Solution of the Wave Equation by Homotopy Perturbation Method

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Abstract

Homotopy perturbation method has been applied to solve many functional equations so far. In this article, we have uses this method to solve the wave equation, which governs numerous engineering experimentations, particularly in geosciences applications. Comparison of the results with those of Adomian's decomposition method [5] leads to significant consequences. Some special cases of the equation are solved as examples, to illustrate the reliable ness of the method.

Keywords: Homotopy perturbation method, Wave equation

1 Introduction

Since the governing equations on many experiments in engineering as well as science leads to the wave equation, this equation has attracted much attention, and solving the equation has been one of the interesting tasks for mathematicians. Analytical methods commonly used for solving the wave equation are very restricted and can be used in very special cases so they can not be used to solve equations of numerous realistic scenarios. Numerical techniques, which are commonly used, encounter difficulties in terms of the size of computational works needed and usually the round-off error causes the loss of accuracy. Homotopy perturbation method well addressed by He in [1-4] has been applied to

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solve many functional equations. This method has proven to be very effective and results in considerable saving in computation time.

2 Basic idea of homotopy perturbation method by He

To illustrate the basic ideas of the method, we consider nonlinear differential equation

$$A(u) = f(r), \quad r \in \Omega, \quad (1)$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω . The operator A can, generally speaking, be divided into two parts L and N , where L is linear, and N is nonlinear operator. Eq. (1), therefore, can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0.$$

By the homotopy technique [1], we construct a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (2)$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eq. (2) we have

$$\begin{aligned} H(v, 0) &= L(v) - L(u_0) = 0, \\ H(v, 1) &= A(v) - f(r) = 0, \end{aligned}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and quantities $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Suppose, the solution of Eq. (2) can be written as a power series in p ,

$$v = v_0 + pv_1 + p^2v_2 + \dots$$

Setting $p=1$ results in approximate solution of Eq. (1)

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots$$

3 The homotopy perturbation to wave equations

Consider the following general form of wave equation with the indicated initial and boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = A(x, y, z, t) \frac{\partial^2 u}{\partial x^2} + B(x, y, z, t) \frac{\partial^2 u}{\partial y^2} + C(x, y, z, t) \frac{\partial^2 u}{\partial z^2} + D(x, y, z, t) \quad (3)$$

With the initial conditions

$$u(x, y, z, 0) = f(x, y, z), \quad \frac{\partial u}{\partial t}(x, y, z, 0) = g(x, y, z) \quad (4)$$

Or the following boundary conditions

$$u(L, y, z, t) = f_1(y, z, t), \quad \frac{\partial u}{\partial x}(L, y, z, t) = g_1(x, y, z) \quad (5)$$

For solving Eq. (3) with initial conditions (4) according to the homotopy perturbation, we construct the following homotopy:

$$(1-p) \left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} \right) + p \left(\frac{\partial^2 v}{\partial t^2} - A \frac{\partial^2 v}{\partial x^2} - B \frac{\partial^2 v}{\partial y^2} - C \frac{\partial^2 v}{\partial z^2} - D \right) = 0$$

or

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(A \frac{\partial^2 v}{\partial x^2} + B \frac{\partial^2 v}{\partial y^2} + C \frac{\partial^2 v}{\partial z^2} + D - \frac{\partial^2 u_0}{\partial t^2} \right) \quad (6)$$

with initial approximation $u_0 = f(x, y, z) + g(x, y, z)t$. Suppose the solution of Eq. (6) has the form

$$v = v_0 + pv_1 + p^2v_2 + \dots \quad (7)$$

Substituting (7) into (6), and equating the terms with the identical powers of p ,

$$\begin{aligned} p^0 : \frac{\partial^2 v_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} &= 0 \\ p^1 : \frac{\partial^2 v_1}{\partial t^2} &= A \frac{\partial^2 v_0}{\partial x^2} + B \frac{\partial^2 v_0}{\partial y^2} + C \frac{\partial^2 v_0}{\partial z^2} + D - \frac{\partial^2 u_0}{\partial t^2}, \quad v_1(x, y, z, 0) = \frac{\partial v_1}{\partial t}(x, y, z, 0) \\ p^2 : \frac{\partial^2 v_2}{\partial t^2} &= A \frac{\partial^2 v_1}{\partial x^2} + B \frac{\partial^2 v_1}{\partial y^2} + C \frac{\partial^2 v_1}{\partial z^2}, \quad v_2(x, y, z, 0) = \frac{\partial v_2}{\partial t}(x, y, z, 0) \\ &\vdots \end{aligned} \quad (8)$$

