Existence of Infinitely Many Brake Orbits of Lagrangian Autonomous Systems on Tori¹

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Abstract. In this note we point out that the arguments by Campos and Tarallo [1] implied the existence of periodic brake orbits with minimal period greater than any prescribed number for real analytic Lagrangian autonomous systems on tori.

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An autonomous Lagrangian system on the Riemannian torus $(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, A)$ with a potential V is described on $\mathbb{R}^n$ by the equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad (1)$$

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where $L(q, \dot{q}) = \frac{1}{2}(A(q)\dot{q}, \dot{q}) - V(q)$, the matrix $A(q)$ is symmetric and positive definite for every $q \in \mathbb{R}^n$ and A and V are 1-period in each variable. For the flat metric A , i.e. A being a positive matrix on $\mathbb{R}^n$ without depending on $q \in \mathbb{R}^n$, and C^3 -smooth V , Yiming Long and the first author proved in [2] that for any $\tau > 0$ the system (1) possesses infinitely many geometrically distinct contractible periodic solution orbits of periods being integer multiples of τ . When both A and V are assumed to be real analytic functions of q , it was also proved in [1] by Campos and Tarallo that for every $\tau_0 > 0$ there exists a periodic solution to (1) with minimal period greater than τ_0 . Here we are interested in the existence of τ -periodic solutions $q : \mathbb{R} \rightarrow \mathbb{R}^n$ of (1) satisfying

$$\dot{q}(0) = \dot{q}(\tau/2) = 0. \quad (2)$$

Such periodic solutions are called **brake orbits** of (1) because they bear some formal analogy with brakes studied by Seifert [4]. Clearly, a C^1 -map $q : \mathbb{R} \rightarrow \mathbb{R}^n$ with

$$q(\tau/2 + t) = q(\tau/2 - t) \quad \forall t \in \mathbb{R}, \quad (3)$$

must satisfy (2). Moreover, a τ -periodic map $q : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfies (3) if and only if

$$q(t) = q(-t) \quad \forall t \in \mathbb{R}. \quad (4)$$

A map $q : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfying (4) is said to be **reversible** in time. So a periodic solution of (1) which is reversible in time gives rise to a brake orbit of (1). Note also that a τ -periodic map $q : \mathbb{R} \rightarrow \mathbb{R}^n$ satisfies (4) if and only if $q(k\tau/2 + t) = q(k\tau/2 - t)$ for any $t \in \mathbb{R}$ and $k \in \mathbb{Z}$. Thus a τ -periodic solution $q : \mathbb{R} \rightarrow \mathbb{R}^n$ of (1) satisfying $q(t) = q(-t)$ automatically produces many other such periodic solutions of (1), such as $q_k(t) = q(t + k\tau/2) + h$ with $k \in \mathbb{Z}$ and $h \in \mathbb{Z}^n$. For τ_i -periodic solutions $q_i : \mathbb{R} \rightarrow \mathbb{R}^n$ of (1) being reversible in time, $i = 1, 2$, they are called **distinct** if one of them cannot be obtained from another in the way above. Suitably modifying the arguments in [1] we get

Theorem 1. *For real analytic A and V , if V is also nonconstant then for every $\tau > 0$ there exists a reversible period solution of (1) with minimal period greater than τ , and thus there exist brake orbits of (1) with minimal period greater than any prescribed number.*

As in Theorem 2 of [1], in the case that A and V are only smooth, if V is also nonconstant then there exist uncountable number of distinct and nonconstant reversible period solutions of (1). Let H be the Hilbert space of the absolutely continuous 1-periodic function $q : \mathbb{R} \rightarrow \mathbb{R}^n$ with L^2 -derivative on $[0, 1]$. There

is a natural $\mathbb{Z}^n$ -action on it. For each $\tau > 0$, a τ -periodic solution q to (1) corresponds to a critical point u of the $\mathbb{Z}^n$ -invariant smooth functional

$$\mathcal{A}_\tau(u) = \frac{1}{2\tau^2} \int_0^1 (A(u(t))\dot{u}(t), \dot{u}(t))dt - \int_0^1 V(u(t))dt$$

on H via $u(t) := q(\tau t)$ (cf. [1]). Denote by H_0 the subspace of H consisting of those $u \in H$ which are reversible in time, i.e., $u(t) = u(-t)$ for any $t \in \mathbb{R}$. It is easily checked that H_0 is a Hilbert subspace of H . As in [3], one can prove that $\mathcal{A}_\tau$ also satisfies the Palais-Smale condition on $\tilde{H}_0 := H_0/\mathbb{Z}^n$.

