On K -Homomorphisms of K -Algebras

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Abstract

The authors have introduced a class of K -algebras in [1] and have further extended its scope of study in literature [2, 3, 4, 5]. In this paper, we introduce the notion of K -homomorphism of K -algebras, and investigate some of their properties and structure.

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1 Introduction

The notion of a K -algebra $(G, \cdot, \odot, e)$ was first introduced by K. H. Dar and M. Akram in [1]. A K -algebra built on a group $(G, \cdot)$ by adjoining induced binary operation $\odot$ defined by $x \odot y = x \cdot y^{-1}$, for all $x, y \in G$, where e is the identity of the group. It is attached to an abstract K -algebra $(G, \odot, e)$, which is non-commutative and non-associative with right identity element e . It is characterized by using its left and right mappings in [2]. Recently, same authors have proved that a class of K -algebras as a generalization of a class of B -algebras [13] and that of a family of $BCH/BCI/BCK$ -algebras [7, 8, 9, 10, 11, 12] in [4]. The notion of fuzzy structures of $K(G)$ -algebras is also

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introduced in [5]. In this paper, we discuss some properties of K -subalgebras and K -ideals in K -algebras in Section 2. The notion of K -homomorphisms of K -algebras is introduced, and some of its related properties are investigated in Section 3.

2 K -subalgebras

In this section, we cite notion of K -algebras and extend study of K -subalgebra.

Definition 2.1. [1] An algebra $(G, \odot, e)$ built on a group G with identity e is called a K -algebra on G (briefly, $K(G)$ -algebra), if G is not elementary abelian 2-group and observes the following $\odot$ -axioms:

$$(k1) \quad (x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x,$$

$$(k2) \quad x \odot (x \odot y) = (x \odot (e \odot y)) \odot x,$$

$$(k3) \quad x \odot x = e,$$

$$(k4) \quad x \odot e = x,$$

$$(k5) \quad e \odot x = x^{-1},$$

for all $x, y, z \in G$.

If the group G is abelian then the axioms (k1) and (k2) of K -algebra change to be $(\overline{k1})$ and $(\overline{k2})$ respectively where

$$(\overline{k1}) \quad (x \odot y) \odot (x \odot z) = z \odot y,$$

$$(\overline{k2}) \quad x \odot (x \odot y) = y,$$

for all $x, y, z \in G$.

Definition 2.2. A $K(G)$ -algebra $(G, \odot, e)$ is abelian if, $g \odot (e \odot x) = x \odot (e \odot g)$, for all $x, g \in G$.

Proposition 2.3. A $K(G)$ -algebra $(G, \odot, e)$ is abelian if and only if, $e \odot x = x$, for all $x \in G$.

Proof. Routine. □

Proposition 2.4. In a $K(G)$ -algebra $(G, \odot, e)$, the following equations are valid if the group G is non-abelian:

$$(k6) \quad (x \odot y) \odot (u \odot v) = (x \odot (e \odot v) \odot (e \odot y)) \odot u.$$

$$(k7) \quad (x \odot y) \odot z = x \odot (z \odot (e \odot y)).$$

$$(k8) \quad e \odot (e \odot x) = x.$$

$$(k9) \quad e \odot (x \odot y) = y \odot x = (e \odot x) \odot (e \odot y).$$

$$(k10) \quad x \odot y = e \quad \text{if and only if} \quad x = y.$$

Proof. The proof is straightforward and we omit it. $\square$

Definition 2.5. [1] A non-empty subset H of a K -algebra $(G, \odot, e)$ is called K -subalgebra if

$$(i) \quad e \in H,$$

$$(ii) \quad h_1 \odot h_2 \in H, \text{ for all } h_1, h_2 \in H.$$

Example 2.6. [1] Consider the $K(S_3)$ -algebra $(S_3, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

We see that K -subalgebra $(A_3, \odot, e)$ is a proper subalgebra having the following table:

$\odot$	e	a	b
e	e	b	a
a	a	e	b
b	b	a	e

If $H = \{e, x\} = C_2$ then $K(S_3)$ -subalgebra $(C_2, \odot, e)$ is an improper in $K(S_3)$ by the Cayley table.

$\odot$	e	x
e	e	x
x	x	e

Remark. (1) For every subgroup H of group G there exists a proper K -subalgebra $(H, \odot, e)$ if H is not an elementary abelian 2-subgroup of G .

