( a, b \in \mathbb{R} \), the function \( f : J \to J \) given by

\[
f(x) = \varphi^{-1}(a\varphi(x) + b), \quad x \in J,
\]

is \( M_p^{\varphi^q} \)-affine. From the assumed implication we infer that this function \( f \) is \( M \)-convex, that is

\[
\varphi^{-1}(a\varphi(M(x,y)) + b) \leq M(\varphi^{-1}(a\varphi(x) + b), \varphi^{-1}(a\varphi(y) + b)), \quad x, y \in J,
\]

for all \( a, b \in \mathbb{R} \). By Remark 2, without any loss of generality, we can assume that \( \varphi \) is strictly increasing. Therefore, replacing \( x \) by \( \varphi^{-1}(u) \) and \( y \) by \( \varphi^{-1}(v) \), we hence get

\[
a\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) + b \leq \varphi(M(\varphi^{-1}(au + b), \varphi^{-1}(av + b))), \quad u, v \in \mathbb{R},
\]

for all \( a, b \in \mathbb{R} \), that is

\[
aM^*(u,v) + b \leq M^*(au + b, av + b), \quad a, b, u, v \in \mathbb{R},
\]

where \( M^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is defined by

\[
M^*(u,v) := \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v)))).
\]

For \( b = 0 \) we hence get

\[
aM^*(u,v) \leq M^*(au, av), \quad a, u, v \in \mathbb{R},
\]

\[\square\]
which, of course, implies that

\[(2) \quad aM^*(u, v) = M^*(au, av), \quad a, u, v \in \mathbb{R},\]

that is \(M^*\) is homogeneous.

Taking \(a = 1\) in (1) we get

\[M^*(u, v) + b \leq M^*(u + b, v + b), \quad b, u, v \in \mathbb{R}.\]

Replacing here \(u\) by \(u - b\) and \(v\) by \(v - b\) we obtain

\[M^*(u - b, v - b) \leq M^*(u, v) - b, \quad b, u, v \in \mathbb{R},\]

whence

\[M^*(u + b, v + b) \leq M^*(u, v) + b, \quad b, u, v \in \mathbb{R}.\]

Consequently,

\[(3) \quad M^*(u + b, v + b) = M^*(u, v) + b, \quad b, u, v \in \mathbb{R}.\]

Applying in turn (3) and (2) we obtain, for all \(u, v \in \mathbb{R},\)

\[M^*(u, v) = M^*((u - v) + v, 0 + v) = M^*(u - v, 0) + v = (u - v)M^*(1, 0) + v = qu + (1 - q)v,\]

where

\[q := M^*(1, 0).\]

Hence, by the definition of \(M^*\),

\[M(x, y) = \varphi^{-1}(q\varphi(x) + (1 - q)\varphi(y)), \quad x, y \in J.\]

Since \(M^*\) is a mean, we have \(q \in [0, 1]\). This completes the proof. \(\square\)

Denote by \(\mathbb{Q}\) the set of rational numbers.

The continuity of the mean \(M\) in Theorem 1 (as well as in Corollary 1) allows to weaken the basic assumption significantly.

**Theorem 2.** Let \(\varphi : J \to \mathbb{R}\) be continuous and strictly monotonic in an open interval \(J \subset \mathbb{R}, \varphi(J) = \mathbb{R},\) and \(p \in (0, 1)\) a fixed number. Suppose that \(M : J^2 \to J\) is a continuous mean.

If there are \(a, b, c, d \in \mathbb{R}\backslash\{0\}, 0 < a < 1 < b, \frac{\log b}{\log a} \notin \mathbb{Q}\) such that the functions

\[\varphi^{-1} \circ (a\varphi), \quad \varphi^{-1} \circ (b\varphi), \quad \varphi^{-1} \circ (c\varphi + d)\]

are \(M\)-convex

and the function

\[\varphi^{-1} \circ (-\varphi)\]