For simplicity we always take $v_0 = u_0 = f(x, y, z) + g(x, y, z)t$. So we have

$$\begin{aligned} v_1 &= \int_0^t \int_0^t \left(A(x, y, z, \xi) \frac{\partial^2 v_0}{\partial x^2} + B(x, y, z, \xi) \frac{\partial^2 v_0}{\partial y^2} + C(x, y, z, \xi) \frac{\partial^2 v_0}{\partial z^2} + D(x, y, z, \xi) \right) d\xi dt \\ v_2 &= \int_0^t \int_0^t \left(A(x, y, z, \xi) \frac{\partial^2 v_1}{\partial x^2} + B(x, y, z, \xi) \frac{\partial^2 v_1}{\partial y^2} + C(x, y, z, \xi) \frac{\partial^2 v_1}{\partial z^2} \right) d\xi dt \\ &\vdots \end{aligned} \quad (9)$$

The approximate solution of (3), therefore, can be obtained by setting $p = 1$

$$u = v_0 + v_1 + v_2 + \cdots$$

Similarly, to solve Eq. (3) with boundary condition (5) we construct the following homotopy

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = p \left(\frac{1}{A} \frac{\partial^2 v}{\partial t^2} - \frac{B}{A} \frac{\partial^2 v}{\partial y^2} - \frac{C}{A} \frac{\partial^2 v}{\partial z^2} - \frac{D}{A} - \frac{\partial^2 u_0}{\partial x^2} \right) \quad (10)$$

with the initial approximation $v_0 = u_0 = f_1(y, z, t) + g_1(y, z, t)(x - L)$. Suppose the solution of Eq. (10) has the form (7), according to scheme mentioned above we have

$$\begin{aligned} p^0 &: \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = 0 \\ p^1 &: \frac{\partial^2 v_1}{\partial x^2} = \frac{1}{A} \frac{\partial^2 v_0}{\partial t^2} - \frac{B}{A} \frac{\partial^2 v_0}{\partial y^2} - \frac{C}{A} \frac{\partial^2 v_0}{\partial z^2} - \frac{D}{A} - \frac{\partial^2 u_0}{\partial x^2}, \quad v_1(L, y, z, t) = \frac{\partial v_1}{\partial x}(L, y, z, t) = 0 \\ p^2 &: \frac{\partial^2 v_2}{\partial x^2} = \frac{1}{A} \frac{\partial^2 v_1}{\partial t^2} - \frac{B}{A} \frac{\partial^2 v_1}{\partial y^2} - \frac{C}{A} \frac{\partial^2 v_1}{\partial z^2}, \quad v_2(L, y, z, t) = \frac{\partial v_2}{\partial x}(L, y, z, t) = 0 \\ &\vdots \end{aligned} \quad (11)$$

Accordingly we have:

$$\begin{aligned} v_1 &= \int_L^x \int_L^x \left(\frac{1}{A(\xi, y, z, t)} \frac{\partial^2 v_0}{\partial t^2} - \frac{B(\xi, y, z, t)}{A(\xi, y, z, t)} \frac{\partial^2 v_0}{\partial y^2} - \frac{C(\xi, y, z, t)}{A(\xi, y, z, t)} \frac{\partial^2 v_0}{\partial z^2} - \frac{D(\xi, y, z, t)}{A(\xi, y, z, t)} \right) d\xi dx \\ v_2 &= \int_L^x \int_L^x \left(\frac{1}{A(\xi, y, z, t)} \frac{\partial^2 v_1}{\partial t^2} - \frac{B(\xi, y, z, t)}{A(\xi, y, z, t)} \frac{\partial^2 v_1}{\partial y^2} - \frac{C(\xi, y, z, t)}{A(\xi, y, z, t)} \frac{\partial^2 v_1}{\partial z^2} \right) d\xi dx \\ &\vdots \end{aligned} \quad (12)$$

Setting $p = 1$, results in approximation solution of E.q (3),

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \cdots$$

4 Examples

To illustrate the method and to show the ability of the method so examples are presented here.