- (2) If H is an elementary abelian 2-subgroup of G then K -subalgebra $(H, \odot, e)$ is improper.

Proposition 2.7. *In a K -algebra $(G, \odot, e)$, let g be a fixed element of G and H be a subgroup of a group G . Then $H_{g^2} = \{g \odot (g \odot x) : x \in G\}$ forms a K -subalgebra.*

Proof. Since $g \odot (g \odot e) = e \in H_{g^2}$, where $e \in H$ and

$$\begin{aligned} (g \odot (g \odot x)) \odot (g \odot (g \odot y)) &= (g \odot ((e \odot y) \odot (e \odot x))) \odot g \\ &= (g \odot (e \odot (y \odot x))) \odot g \\ &= (g \odot (x \odot y)) \odot g \\ &= g \odot (g \odot (e \odot (x \odot y))) \\ &= g \odot (g \odot (y \odot x)) \in H_{g^2} \end{aligned}$$

for all $x, y \in H$ and $g \odot (g \odot x), g \odot (g \odot y) \in H_{g^2}$. This complete the proof. $\square$

Definition 2.8. A proper K -subalgebra $(H, \odot, e)$ of a K -algebra $(G, \odot, e)$ is called a K -ideal if for for all $x \in G, g \in G, x \odot g$ and $g \odot (g \odot x) \in H$ implies $x \in H$.

In example 2. 6, the K -subalgebra $(A_3, \odot, e)$ is a K -ideal of K -algebra $(G, \odot, e)$.

Proposition 2.9. *Let H_1 and H_2 be two K -subalgebras(K -ideals) of a K -algebra $(G, \odot, e)$ then*

- (a) $H_1 \cap H_2$ is a K -subalgebra(K -ideal) of $(G, \odot, e)$.
- (b) $H_1 \odot H_2$ is a K -subalgebra(K -ideal) of $(G, \odot, e)$ if and only if $H_2 \odot H_1 = H_1 \odot H_2$ (either H_1 or H_2 is ideal).

Proof. Straightforward. $\square$

Proposition 2.10. *For each proper normal subgroup H of G there corresponds a proper K -ideal $(H, \odot, e)$ of $(G, \odot, e)$.*

Proof. If H is a normal subgroup of G then $gxg^{-1} \in H$ for all $g \in G$ implies $x \in H$. Thus $xg^{-1} = x \odot g$ and $g \odot (g \odot x) \in (H, \odot, e)$ imply $x \in H$, which proves that $(H, \odot, e)$ is a K -ideal of K -algebra $(G, \odot, e)$.

Conversely, if $(H, \odot, e)$ is a K -ideal of K -algebra then $g \odot (g \odot x)$ and $x \odot g \in H$. This implies that $gxg^{-1} \in H$ for all $g \in G$ and $x \in H$. This completes the proof $\square$

Corollary 2.11. *The improper K -subalgebras of a K -algebra correspond to the improper K -ideals of the K -algebra $(G, \odot, e)$.*

Corollary 2.12. *If G is a simple group then the K -algebra is a simple algebra having no proper K -ideal.*

Theorem 2.13. *Let $K_1 = (G, \odot, e)$ be a K -algebra on G . If $Z(G)$ is the centre of the group G then $(Z(G), \odot, e) = Z(K_1)$, the centre of K_1 .*

Proof. From definition 2.2, if h commutes with g in K_1 then $g \odot (e \odot h) = h \odot (e \odot g)$. If $z_1, z_2 \in Z(G)$ then $z_1 \odot (e \odot g) = g \odot (e \odot z_1)$ and $z_2 \odot (e \odot g) = g \odot (e \odot z_2)$ for all $g \in G$. Then $z_1 \odot z_2 \in Z(K_1)$ since, for all $g \in G$,

$$\begin{aligned}
 (z_1 \odot z_2) \odot (e \odot g) &= z_1 \odot ((e \odot g) \odot (e \odot z_2)) \\
 &= z_1 \odot (e \odot (g \odot z_2)) \\
 &= z_1 \odot (z_2 \odot g) \\
 &= (z_1 \odot (e \odot g)) \odot z_2 \\
 &= (g \odot (e \odot z_1)) \odot z_2 \\
 &= g \odot (z_2 \odot (e \odot (e \odot z_1))) \\
 &= g \odot (z_2 \odot z_1) \\
 &= g \odot (e \odot (z_1 \odot z_2)) \quad \text{for all } g \in K_1.
 \end{aligned}$$