is \(M\)-affine,

then \(M = M^*[q]\) for some \(q \in [0; 1]\). If moreover \(M\) is strict, then \(q \in (0; 1)\).
Proof. We can assume that $\varphi$ is strictly increasing. From the $M$-convexity of the function $\varphi^{-1} \circ (a\varphi)$ we have
\[
\varphi^{-1}(a\varphi(M(x, y))) \leq M(\varphi^{-1}(a\varphi(x)), \varphi^{-1}(a\varphi(y))), \quad x, y \in J.
\]
Replacing $x$ by $\varphi^{-1}(u)$ and $y$ by $\varphi^{-1}(v)$, we hence get
\[
a\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) \leq \varphi(M(\varphi^{-1}(au), \varphi^{-1}(av))), \quad u, v \in \mathbb{R},
\]
whence, by induction,
\[
a^n\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) \leq M(\varphi^{-1}(a^n u), \varphi^{-1}(a^n v)), \quad u, v \in \mathbb{R}, \ n \in \mathbb{N}.
\]
(From the $M$-convexity of the function $\varphi^{-1} \circ (b\varphi)$, in the same way, we get
\[
b^n\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) \leq M(\varphi^{-1}(b^n u), \varphi^{-1}(b^n v)), \quad u, v \in \mathbb{R}, \ n \in \mathbb{N}.
\]
Making use both these inequalities we obtain
\[
a^n b^m \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) \leq \varphi(M(\varphi^{-1}(a^n b^m u), \varphi^{-1}(a^n b^m v)))
\]
for all $u, v \in \mathbb{R}, \ n \in \mathbb{N}$. Since, by assumption, $0 < a < 1 < b, \log \frac{a}{\log a} \notin \mathbb{Q}$, the set
\[
\{a^n b^m : n, m \in \mathbb{N}\}
\]
is dense in $(0, \infty)$. The last inequality and the continuity of $M$ imply that
\[
t \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) \leq \varphi(M(\varphi^{-1}(tu), \varphi^{-1}(tv))), \quad u, v \in \mathbb{R}, \ t > 0.
\]
Replacing $u$ and $v$ by $u/t$ and $v/t$, respectively, we obtain the reversed inequality, whence
\[
t \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) = \varphi(M(\varphi^{-1}(tu), \varphi^{-1}(tv))), \quad u, v \in \mathbb{R}, \ t > 0.
\]
This proves that the mean $M^*: \mathbb{R}^2 \to \mathbb{R}$ defined by
\[
(4) \quad M^*(u, v) := \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))), \quad u, v \in \mathbb{R},
\]
is positively homogeneous. Since, by assumption, $\varphi^{-1} \circ (-\varphi)$ is $M$-affine, we have
\[
\varphi^{-1}(-\varphi(M(x, y))) = M(\varphi^{-1}(-\varphi(x)), \varphi^{-1}(-\varphi(y))), \quad x, y \in J,
\]
which can be written in the form
\[
-\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) = \varphi(M(\varphi^{-1}(-u), \varphi^{-1}(-v))), \quad u, v \in \mathbb{R},
\]
whence, by the definition of $M^*$,
\[
(5) \quad M^*(-u, -v) := -M^*(u, v), \quad u, v \in \mathbb{R}.
\]
This relation and the positively homogeneity of $M^*$ imply that $M^*$ is homogeneous, i.e.
\[
(6) \quad M^*(tu, tv) := tM^*(u, v), \quad t, u, v \in \mathbb{R}.
\]
By assumption, there are real nonzero $c, d$ such that the function $\varphi^{-1} \circ (c\varphi + d)$ is $M$-convex. Consequently,
\[
\varphi^{-1}(c\varphi(M(x, y)) + d) \leq M(\varphi^{-1}(c\varphi(x) + d), \varphi^{-1}(c\varphi(y) + d)), \quad x, y \in J,
\]
which can be written in the form
\[ c\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) + d \leq \varphi(M(\varphi^{-1}(cu + d), \varphi^{-1}(cv + d))), \quad u, v \in \mathbb{R}, \]
whence
\[ cM^*(u, v) + d \leq M^*(cu + d, cv + d), \quad u, v \in \mathbb{R}. \]
Hence, applying (6), the homogeneity of \( M^* \), we get
\[ M^*(cu, cv) + d \leq M^*(cu + d, cv + d), \quad u, v \in \mathbb{R}, \]
and, consequently,
\[ (7) \quad M^*(u, v) + d \leq M^*(u + d, v + d), \quad u, v \in \mathbb{R}. \]
Replacing here \( u \) by \( u - d \) and \( v \) by \( v - d \) we get
\[ M^*(u - d, v - d) \leq M^*(u, v) - d, \quad u, v \in \mathbb{R}, \]
whence, by replacing \( u \) by \( -u \) and \( v \) by \( -v \),
\[ M^*(-u - d, -v - d) \leq M^*(-u, -v) - d, \quad u, v \in \mathbb{R}. \]
Making use of (5), we obtain
\[ -M^*(u + d, v + d) \leq -M^*(u, v) - d, \quad u, v \in \mathbb{R}, \]
that is
\[ (8) \quad M^*(u + d, v + d) \geq M^*(u, v) + d, \quad u, v \in \mathbb{R}, \]
The inequalities (7) and (8) imply that
\[ M^*(u + d, v + d) = M^*(u, v) + d, \quad u, v \in \mathbb{R}. \]
Hence, by the homogeneity of \( M^* \),
\[ M^*(tu + td, tv + td) = M^*(tu, tv) + td, \quad t, u, v \in \mathbb{R}, \]
whence
\[ M^*(u + td, v + td) = M^*(u, v) + td, \quad t, u, v \in \mathbb{R}. \]
Since \( t \in \mathbb{R} \) is arbitrary, we conclude that
\[ (9) \quad M^*(u + w, v + w) = M^*(u, v) + w, \quad u, v, w \in \mathbb{R}. \]
Now, similarly as in the proof of Theorem 1, applying (6) and (9) we obtain
\[ M^*(u, v) = M^*((u - v) + v, 0 + v) = M^*(u - v, 0) + v \]
\[ = (u - v)M^*(1, 0) + v = qu + (1 - q)v \]
for all \( u, v \in \mathbb{R} \), and the result is a consequence of (4).

