An Accurate Estimate of Mathieu's Series

Vito Lampret

University of Ljubljana
Faculty of Civil and Geodetic Engineering
Department of Mathematics and Physics
Jamova 2, Ljubljana 1000, Slovenia 386
Vito.Lampret@fgg.uni-lj.si

We dedicate this article to Leonard Euler, the father of the Euler-Maclaurin formula, on the occasion of three hundreds anniversary of his birthday, April 15, 1707.

Abstract

Using Hermite's, i.e. the Euler-Maclaurin summation formula of order four, new approximations to Mathieu's series $S(x) \equiv \sum_{k=1}^{\infty} \frac{2k}{(k^2+x^2)^2}$ are obtained, which are more accurate than the approximations presented recently in the literature.

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1 Introduction

In 1890 Mathieu [11] defined $S(x)$, now called Mathieu's series, as

$$S(x) \equiv \sum_{k=1}^{\infty} \frac{2k}{(k^2 + x^2)^2}. \quad (1)$$

Since that time Mathieu's series has been investigated intensively many times, see for example [4], [5], [15], [16], [14], and [17]. Alzer [2] found that, for every real x ,

$$\frac{1}{x^2 + 1/(2\zeta(3))} < S(x) < \frac{1}{x^2 + 1/6}. \quad (2)$$

Moreover, recently, F. Qi et al. [13, Th. 4.2, p. 2550], have estimated Mathieu's series as

$$A(x) \leq S(x) \leq B(x), \quad (3)$$

for $x > 0$, where

$$A(x) \equiv \frac{4(1+x^2)(e^{-\pi/x} + e^{-\pi/(2x)}) - 4x^2 - 1}{(e^{-\pi/x} - 1)(1+x^2)(1+4x^2)}$$

$$B(x) \equiv \frac{(1+4x^2)(e^{-\pi/x} - e^{-\pi/(2x)}) - 4(x^2+1)}{(e^{-\pi/x} - 1)(1+x^2)(1+4x^2)}.$$

Although the estimate (2) is asymptotically sharp, it fails for $x \approx 0$, since the upper bound in (2) approximates $S(x)$ very roughly. However, the lower bound in (2) is considerably closer to $S(x)$, as is evident from Figure 5. Unfortunately, for larger values of x the estimate (3) is worse than (2). This is illustrated in Figure 1, where we plotted the graphs of functions $x \mapsto 1/\left(x^2 + \frac{1}{2\zeta(3)}\right)$ and $x \mapsto 1/\left(x^2 + \frac{1}{6}\right)$ (dashed lines), together with the graphs of the bounds $x \mapsto A(x)$ and $x \mapsto B(x)$ (solid lines). In addition, the lower bound $A(x)$ becomes negative, consequently useless for larger values of x . This is illustrated in Figure 2. In this note we wish to find an estimate of $S(x)$ applicable on the entire $\mathbb{R}$.

Figure 1: The graphs of functions $x \mapsto 1/\left(x^2 + \frac{1}{2\zeta(3)}\right)$ and $x \mapsto 1/\left(x^2 + \frac{1}{6}\right)$ (dashed lines), together with the graphs of the bounds $x \mapsto A(x)$ and $x \mapsto B(x)$ (solid lines).

2 A summation formula

According to [10, Theorem, p. 320] or¹ [9, Theorem 2, p. 119] we have at our disposal a theorem comparing the convergence of a series $\sum_{k=1}^{\infty} f(k)$ and an integral $\int_1^{\infty} f(x) dx$.

¹Additional literature: [1], [3], [6], [7], [8].

Figure 2: The graph of lower bound $A(x)$.

THEOREM. *If $f \in C^4[1, \infty)$, $\int_1^\infty |f^{(4)}(x)| dx$ converges, and the finite limits $\lambda_0 := \lim_{\substack{n \in \mathbb{N} \\ n \rightarrow \infty}} f(n)$ and $\lambda_1 := \lim_{\substack{n \in \mathbb{N} \\ n \rightarrow \infty}} f'(n)$ exist, then:*