Hence $z_1 \odot z_2 \in Z(K_1)$, the centre of K_1 and

$$(Z(G), \odot, e) \subseteq Z(K_1) \quad (a)$$

For the converse, if $z_1, z_2 \in Z(K_1)$, then $z_1 \odot (e \odot g) = g \odot (e \odot z_1)$ and $z_2 \odot (e \odot g) = g \odot (e \odot z_2)$. Thus $z_1 \odot z_2 \in (Z(G), \odot, e)$ since

$$(z_1 \odot (e \odot g)) \odot (z_2 \odot (e \odot g)) = z_1 \odot z_2 = z_1 z_2^{-1} \in Z(G).$$

Hence

$$Z(K_1) \subseteq (Z(G), \odot, e) \quad (b)$$

Inequalities (a) and (b) imply the equality and the proof is completed. $\square$

3 K -homomorphisms of K -algebras

In a K -algebra $K(G) = (G, \odot, e)$ which is built on a group G , we see that set $K(G)$ = set G and hence the induced operation $\odot$ on G is defined. We see easily that the mapping: endomorphisms and automorphisms of $K(G)$ -algebra coincide set-wise [1] via actions on the respective group G .

In this section we generalize the notion of a mapping from one K -algebra $K_1 = (G_1, \odot, e_1)$ into another K -algebra $K_2 = (G_2, \odot, e_2)$. Thus we define a K -homomorphism in a usual way.

Definition 3.1. A mapping ψ from a K -algebra K_1 into K_2 is called a K -homomorphism if, for every $x_1, y_1 \in K_1$, $\psi(x_1 \odot y_1) = \psi(x_1) \odot \psi(y_1)$, where $\psi(x_1), \psi(y_1) \in K_2$.

Example 3.2. If $H = \{g \odot (g \odot x) : x \in G\}$ then mapping $\psi : (G, \odot) \rightarrow (H, \odot)$ is a K -homomorphism where, for a fixed g in G , $\psi(x) = g \odot (g \odot x)$ is a homomorphism from $(G, \odot)$ into $(H, \odot)$. It is easy to see that ψ is a bijection.

Remark. (1) A K -homomorphism $\psi : K_1 \rightarrow K_2$, from K_1 into K_2 is a group-homomorphism, from G_1 into G_2 and the vice versa.

- (2) A K -homomorphism ψ is called as usual, a monomorphism, epimorphism and isomorphism if ψ is injective, surjective and bijective respectively.
- (3) If a K -homomorphism ψ from K_1 into K_2 is a K -isomorphism then the algebras K_1 and K_2 are called isomorphic and are written by $K_1 \cong K_2$.
- (4) The relation of isomorphism ($\cong$) defined on the set of all K -algebras is an equivalence relation which subdivides K -algebras into disjoint equivalence classes.

Proposition 3.3. Let $K_1 = (G_1, \odot, e_1)$ and $K_2 = (G_2, \odot, e_2)$ be two K -algebras and $\psi \in \text{Hom}(K_1, K_2)$. Then, for $x_1, y_1 \in K_1$ and $\psi(x_1), \psi(y_1) \in K_2$, we conclude that:

- (1) $\psi(e_1) = e_2$.
- (2) $\psi(x) = \psi(x^{-1})$.
- (3) $\psi(e_1 \odot x_1) = e_2 \odot \psi(x_1)$.
- (4) $\psi(x_1 \odot x_2) = e_2$, if and only if $\psi(x_1) = \psi(x_2)$.
- (5) If H_1 is a subalgebra of K_1 then $\psi(H_1)$ is a subalgebra of K_2 .
- (6) If H_1 is an ideal of K_1 then $\psi(H_1)$ is an ideal of K_2 .

Proof. The proofs of (1), (2), (3) and (4) follow easily from the definition 3.1.