Example 1. Consider the wave equation with the following initial conditions [5]:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial y^2} + C \frac{\partial^2 u}{\partial z^2} + D \\ u(x, y, z, 0) = a_1 x + a_2 x^2 + b_1 y + b_2 y^2 + c_1 z + c_2 z^2 \\ \frac{\partial u}{\partial t}(x, y, z, 0) = a'_1 x + a'_2 x^2 + b'_1 y + b'_2 y^2 + c'_1 z + c'_2 z^2 \end{cases}$$

Where A, B, C and D are constant. Suppose $u = u_0 + pu_1 + p^2 u_2 + \dots$, according to (6), we have

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial y^2} + C \frac{\partial^2 u}{\partial z^2} + D - \frac{\partial^2 u_0}{\partial t^2} \right)$$

Beginning with

$$u_0 = a_1 x + a_2 x^2 + b_1 y + b_2 y^2 + c_1 z + c_2 z^2 + (a'_1 x + a'_2 x^2 + b'_1 y + b'_2 y^2 + c'_1 z + c'_2 z^2) t$$

From (8) we have

$$\frac{\partial^2 u_1}{\partial t^2} = 2(Aa_2 + Bb_2 + Cc_2) + 2(Aa'_2 + Bb'_2 + Cc'_2) t + D,$$

$$u_1(x, y, z, 0) = \frac{\partial u_1}{\partial t}(x, y, z, 0) = 0$$

And we derive the following results,

$$u_1(x, y, z, t) = (Aa_2 + Bb_2 + Cc_2 + \frac{1}{2} D) t^2 + (Aa'_2 + Bb'_2 + Cc'_2) \frac{t^3}{3}$$

So the first-order approximate solution is

$$u = u_0 + u_1 = a_1 x + a_2 x^2 + b_2 y^2 + c_1 z + c_2 z^2 + (a'_1 x + a'_2 x^2 + b'_1 y + b'_2 y^2 + c'_1 z + c'_2 z^2) t + (Aa_2 + Bb_2 + Cc_2 + \frac{1}{2} D) t^2 + (Aa'_2 + Bb'_2 + Cc'_2) \frac{t^3}{3}$$

which is an exact solution.

Example 2. Let us solve the following partial differential equation [5]:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{1}{6} \left(x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} \right) \\ u(x, y, z, 0) = x^2 y^2 z^2 \\ \frac{\partial u}{\partial t}(x, y, z, 0) = -x^2 y^2 z^2 \end{cases}$$

Its homotopy can be written down as follows

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(\frac{1}{6} \left(x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} \right) - \frac{\partial^2 u_0}{\partial t^2} \right)$$

Beginning with $u_0 = x^2 y^2 z^2 - x^2 y^2 z^2 t$, from (6), we have

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} &= x^2 y^2 z^2 - x^2 y^2 z^2 t, & u_1(x, y, z, 0) &= \frac{\partial u_1}{\partial t}(x, y, z, 0) = 0 \\ \frac{\partial^2 u_2}{\partial t^2} &= x^2 y^2 z^2 \frac{t^2}{2!} - x^2 y^2 z^2 \frac{t^3}{3!}, & u_2(x, y, z, 0) &= \frac{\partial u_2}{\partial t}(x, y, z, 0) = 0 \\ &\vdots \\ \frac{\partial^2 u_n}{\partial t^2} &= x^2 y^2 z^2 \frac{t^{2n-2}}{(2n-2)!} - x^2 y^2 z^2 \frac{t^{2n-1}}{(2n-1)!}, & u_n(x, y, z, 0) &= \frac{\partial u_n}{\partial t}(x, y, z, 0) = 0 \end{aligned}$$

Applying the inverse of differentiation operator, we have

$$\begin{aligned} u_1(x, y, z, t) &= x^2 y^2 z^2 \left(\frac{t^2}{2!} - \frac{t^3}{3!} \right) \\ u_2(x, y, z, t) &= x^2 y^2 z^2 \left(\frac{t^4}{4!} - \frac{t^5}{5!} \right) \\ &\vdots \\ u_n(x, y, z, t) &= x^2 y^2 z^2 \left(\frac{t^{2n}}{(2n)!} - \frac{t^{2n+1}}{(2n+1)!} \right) \end{aligned}$$

Thus we have $u(x, y, z, t) = x^2 y^2 z^2 \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} = x^2 y^2 z^2 e^{-t}$, which is an exact solution.