In the case \( \varphi(J) \nsubseteq \mathbb{R} \) the following result holds true:
Theorem 3. Let $\varphi : J \to \mathbb{R}$ be continuous and strictly monotonic in an open interval $J \subset \mathbb{R}$ and let $p \in (0, 1)$ be a fixed number. Suppose that $M : J^2 \to J$ is a mean. If for all compact intervals $I \subset J$ and for all continuous functions $f : I \to J$,

$$f \text{ is } M_p^{[q]}\text{-affine } \implies \text{ } f \text{ is } M\text{-convex},$$

then $M = M_q^{[p]}$ for some $q \in [0; 1]$. If moreover $M$ is a strict mean, then $q \in (0; 1)$. Furthermore, if $M$ is symmetric, then $M = M^{[p]}$.

Proof. By Remark 2 we may assume that $0 \in \text{int} \varphi(J)$. By Remark 3 we can also assume that $0 \in \text{int} J$. Take a compact subinterval $I \subset J$ such that $0 \in \text{int}(I)$ and a continuous function $f : I \to J$. By Lemma 1, if $f$ is $M_p^{[q]}$-affine, then there are $a, b \in \mathbb{R}$ such that

$$f(x) = \varphi^{-1}(a\varphi(x) + b), \quad x \in I.$$  

Conversely, for all $a, b \in \mathbb{R}$ such that

$$(a\varphi(I) + b) \subset \varphi(J),$$

the function $f$ given by this formula is $M_p^{[q]}$-affine. Since $I \subset J$ is compact, there are $\alpha = \alpha(I) > 1$ and $\beta = \beta(I) > 0$ such that this inclusion holds true for all $a, b \in \mathbb{R}$ such that $0 \leq a < \alpha$ and $|b| < \beta$. Suppose that condition 1 is satisfied. Then, for all $a, b \in \mathbb{R}$, such that $0 \leq a < \alpha$ and $|b| < \beta$, we have

$$\varphi^{-1}(a\varphi(M(x, y)) + b) \leq M(\varphi^{-1}(a\varphi(x) + b), \varphi^{-1}(a\varphi(y) + b)), \quad x, y \in I,$$

or equivalently, for all $a, b \in \mathbb{R}$, such that $0 \leq a < \alpha$ and $|b| < \beta$,

$$a\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) + b \leq \varphi(M(\varphi^{-1}(au + b), \varphi^{-1}(av + b))), \quad u, v \in \varphi(I).$$

Thus, for all $a, b \in \mathbb{R}$, such that $0 \leq a < \alpha$ and $|b| < \beta$,

$$aM_1(u, v) + b \leq M_1(au + b, av + b), \quad u, v \in \varphi(I),$$

where $M_1 : \varphi(J) \times \varphi(J) \to \varphi(J)$ is defined by

$$M_1(u, v) := \varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))), \quad u, v \in \varphi(J).$$

Taking $b = 0$ we hence get

$$aM_1(u, v) \leq M_1(au, av), \quad u, v \in \varphi(I),$$

for all $a \in \mathbb{R}$ such that $0 \leq a < \alpha$, which implies that

$$aM_1(u, v) = M_1(au, av), \quad u, v \in \varphi(I),$$

for all $a \in \mathbb{R}$ such that $0 \leq a < \alpha$.