- (5) If $x_1, y_1 \in H_1$ then $x_1 \odot y_1 \in H_1$ as H_1 is a K -subalgebra of K_1 then $\psi(x_1), \psi(y_1)$ and $\psi(x_1 \odot y_1) \in \psi(H_1)$, a subset of K_2 . $\psi(H_1)$ is a K -subalgebra of K_2 since $\psi(x_1 \odot y_1) = \psi(x_1) \odot \psi(y_1) \in \psi(H_1)$. By (1) it follows that $\psi(H_1)$ is a K -subalgebra of K_2 .
- (6) Suppose that H_1 is an ideal of K_1 and $\psi(e_1) = e_2 \in \psi(H_1)$. Then, for all $g, x \in K_1$, $g \odot (g \odot x) \in H_1$ and $x \odot g \in H_1$ it implies that $x \in H_1$. Then $\psi(g) \odot (\psi(g) \odot \psi(x)) \in \psi(H_1)$ and $\psi(x \odot g) \in \psi(H_1) \rightarrow \psi(x) \in \psi(H_1)$, which implies that $\psi(H_1)$ is an ideal of K_2 .

□

Corollary 3.4. $\psi(x_1 \odot y_1) = \psi(y_1 \odot x_1)$, for all $x_1, y_1 \in K_1$.

Corollary 3.5. $\psi(x_1) = \psi(e_1 \odot x_1)$, for all $x_1 \in K_1$.

Corollary 3.6. If $\psi : K_1 \rightarrow K_2$ is an epimorphism then $\psi^{-1} \in \text{Hom}(K_2, K_1)$ and the Propositions (5) and (6) hold from K_2 into K_1 under $\psi^{-1} \in \text{Hom}(K_2, K_1)$.

Definition 3.7. Let $K_1 = (G_1, \odot, e_1)$ and $K_2 = (G_2, \odot, e_2)$ be two K -algebras and ψ be a homomorphism from K_1 into K_2 . The subset $\text{Ker}\psi = \{x \in K_1 : \psi(x) = e_2\}$ of K_1 is called the kernel of ψ .

We see that:

- (a) For each $\psi \in \text{Hom}(K_1, K_2)$ there exists $\text{Ker}\psi$ which contains at least the identity element e_1 of K_1 .
- (b) $\text{Ker}\psi = \{e_1\}$ if and only if ψ is a monomorphism.
- (c) For every $x_1, y_1 \in K_1$, $x_1 \odot y_1 \in \text{Ker}\psi$ if and only if $y_1 \odot x_1 \in \text{Ker}\psi$.
- (d) $\text{Ker}\psi$ is a subalgebra of K_1 .
- (e) $\text{Ker}\psi$ is an ideal of K_1 .

Theorem 3.8. Any two groups G_1 and G_2 build their respective isomorphic K -algebras $K_1 = (G_1, \odot, e_1)$ and $K_2 = (G_2, \odot, e_2)$ if and only if the groups G_1 and G_2 are isomorphic.

Proof. Let $G_1 \cong G_2$ and ψ be an isomorphism between G_1 and G_2 . Then $\psi(G_1) = G_2 \cong G_1$ and $\psi(G_1, \odot) = (G_2, \odot) \cong (G_1, \odot) \rightarrow K_1 \cong K_2$. The commutative property $\psi \circ \odot = \odot \circ \psi$ establishes the theorem since

$$\begin{aligned}
 (\psi \circ \odot)(x, y) &= \psi(\odot(x, y)) = \psi(x \odot y) \\
 &= \psi(x) \odot \psi(y) \\
 &= \odot(\psi(x), \psi(y)) \\
 &= (\odot \circ \psi)(x, y), \text{ for every } (x, y) \in G_1 \times G_2,
 \end{aligned}$$

and the following diagram commutes.

$$\begin{array}{ccc}
 G_1 & \xrightarrow{\psi} & G_2 \\
 \downarrow \odot & \swarrow \odot \circ \psi = \psi \circ \odot & \downarrow \odot \\
 K_1 & \xrightarrow{\psi} & K_2
 \end{array}$$

□

Theorem 3.9. Let K_1 and K_2 be two K -algebras. Let $\psi \in \text{Hom}(K_1, K_2)$. A relation ' $\sim$ ' defined on K_1 by, $x \sim y$, if and only if $x_1 \odot y_1 \in \ker\psi$, for all $x_1, y_1 \in K_1$, is an equivalence relation on K_1 .