- (a) *The series $\sum_{k=1}^\infty f(k)$ converges. $\iff$ The sequence $n \mapsto \int_1^n f(x) dx$ converges.*
- (b) *If the series $\sum_{k=1}^\infty f(k)$ converges, then $\lambda_0 = 0$ and*

$$\sum_{k=1}^\infty f(k) = \sum_{k=1}^{m-1} f(k) + \lim_{\substack{n \in \mathbb{N} \\ n \rightarrow \infty}} \int_m^n f(x) dx + \frac{f(m)}{2} + \frac{\lambda_1}{12} - \frac{f'(m)}{12} + \rho(m), \quad (4)$$

for every integer $m \geq 1$, where

$$|\rho(m)| \leq \frac{1}{384} \int_m^\infty |f^{(4)}(x)| dx. \quad (5)$$

3 An approximation

Mathieu's series originates from the function

$$f_x(t) \equiv \frac{2t}{(t^2 + x^2)^2}, \quad x \in \mathbb{R} \text{ being a parameter}, \quad (6)$$

having the derivatives,

$$f'_x(t) \equiv -2 \frac{3t^2 - x^2}{(t^2 + x^2)^3}, \quad (7)$$

$$f^{(4)}_x(t) \equiv 240 \frac{3t^5 - 10t^3x^2 + 3tx^4}{(t^2 + x^2)^6}, \quad (8)$$

and the integral,

$$\int_m^\infty f_x(t) dt = \left[-\frac{1}{t^2 + x^2} \right]_{t=m}^\infty = \frac{1}{m^2 + x^2}. \quad (9)$$

To use the quoted summation theorem we have to estimate the error term $\rho(m, x)$. Thanks to (5) and (8), we estimate,

$$\begin{aligned} |\rho(m, x)| &\leq \frac{1}{384} \int_m^\infty |f_x^{(4)}(t)| dt \\ &\leq \frac{240}{384} \int_m^\infty \frac{3t^5 + 10t^3x^2 + 3tx^4}{(t^2 + x^2)^6} dt \\ &= \delta(m, x) := \frac{5m^4 + 15m^2x^2 + 6x^4}{16(m^2 + x^2)^5} \end{aligned} \quad (10)$$

$$< \Delta(m, x) := \frac{1}{2(m^2 + x^2)^3}, \quad (11)$$

valid for every integer $m \geq 1$ and any real x . Therefore, using the formula (4), and relations (6)–(7), we find the expression

$$S(x) = \sigma(m, x) + \rho(m, x), \quad (12)$$

true for every real x and any integer $m \geq 1$, where

$$\sigma(m, x) \equiv \sum_{k=1}^{m-1} \frac{2k}{(k^2 + x^2)^2} + \frac{1}{m^2 + x^2} + \frac{m}{(m^2 + x^2)^2} + \frac{3m^2 - x^2}{6(m^2 + x^2)^3}. \quad (13)$$

We have, for $m \geq 1$,

$$\begin{aligned} \frac{m}{(m^2 + x^2)^2} + \frac{3m^2 - x^2}{6(m^2 + x^2)^3} &= \frac{6m^3 + 3m^2 + (6m - 1)x^2}{6(m^2 + x^2)^3} \\ &\geq \frac{6m^3 + 3m^2}{6(m^2 + x^2)^3} = \frac{2m^3 + m^2}{2(m^2 + x^2)^3} > 0. \end{aligned}$$

Therefore, the relative error,

$$r(m, x) := \frac{S(x) - \sigma(m, x)}{\sigma(m, x)}, \quad (14)$$

of the approximation $S(x) \approx \sigma(m, x)$ can be estimated, using the relations (11)–(13), as

$$\begin{aligned} |r(m, x)| &= \left| \frac{\rho(m, x)}{\sigma(m, x)} \right| \\ &< \frac{\frac{1}{2(m^2 + x^2)^3}}{\frac{1}{m^2 + x^2} + \frac{2m^3 + m^2}{2(m^2 + x^2)^3}} \\ &= r^*(m, x) := \frac{1}{2(m^2 + x^2)^2 + 2m^3 + m^2} \leq \frac{1}{2m^4 + 2m^3 + m^2}, \end{aligned} \quad (15)$$

Figure 3: The graphs of functions $x \mapsto r^*(m, x)$.

for any integer $m \geq 1$. Figure 3 shows the graphs of the functions $x \mapsto r^*(m, x)$.