For an arbitrary $(u, v) \in \mathbb{R} \times \mathbb{R}$ take $t > 0$ such that $\frac{u}{t}, \frac{v}{t} \in \varphi(I)$ and put

$$M^*(u, v) := tM_1\left(\frac{u}{t}, \frac{v}{t}\right).$$
To show that $M^*(u, v)$ does not depend on the choice of $t$, take an $s > 0$ such that $\frac{u}{s}, \frac{v}{s} \in \varphi(I)$. Without any loss of generality we can assume that $s > t$. Since $0 < \frac{s}{t} < 1$, from (6) we have

$$tM_1\left( \frac{u}{t}, \frac{v}{t} \right) = \frac{t}{s}M_1\left( \frac{u}{s}, \frac{v}{s} \right) = \frac{t}{s}M_1\left( \frac{u}{t}, \frac{v}{t} \right) = \frac{t}{s}M_1\left( \frac{u}{s}, \frac{v}{s} \right).$$

This proves that the function $M^* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is correctly defined. Clearly, $M^*$ is positively homogeneous, that is

$$M^*(tu, tv) = tM^*(u, v), \quad u, v \in \mathbb{R}, \quad t \geq 0.$$ 

Taking $a = 1$ in (1) we get, for all $b \in \mathbb{R}$, $|b| < \beta$,

$$M_1(u, v) + b \leq M_1(u + b, v + b), \quad u, v \in \varphi(I).$$

Replacing here $u$ by $u - b$ and $v$ by $v - b$ we obtain

$$M_1(u - b, v - b) \leq M_1(u, v) - b, \quad u, v \in \varphi(I),$$

for all $b \in \mathbb{R}$, $|b| < \beta$. Therefore

$$M_1(u + b, v + b) = M_1(u, v) + b, \quad |b| < \beta; \quad u, v \in \varphi(I).$$

Take arbitrary $u, v \in \mathbb{R}$, $b \in \mathbb{R}$ and choose a $t > 0$ such that $\frac{u}{t}, \frac{v}{t} \in \varphi(I)$ and $|\frac{b}{t}| < \beta$. Then, by the definition of $M^*$ and its homogeneity, we have

$$M^*(u + b, v + b) = tM_1\left( \frac{u}{t} + \frac{b}{t}, \frac{v}{t} + \frac{b}{t} \right) = t\left[ M_1\left( \frac{u}{t}, \frac{v}{t} \right) + \frac{b}{t} \right]$$

$$= tM_1\left( \frac{u}{t}, \frac{v}{t} \right) + b = M^*(u, v) + b.$$ 

Now the same reasoning as in the proof of Theorem 1 shows that there is a $q \in [0, 1]$ such that

$$M^*(u, v) = qu + (1 - q)v, \quad u, v \in \mathbb{R}.$$ 

Since the function $M^*$ is uniquely determined and the definition does not depend on the choice of the compact interval $I$, we infer that

$$\varphi(M(\varphi^{-1}(u), \varphi^{-1}(v))) = M^*(u, v), \quad u, v \in \varphi(J).$$

Consequently,

$$M(x, y) = \varphi^{-1}(q\varphi(u) + (1 - q)\varphi(y)), \quad x, y \in J.$$ 

Since remaining statements are obvious, the proof is complete. □
4. A REMARK AND OPEN PROBLEM

Remark 5. In the basic supposition of the above results that "$N$-affinity of some $f$ implies its $M$-convexity", we assume that $N$ is a quasi-arithmetic mean. This assumption is essential because there are different non-quasi-arithmetic means with the same classes of affine functions. For instance, the logarithmic mean $L : (0, \infty)^2 \to (0, \infty)$,

$$L(x, y) = \begin{cases} \frac{x - y}{\log x - \log y}, & x \neq y \\ x, & x = y \end{cases}$$

and the mean $M : (0, \infty)^2 \to (0, \infty)$,

$$M(x, y) := \frac{2}{3} \frac{x^2 + xy + y^2}{x + y},$$

are symmetric and have the same classes continuous $L$-affine functions and $M$-affine functions coincide (cf. [9], [10]).

We end up this paper with the following open

Problem 1. Let $M$ and $N$ be strict, symmetric and continuous means in an interval $J$. Suppose that for all intervals $I \subset J$ and for all continuous functions $f : I \to J$, the implication

$$f \text{ is } N\text{-convex} \implies f \text{ is } M\text{-convex}$$

holds true. Is then $M = N$?

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5. REFERENCES


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