Lemma 2. *A map $q : \mathbb{R} \rightarrow \mathbb{R}^n$ which is reversible in time is a τ -periodic solution to (1) if and only if $u(t) := q(\tau t)$ is a critical point of $\mathcal{A}_\tau$ on H_0 .*

Proof. We only need to prove that a critical point u_0 of $\mathcal{A}_\tau$ on H_0 is also one of $\mathcal{A}_\tau$ on H . Let $\Xi : H \rightarrow H$ be the Hilbert space isomorphism defined by $u \rightarrow \Xi(u)$, $\Xi(u)(t) = u(-t)$. It is actually involution with fixed point set $\text{Fix}(\Xi) = H_0$. One easily checks that $\mathcal{A}_\tau(\Xi(u)) = \mathcal{A}_\tau(u)$ for any $u \in H$. It follows that $\Xi(\mathcal{A}'_\tau(u)) = \mathcal{A}'_\tau(\Xi(u))$ for any $u \in H$. Consider the orthogonal decomposition $\mathcal{A}'_\tau(u) = \mathcal{A}'_\tau(u)_0 \oplus \mathcal{A}'_\tau(u)_0^\perp \in H_0 \oplus H_0^\perp$. Since $\Xi(u_0) = u_0$ and $\Xi(\mathcal{A}'_\tau(u_0)_0) = \mathcal{A}'_\tau(u_0)_0$, we deduce that $\Xi(\mathcal{A}'_\tau(u_0)_0^\perp) = \mathcal{A}'_\tau(u_0)_0^\perp$ and thus $\mathcal{A}'_\tau(u_0)_0^\perp = 0$. Hence $\mathcal{A}'_\tau(u_0) = \mathcal{A}'_\tau(u_0)_0 \in H_0$. By the definition,

$$((\mathcal{A}_\tau|_{H_0})'(u_0), v) = (\mathcal{A}_\tau|_{H_0})'(u_0)(v) = (\mathcal{A}'_\tau(u_0)(v) = (\mathcal{A}'_\tau(u_0), v) \forall v \in H_0.$$

Since $(\mathcal{A}_\tau|_{H_0})'(u_0) = 0$, $(\mathcal{A}'_\tau(u_0), v) = 0 \forall v \in H_0$, i.e., $\mathcal{A}'_\tau(u_0) \in H_0^\perp$. It follows that $\mathcal{A}'_\tau(u_0) = 0$. The desired result is proved. $\square$

Combing this lemma with Lemma 5 in [1] we have the claim that for every $S \in \mathbb{R}$ there exists a $T > 0$ such that

$$u \in H_0, \mathcal{A}'_\tau(u) = 0 \text{ and } \mathcal{A}_\tau(u) \leq S \Rightarrow \text{either } u(t) \text{ is a constant or } \tau \geq T. \quad (5)$$

As in [1] we always assume that $\max V = 0$ so that $\mathcal{A}_\tau(u) \geq 0 \forall u \in H$ and that the ground level zero is attained only at the constant functions that maximize V . Then the critical value sets $C_\tau = \{\mathcal{A}_\tau(u) \mid \mathcal{A}'_\tau(u) = 0, u \in H_0\} \neq \emptyset$ and $C_\tau \supset C_0 = \{-V(z) \mid \nabla V(z) = 0\}$ for any $\tau > 0$. Let $\overline{H}_0 = \{u \in H_0 \mid \bar{u} = \int_0^1 u(t)dt = 0\}$ and