Example 1. Setting $m = 1$ in (13) we obtain

$$\sigma(1, x) \equiv \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \frac{3-x^2}{6(1+x^2)^3}.$$

According to (15), the approximation $S(x) \approx \sigma(1, x)$ has the relative error estimated as

$$|r(1, x)| < \frac{1}{3+2(1+x^2)^2} \leq \frac{1}{5}, \quad x \in \mathbb{R}.$$

Example 2. Putting $m = 4$ in (13) we get

$$\sigma(4, x) \equiv \frac{1}{16+x^2} + \frac{2}{(1+x^2)^2} + \frac{4}{(4+x^2)^2} + \frac{6}{(9+x^2)^2} + \frac{4}{(16+x^2)^2} + \frac{48-x^2}{6(16+x^2)^3}.$$

Thanks to (15), the approximation $S(x) \approx \sigma(4, x)$ has the relative error estimated as,

$$|r(4, x)| < \frac{1}{144+2(16+x^2)^2} \leq \frac{1}{656} < 0.16\%, \quad x \in \mathbb{R}. \quad (16)$$

Figure 4 shows the graph of the function $\sigma(4, x)$ and, according to (16), it represents also, for all practical purposes, the graph of Mathieu's function $S(x)$.

Figure 4: The graph of Mathieu's function $S(x)$.

Invoking (14) and (15), we estimate

$$a_0(m, x) \leq a(m, x) < S(x) < b(m, x) \leq b_0(m, x) \quad (17)$$

for every integer $m \geq 1$ and every real x , where

$$a_0(m, x) \equiv \left(1 - \frac{1}{2m^4 + 2m^3 + m^2}\right) \cdot \sigma(m, x) \quad (18)$$

$$b_0(m, x) \equiv \left(1 + \frac{1}{2m^4 + 2m^3 + m^2}\right) \cdot \sigma(m, x) \quad (19)$$

$$a(m, x) \equiv \left(1 - \frac{1}{2(m^2 + x^2)^2 + 2m^3 + m^2}\right) \cdot \sigma(m, x) \quad (20)$$

$$b(m, x) \equiv \left(1 + \frac{1}{2(m^2 + x^2)^2 + 2m^3 + m^2}\right) \cdot \sigma(m, x) \quad (21)$$

In particular, we have

$$a(1, x) \equiv \frac{(x^4 + 2x^2 + \mathbf{2})(6x^4 + 17x^2 + 15)}{3(1 + x^2)^3(2x^4 + 4x^2 + 5)} \quad (22)$$

$$b(1, x) \equiv \frac{(x^4 + 2x^2 + \mathbf{3})(6x^4 + 17x^2 + 15)}{3(1 + x^2)^3(2x^4 + 4x^2 + 5)} \quad (23)$$

$$a_0(4, x) \equiv \frac{655}{656} \left(\frac{1}{16+x^2} + \frac{2}{(1+x^2)^2} + \frac{4}{(4+x^2)^2} + \frac{6}{(9+x^2)^2} + \frac{4}{(16+x^2)^2} + \frac{48-x^2}{6(16+x^2)^3} \right) \quad (24)$$

$$b_0(4, x) \equiv \frac{657}{656} \left(\frac{1}{16+x^2} + \frac{2}{(1+x^2)^2} + \frac{4}{(4+x^2)^2} + \frac{6}{(9+x^2)^2} + \frac{4}{(16+x^2)^2} + \frac{48-x^2}{6(16+x^2)^3} \right). \quad (25)$$

Figure 5 shows, on the left, the graphs of functions $a(1, x)$ and $b(1, x)$ (solid lines) and, on the right, the graphs of functions $a_0(4, x)$ and $b_0(4, x)$ (solid lines, which practically coincide), together with the graphs of the bounds $A(x)$ and $B(x)$ (dashed lines), included in both parts of the figure.

Figure 5: The graphs of the bounds $A(x)$ and $B(x)$ (dashed lines), together with the graphs of functions $a(1, x)$ and $b(1, x)$ (solid lines) on the left, and the graphs of functions $a_0(4, x)$ and $b_0(4, x)$ (solid lines, which practically coincide), on the right.

4 Conclusion

Using the Euler-Maclaurin summation formula we found new approximations to Mathieu's series $S(x)$, which are more accurate than the approximations

available in the literature. It is worthwhile to stress that we succeeded using only elementary tools, and the fact that the approximations have high global accuracy. The Euler series accelerating convergence tool, which is quite an old idea, proved to be very efficient in pure and applied mathematical analysis. Even in the age of computers the Euler-Maclaurin summation formula is still valuable; it is included in the softwares of most modern computers. We dedicate this article to Euler, celebrating his birth, April 15, 1707.

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