Example 3. Consider the following partial equation with specified initial conditions [5] :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = e^{-xt} \frac{\partial^2 u}{\partial x^2} + e^{-yt} \frac{\partial^2 u}{\partial y^2} + e^{-zt} \frac{\partial^2 u}{\partial z^2} \\ u(x, y, z, 0) = x^2 + y^2 + z^2 \\ \frac{\partial u}{\partial t}(x, y, z, 0) = 0 \end{cases}$$

Homotopy perturbation method consist of the following scheme:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2} = p \left(e^{-xt} \frac{\partial^2 u}{\partial x^2} + e^{-yt} \frac{\partial^2 u}{\partial y^2} + e^{-zt} \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u_0}{\partial t^2} \right)$$

Beginning with $u_0(x, y, z, t) = x^2 + y^2 + z^2$, by using (8) and (9) we have

$$u_1(x, y, z, t) =$$

$$\frac{2((e^{-xt}(1+x) - 1) y^2 z^2 + (e^{-yt}(1+y) - 1) x^2 z^2 + (e^{-zt}(1+z) - 1) y^2 x^2)}{x^2 y^2 z^2}.$$

And u_2 can be calculated similarly. Considering $u \approx u_0 + u_1 + u_2$, therefore,

$$\begin{aligned} u(x, y, z, t) &\approx \frac{1}{y^6 x^6 z^6} (-2tz^6 y^6 x^5 - 24e^{(-yt)} z^6 x^6 y + 2z^5 y^6 x^6 + 2e^{(-2xt)} t^2 y^6 x^2 z^6 + 24e^{(-2zt)} y^6 x^6 z \\ &+ 24e^{(-2xt)} y^6 x^6 z^6 + 24te^{(-zt)} y^6 x^6 z - 40e^{(-zt)} y^6 x^6 - 2tz^5 y^6 x^6 + 40e^{(-2yt)} z^6 x^6 \\ &+ 40e^{(-2xt)} z^6 y^6 + 12te^{(-2zt)} y^6 x^6 z^2 + 12te^{(-2xt)} y^6 z^6 x^2 + 16te^{(-2xt)} y^6 z^6 x \\ &+ 16te^{(-2zt)} y^6 x^6 z + 12te^{(-2yt)} y^2 z^6 x^6 + y^8 z^6 x^6 + z^8 y^6 x^6 + x^8 z^6 y^6 \\ &+ 16te^{(-2yt)} z^6 x^6 y + 24e^{(-2yt)} z^6 x^6 y - 6ty^6 z^6 x^6 + 2y^5 z^6 x^6 + y^6 z^6 x^6 + 12te^{(-zt)} z^2 x^6 y^2 \\ &- 2ty^5 z^6 x^6 + 24te^{(-xt)} z^6 y^6 x + 40e^{(-2zt)} x^6 y^6 + 12te^{(-xt)} z^6 y^6 x^2 + 2e^{(-2yt)} t^2 z^6 y^2 x^6 \\ &+ 2e^{(-2yt)} t^2 z^6 y^3 x^6 - 40e^{(-xt)} z^6 y^6 + 2z^6 y^6 x^5 - 40e^{(-yt)} z^6 x^6 - 24e^{(-xt)} z^6 y^6 x \\ &- 24e^{(-zt)} x^6 y^6 z + 12te^{(-yt)} z^6 y^2 x^6 + 24te^{(-yt)} z^6 y x^6 - 6t^2 e^{(-zt)} z^2 y^6 x^6 \\ &- 6t^2 e^{(-xt)} z^6 y^6 x^2 - 6t^2 e^{(-yt)} z^3 y^6 x^6 + 6t^2 e^{(-yt)} z^6 y^6 x^6 + t^2 e^{(-zt)} z^6 y^6 x^6 \\ &+ t^2 e^{(-yt)} z^6 y^6 x^6 + 2t^2 e^{(-2zt)} z^3 y^6 x^6 + 2t^2 e^{(-2zt)} z^2 y^6 x^6 + 2t^2 e^{(-2xt)} z^6 y^6 x^3 \end{aligned}$$