Proof. The relation $\sim$ is reflexive on K_1 by definition 2.1. The relation is symmetric by Corollary 3.5 of Proposition 3.4.

For transitivity, take $x \sim y$ and $y \sim z$. Then

$$\begin{aligned}\psi(x \odot y) &= e_2 = \psi(y \odot z) \\ \psi((x \odot y) \odot (y \odot z)) &= e_2 \\ \psi((x \odot (y \odot z) \odot (e_1 \odot y))) &= e_2 \quad [\text{by } K_7] \\ \psi((x \odot (y \odot y) \odot (e_1 \odot z))) &= e_2 \quad [\text{by } K_7] \\ \psi((x \odot (e_1 \odot (e_1 \odot z)))) &= e_2 \quad [\text{by } K_3] \\ \psi(x \odot z) &= e_2 \quad [\text{by } K_8] \\ x &\sim z.\end{aligned}$$

Thus it proves the assertion. $\square$

For any homomorphism $\psi \in \text{Hom}(K_1, K_2)$ there corresponds an equivalence relation on K_1 which subdivides K_1 into a set of disjoint equivalence classes whose union is the whole of K_1 . Since the subdivision of K_1 is by $\ker\psi$, therefore, the set of all classes is called the quotient set of K_1 by $\ker\psi$ consisting of equivalence classes(cosets) denoted by

$$\frac{K_1}{\ker\psi} = \{C_x : x \in K_1 - \ker\psi\}$$

where

$$C_x = \ker\psi \odot x = x \odot \ker\psi,$$

$$C_x \odot C_y = C_{x \odot y}, \quad \text{for all } x, y \in K_1$$

and

$$C_{e_1} = \ker\psi.$$

Theorem 3.10. The quotient set $\frac{K_1}{\ker\psi}$ consisting of all equivalence classes of K_1 by $\ker\psi$ forms a K -algebra $(\frac{K_1}{\ker\psi}, \odot, \ker\psi)$, where $C_x \odot C_y = C_{x \odot y}$, for all $x, y \in K_1$ and $\ker\psi = e$ in the quotient algebra $\frac{K_1}{\ker\psi}$.

Proof. Let $C_x, C_y \in \frac{K_1}{\ker\psi}$ and $C_x \odot C_y = C_{x \odot y}$, for all $x, y \in K_1$. The properties (k_1) to (k_5) of definition 2.1 follow by the routine manipulation. Thus the quotient algebra $(\frac{K_1}{\ker\psi}, \odot, \ker\psi)$ verifies to be a K -algebra. $\square$

Corollary 3.11. *For each endomorphism ψ on K -algebra K_1 , the quotient algebra $(\frac{K_1}{\ker\psi}, \odot, \ker\psi)$ is formed of order $|\frac{K_1}{\ker\psi}|$, if K_1 is a finite K -algebra.*

Corollary 3.12. *For each ideal H of a K -algebra $K_1 = (G, \odot, e)$ there exists a quotient algebra $(\frac{K_1}{H}, \odot, H)$ as a homomorphic image of K_1 . That is, there exists a homomorphism $\theta : K_1 \rightarrow \frac{K_1}{H}$, defined by $\theta(x) = x \odot H$, where*

$$\begin{aligned}\theta(x \odot y) &= (x \odot y) \odot H \\ &= (x \odot H) \odot (y \odot H) \\ &= \theta(x) \odot \theta(y), \text{ for all } x, y \in K_1\end{aligned}$$

having $\ker\psi = H$. The homomorphism θ is called natural homomorphism and denoted by n_θ .

If ψ in $\text{Hom}(K_1, K_2)$ is an K -epimorphism then:

Theorem 3.13. *(Fundamental homomorphism theorem) Let K_1 and K_2 be any two K -algebras and ψ be an epimorphism from K_1 into K_2 . Then $\frac{K_1}{\ker\psi} \cong K_2$*