$$P : H_0 \rightarrow \mathbb{T}^n \times \overline{H}_0, u \mapsto ([\bar{u}], u - \bar{u}),$$

where $[\bar{u}]$ is the equivalence class of $\bar{u} \in \mathbb{R}^n$ in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Then P is surjective because $\{u \in H_0 \mid u = \bar{u}\}$ is isomorphic to $\mathbb{R}^n$ and there exists a

natural orthogonal decomposition $H_0 = \mathbb{R}^n \oplus \overline{H}_0$ given by $u = \bar{u} + (u - \bar{u})$. The Lusternik-Schnirelmann category can be used to associate an index to every closed subset S of H_0 as follows: $i(S) = \text{cat}_{\mathbb{T}^n \times \overline{H}_0}(\mathbf{P}(S))$. Since $\text{cat}_{\mathbb{T}^n} \mathbb{T}^n = n+1$ and $\overline{H}_0$ is contractible, one has $i(H_0) = n+1$, and $i(S) = n+1$ if and only if $\{[\bar{u}] \mid u \in S\} \equiv \mathbb{T}^n$. In particular, $\Gamma = \{S \subset H_0 \mid S \text{ is closed and } i(S) = n+1\} \neq \emptyset$. The standard arguments (using [3, Prop.1.10]) can prove that $c(\tau) := \inf_{S \in \Gamma} \sup_S \mathcal{A}_\tau$ is a critical value for $\mathcal{A}_\tau$ on H_0 (cf.[1]). Corresponding to Proposition 6 in [1] we have

Lemma 3. *If V is nonconstant, then for every $\tau > 0$ there exists a critical value $c(\tau) \in C_\tau$ of $\mathcal{A}_\tau$ on H_0 satisfying:*

- (i) $0 < c(\tau) \leq -\min_{\mathbb{R}^n} V$ for every τ ;
- (ii) $\tau \rightarrow c(\tau)$ is decreasing and locally Lipschitz continuous in $(0, +\infty)$;
- (iii) $c(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$.

Proof The proofs of (i) and (ii) are completely same as that of [1]. (iii) can also be obtained by slightly modifying proof in [1]. The goal is to construct a closed subset $S_\tau \in \Gamma$ for each τ such that $\sup_{S_\tau} \mathcal{A}_\tau \rightarrow 0$ as $\tau \rightarrow \infty$. For any τ and $\lambda \in [0, 1]$, define $\psi_\lambda^\tau : [0, 1] \rightarrow \mathbb{R}$, $t \mapsto (1+\tau)\lambda - 2\tau|t - 1/2|$, then $\psi_0^\tau(t) \leq 0$ and $\psi_1^\tau(t) \geq 1$. It follows that maps

$$\theta_\lambda^\tau : [0, 1] \rightarrow [0, 1], \quad t \mapsto \min\{1, \max\{0, \psi_\lambda^\tau(t)\}\}$$

satisfy: $\theta_0^\tau \equiv 0$, $\theta_1^\tau \equiv 1$, $\theta_\lambda^\tau(0) = \theta_\lambda^\tau(1)$ and $\theta_\lambda^\tau(t + 1/2) = \theta_\lambda^\tau(-t + 1/2)$ for any $0 \leq t \leq 1/2$. The direct computation shows that

$$\overline{\theta}_\lambda^\tau = \int_0^1 \theta_\lambda^\tau(t) dt = \begin{cases} \frac{(1+\tau^2)\lambda^2}{2\tau}, & 0 \leq \lambda \leq \frac{1}{1+\tau}, \\ \frac{2(1+\tau)\lambda}{2\tau}, & \frac{1}{1+\tau} < \lambda \leq \frac{\tau}{1+\tau}, \\ 1 - \frac{(1+\tau)^2(1-\lambda)^2}{2\tau}, & \frac{\tau}{1+\tau} < \lambda \leq 1. \end{cases}$$