Example 4. Consider the following partial differential equation with the following boundary conditions:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial y^2} + C \frac{\partial^2 u}{\partial z^2} + D \\ u(0, y, z, t) = a_1 y + a_2 y^2 + b_1 z + b_2 z^2 \\ \frac{\partial u}{\partial x}(0, y, z, t) = (a'_1 y + a'_2 y^2 + b'_1 z + b'_2 z^2) t \end{cases}$$

Where A, B, C and D are constant. From (10) we derive

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = p \left(\frac{1}{A} \frac{\partial^2 u}{\partial t^2} + \frac{B}{A} \frac{\partial^2 u}{\partial y^2} - \frac{C}{A} \frac{\partial^2 u}{\partial z^2} - \frac{D}{A} - \frac{\partial^2 u_0}{\partial x^2} \right)$$

By imposing $u_0(x, y, z, t) = a_1 y + a_2 y^2 + b_1 z + b_2 z^2 + (a'_1 y + a'_2 y^2 + b'_1 z + b'_2 z^2) x t$
From (11), we have

$$\frac{\partial^2 u_1}{\partial x^2} = -\frac{1}{A} (2(a_2 B + b_2 C) + D - 2(a'_2 B + b'_2 C) x t),$$

$$u_1(x, y, z, t) = \frac{\partial u_1}{\partial x}(0, y, z, t) = 0$$

The solution reads

$$u_1(x, y, z, t) = -\frac{1}{A} \left((a_2 B + b_2 C + \frac{1}{2} D) x^2 + (a'_2 B + b'_2 C) \frac{t x^3}{3} \right).$$

It can be easily shown $u_i = 0$, for $i = 2, 3, \dots$ and so an exact solution as follows

$$u(x, y, z, t) = u_0 + u_1 = a_1 y + a_2 y^2 + b_1 z + b_2 z^2 + (a'_1 y + a'_2 y^2 + b'_1 z + b'_2 z^2) x t - \frac{1}{A} \left((a_2 B + b_2 C + \frac{1}{2} D) x^2 + a'_2 B + b'_2 C \right) \frac{t x^3}{3}.$$

Example 5. Let us solve the following partial differential equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} + z \frac{\partial^2 u}{\partial z^2} \\ u(1, y, z, t) = y z t^2 \\ \frac{\partial u}{\partial t}(1, y, z, t) = -y z t^2 \end{cases}$$

Its homotopy can be written down as follows:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_0}{\partial x^2} = p \left(\frac{1}{x} \frac{\partial^2 u}{\partial t^2} - \frac{y}{x} \frac{\partial^2 u}{\partial y^2} - \frac{z}{x} \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u_0}{\partial x^2} \right)$$

Beginning with $u_0 = 2yzt^2 - xyz t^2$, and using (11), (12) and (13) we get

$$u_1(x, y, z, t) = 4xyz \ln x - 2xyz - x^2yz + 3yz, u_i(x, y, z, t) = 0, i = 2, 3, \dots$$

Thus we have

$$u(x, y, z, t) = u_0 + u_1 = 2yzt^2 - xyz t^2 + 4xyz \ln x - x^2yz - 2xyz + 3yz,$$

which is an exact solution. It's worth to point out that, the results in examples 1, 2 and 3 are exactly the same as Adomian decomposition method [5].

5 Conclusions

The main goal of this article, was derive an analytical approximation solution for the wave equation. We have achieved this goal by applying homotopy perturbation method. In some examples we have derive an exact solution but in others a very accurate analytical approximation. Compared with Adomian decomposition method [5], the present method there is no need to calculate Adomian polynomials. The small size of computations, in comparison with the computational size required in numerical methods and the rapid convergence shows that the homotopy perturbation method is reliable, and introduces a significant improvement in solving the wave equation over existing methods. Maple 10 is used to carry computations.

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