Proof. Let there be a trio of K -algebras; K_1 , K_2 and $\frac{K_1}{\ker\psi}$ for homomorphisms $\psi : K_1 \rightarrow K_2$ and $n_\theta : K_1 \rightarrow \frac{K_1}{\ker\psi}$, where ψ is an epimorphism. Let η be a mapping from $\frac{K_1}{\ker\psi}$ into K_2 defined by $\eta(C_x) = \psi(x)$ for all $x \in K_1$. Then η is a homomorphism since

$$\begin{aligned}\eta(C_x \odot C_y) &= \eta(C_{x \odot y}) = \psi(x \odot y) \\ &= \psi(x) \odot \psi(y) = \eta(C_x) \odot \eta(C_y) \quad \forall \quad C_x, C_y \in \frac{K_1}{\ker\psi}.\end{aligned}$$

In fact, η is an isomorphism because $\ker\eta = \ker\psi = C_{e_1}$ and ψ is an epimorphism onto K_2 . Thus

$$\frac{K_1}{\ker\psi} \cong K_2.$$

Consequently, the following diagram commutes.

$$\begin{array}{ccc} K_1 & \longrightarrow & K_2 \\ \downarrow & \nearrow & \\ \frac{K_1}{\ker\psi} & & \end{array}$$

□

4 Structure of homomorphisms of K -algebras

Let there be three K -algebras K_1 , K_2 and K_3 . If $\phi \in \text{Hom}(K_1, K_2)$ and $\psi \in \text{Hom}(K_2, K_3)$ then their composition $\psi \circ \phi \in \text{Hom}(K_1, K_3)$ which observes the closure property if $\psi, \phi \in \text{Hom}(K_1, K_1)$. The set $\text{End}(K_1)$, consisting of all endomorphisms of K_1 , forms a semigroup with the identity element under the operation of their composition ($\circ$) which is defined by the rule $(\psi \circ \phi)(x) = \psi(\phi(x))$, for all $x \in X$. It is known in [1] that $\text{Aut}(K_1) = \text{Aut}(G)$ is a group structure and K -algebra $(\text{Aut}(K_1), \odot, Id)$ built on $\text{Aut}(K_1)$ is proper if the automorphisms of K_1 are not all of order 2.

Theorem 4.1. *Let $K_1 = (G, \odot, e)$ be a $K(G)$ -algebra on a group G . Then $\text{Aut}(K_1) = (\text{Aut}(G), \odot, Id)$ is K -algebra on the group $(\text{Aut}(G), \circ)$, i.e., $\text{Aut}(K_1) = (\text{Aut}(G), \odot, Id)$.*

Proof. If $\phi, \psi \in \text{Aut}(G)$ and $\phi \circ \psi = \phi \circ \psi^{-1}$ then $\phi \odot Id = \phi$, $Id \odot \phi = \phi^{-1}$ and $\psi \circ \psi = Id$ confirm to the properties K4, K5 and K3 respectively of Definition 2.1. The properties K1 and K2 follow in routine computation. $\square$

Note that the homomorphic/ isomorphic K -algebraic models have similar structures as those of its group theoretic basis. Thus we state the following Theorems without proofs.

Theorem 4.2. *(The second isomorphism theorem) Let A be a K -subalgebra of K -algebra and let B be a K -ideal of K -algebra. Then*

- (i) B is a K -ideal of $A \odot B$,
- (ii) $A \cap B$ is a K -ideal of A ,
- (iii) $\frac{A \odot B}{B} \cong \frac{A}{A \cap B}$.

Theorem 4.3. *(The third isomorphism theorem) Let K_1 be a K -algebra having K -ideals A and B with $A \subseteq B$. Then*

- (i) $\frac{B}{A}$ is K -ideal of $\frac{K_1}{A}$,
- (ii) $\frac{K_1/A}{B/A} \cong \frac{K_1}{B}$.

Corollary 4.4. *The K -algebra $(G, \odot, e)$ is simple if and only if the group G is simple.*

Lemma 4.5. *(Zassenhaus lemma) Let A and B be K -subalgebras of a K -algebra $(G, \odot, e)$ and let A_1 and B_1 be K -ideals of A and B respectively. Then*

- (a) $A_1 \odot (A \cap B_1)$ is a K -ideal of $A_1 \odot (A \cap B)$,
- (b) $B_1 \odot (A_1 \cap B)$ is a K -ideal of $B_1 \odot (A \cap B)$,
- (c) $\frac{A_1 \odot (A \cap B)}{A_1 \odot (A \cap B_1)} \cong \frac{B_1 \odot (A \cap B)}{B_1 \odot (A_1 \cap B)}$.

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