So the map $[0, 1] \rightarrow [0, 1], \lambda \mapsto \overline{\theta}_\lambda^\tau$ is homotopic to the identity from $[0, 1]$ to $[0, 1]$. Define

$$\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau : [0, 1] \rightarrow [0, 1]^n, \quad t \mapsto (\theta_{\lambda_1}^\tau, \theta_{\lambda_2}^\tau, \dots, \theta_{\lambda_n}^\tau),$$

where $\lambda_i \in [0, 1]$, $i = 1, 2, \dots, n$. Then the properties of θ_λ^τ above imply

$$\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau \equiv (\lambda_1, \lambda_2, \dots, \lambda_n) \quad \text{if } (\lambda_1, \lambda_2, \dots, \lambda_n) \in \partial[0, 1]^n,$$

and

$$\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau(t + 1/2) = \Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau(-t + 1/2) \quad \forall 0 \leq t \leq 1/2.$$

The latter shows that we can 1-periodically extend $\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau$ to $\mathbb{R}$, still denoted by $\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau$. Now $\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau \in H_0$, and

$$f : [0, 1]^n \rightarrow [0, 1]^n, (\lambda_1, \lambda_2, \dots, \lambda_n) \mapsto \overline{\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau} = (\overline{\theta_{\lambda_1}^\tau}, \overline{\theta_{\lambda_2}^\tau}, \dots, \overline{\theta_{\lambda_n}^\tau})$$

is homotopic to the identity map $[0, 1]^n \rightarrow [0, 1]^n$.

Fix a point $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in V^{-1}(0)$. Define $S_\tau = \{\eta + \Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau \mid \lambda_i \in [0, 1], i = 1, 2, \dots, n\}$. Then S_τ is closed in H_0 and $P(S_\tau) = B_1 \times B_2$, where

$$\begin{aligned} B_2 &= \{\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau - \overline{\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau} \mid \lambda_i \in [0, 1], i = 1, 2, \dots, n\}, \\ B_1 &= \{[\eta + \overline{\Theta_{(\lambda_1, \lambda_2, \dots, \lambda_n)}^\tau}] \mid \lambda_i \in [0, 1], i = 1, 2, \dots, n\} \\ &= \{[\eta + \zeta] \mid \zeta \in f([0, 1]^n)\} \\ &= \{[\zeta] \mid \zeta \in f([0, 1]^n)\} = \mathbb{T}^n \end{aligned}$$

because $f([0, 1]^n) = [0, 1]^n$ and $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = [0, 1]^n / \partial[0, 1]^n$. From $\overline{H_0}$ being contractible, it follows that $i(S_\tau) = \text{cat}_{\mathbb{T}^n \times \overline{H_0}} B_1 \times B_2 = \text{cat}_{\mathbb{T}^n} B_1 = n + 1$. Hence $S_\tau \in \Gamma$. Repeating the arguments at the end of proof of Proposition 6 in [1], one can prove that S_τ satisfies our requirements, that is, $\sup_{S_\tau} \mathcal{A}_\tau \rightarrow 0$ as $\tau \rightarrow \infty$. $\square$

Lemma 4 ([1, Prop.12]). *Assume that M is a real analytic manifold, $\varphi : M \rightarrow \mathbb{R}$ is a real analytic function, $C = \{x \in M \mid \varphi'(x) = 0\}$ and $\varphi''(x)$ is a Fredholm map for every $x \in C$. Then φ is locally constant on C . Hence, if K is a compact subset of C , then $\varphi(K)$ is finite.*

Now we are a position to give a corresponding result with Proposition 8 in [1].

Lemma 5. *Under the assumptions that A and V are analytic, it holds that*

- (i) C_0 is finite;
- (ii) C_τ is discrete for any $\tau > 0$;
- (iii) zero is isolated in $\cup_{0 < s < \tau_0} C_s$ for every $\tau_0 > 0$.

Let $C = \{q \in \mathbb{R}^n \mid V'(q) = 0\}$ and $K = C \cap [0, 1]^n$. Applying Lemma 4 to V , one gets $V(K) = V(C) = -C_0$ is finite. (ii) and (iii) can be proved with (5) and Lemma 4 as in [1, §5].

Finally, repeating the proof of [1, Theorem 1] we can immediately derive Theorem 1 from Lemma 3 and Lemma 5 above.

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