International Mathematical Forum, 2, 2007, no. 48, 2369 - 2378

On an Extension of Hardy-Hilbert's Integral Inequality with Three Parameters

Laith Emil Azar

Department of Mathematics
Al al-Bayt University
P.O. Box: 130095 Mafraq, Jordan
azar_laith@yahoo.com

Abstract. In this paper by introducing three parameters we establish anextension of Hardy-Hilbert's integral inequality. As an application we give it's equivalent form.

Mathematics Subject Classifications: 26D15

Keywords: Hardy-Hilbert's integral inequality,weight function, equivalent form

1. Introduction

If $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x), g(x) > 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following two equivalent inequalities as (see[1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}} \qquad (1.1)$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^p dy < \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(x) dx \tag{1.2}$$

where the constant factors $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ and $\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right)^p$ are the best possible in (1.1) and (1.2) respectively. Inequality (1.1) is called Hardy-Hilbert's integral inequality, which is important in analysis and it's applications (see[2]). Recently, various extensions on inequality (1.1) have appeared in some papers such as [3],[4], [5] and [6].

In 1998, by introducing a parameter $\lambda \in (0, 1]$ and the Beta function $B(u, v)$ as

$$B(u,v) = \int_0^\infty \frac{x^{u-1}}{(x+1)^{u+v}} dx = B(v,u) \ (u,v > 0), \tag{1.3}$$

Yang [4,5] gave a generalization of (1.1) and (1.2) as: If $\lambda > 2 - \min\{p, q\}$, f, g are non-negative functions such that

$$0 < \int_0^\infty x^{1-\lambda} f^p(x) dx < \infty, \text{ and } 0 < \int_0^\infty x^{1-\lambda} g^q(x) dx < \infty,$$

then the following two inequalities are equivalent

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)\,\lambda} dxdy < k\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} f^q(x) dx \right\}^{\frac{1}{q}}$$

and

$$\int_0^\infty y^{(p-1)(\lambda-1)} \left[\int_0^\infty \frac{f(x)}{(x+y)\,\lambda} dx \right]^p dy < [k\lambda(p)]^p \int_0^\infty x^{1-\lambda} f^p(x) dx$$

where the constant factors $k\lambda(p) = B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})$ and $[k\lambda(p)]^p$ are the best possible.

In 1999, Kuang [3] gave a generalization with a parameter λ of (1.1) as follows:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x\alpha+y\alpha}dxdy < h\alpha(p)\left\{\int_0^\infty x^{1-\alpha}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{1-\alpha}g^q(x)dx\right\}^{\frac{1}{q}}$$

where $\max\left\{\frac{1}{p},\frac{1}{q}\right\}< \alpha \leq 1,\ h\alpha(p) = \pi\left\{\alpha\sin^{\frac{1}{p}}\left(\frac{\pi}{p\alpha}\right)\sin^{\frac{1}{q}}\left(\frac{\pi}{q\alpha}\right)\right\}^{-1}$. Because of the constant factor $h\alpha(p)$ being not the best possible, Yang [6] gave a new generalization of (1.1) as:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x\alpha+y\alpha}dxdy < \frac{\pi}{\alpha\sin\left(\frac{\pi}{p}\right)}\left\{\int_0^\infty x^{(1-\alpha)(p-1)}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{(1-\alpha)(q-1)}g^q(x)dx\right\}^{\frac{1}{q}}$$

it's equivalent form is:

$$\int_0^\infty y^{\alpha-1}\left[\int_0^\infty \frac{f(x)}{x\alpha+y\alpha}dx\right]^p dy < \left[\frac{\pi}{\alpha\sin\left(\frac{\pi}{p}\right)}\right]^p\int_0^\infty x^{(1-\alpha)(p-1)}f^p(x)dx$$

where the constant factors $\frac{\pi}{\alpha\sin\left(\frac{\pi}{p}\right)}$, $\left[\frac{\pi}{\alpha\sin\left(\frac{\pi}{p}\right)}\right]^p$ are the best possible.

In this paper by introducing three parameters, our aim is to estimate the double integral:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x\alpha+y\beta)\,\lambda}dxdy.$$

2. Main Results

Theorem 1. *If $f,\ g \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \alpha, \beta, \lambda > 0$ such that $0 < \int\limits_0^\infty x^{p-\alpha\lambda-1} f^p(x)dx < \infty$, and $0 < \int\limits_0^\infty x^{q-\beta\lambda-1} g^q(x)dx < \infty$, then*

$$\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{(x\alpha + y\beta)\,\lambda} dxdy \leq \frac{B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}} \left\{ \int\limits_0^\infty x^{p-\alpha\lambda-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int\limits_0^\infty x^{q-\beta\lambda-1} g^q(x)dx \right\}^{\frac{1}{q}} \tag{2.1}$$

where the constant factor $\frac{B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}} \beta^{\frac{1}{p}}}$ is the best possible when $\alpha = \beta$.

Proof. Define two functions,

$$\mu = \frac{f(x)\left(x^{\frac{\alpha\lambda}{p}} y^{\frac{\beta\lambda}{q}-1}\right)^{\frac{1}{p}} x^{\frac{\alpha\lambda}{p}-\frac{1}{q}}}{(x\alpha + y\beta)^{\frac{\lambda}{p}}} \text{ and } \varphi = \frac{g(y)\left(x^{\frac{\alpha\lambda}{p}-1} y^{\frac{\beta\lambda}{q}}\right)^{\frac{1}{q}} y^{\frac{\beta\lambda}{q}-\frac{1}{p}}}{(x\alpha + y\beta)^{\frac{\lambda}{q}}}.$$

Let us estimate the left-hand side of (2.1) by Holder's inequality, we obtain

$$\begin{aligned}
\int\limits_0^\infty \int\limits_0^\infty \frac{f(x)g(y)}{(x\alpha + y\beta)\,\lambda} dxdy &= \int\limits_0^\infty \int\limits_0^\infty \mu\varphi dxdy \\
&\leq \left\{ \int\limits_0^\infty \int\limits_0^\infty \mu^p dxdy \right\}^{\frac{1}{p}} \left\{ \int\limits_0^\infty \int\limits_0^\infty \varphi^q dxdy \right\}^{\frac{1}{q}} \\
&= \left\{ \int\limits_0^\infty \int\limits_0^\infty \frac{x^{\frac{\alpha\lambda}{p}} y^{\frac{\beta\lambda}{q}-1}}{(x\alpha + y\beta)\,\lambda} x^{p-\alpha\lambda-1} f^p(x)dxdy \right\}^{\frac{1}{p}} \times \\
&\quad \times \left\{ \int\limits_0^\infty \int\limits_0^\infty \frac{x^{\frac{\alpha\lambda}{p}-1} y^{\frac{\beta\lambda}{q}}}{(x\alpha + y\beta)\,\lambda} y^{q-\beta\lambda-1} g^q(y)dxdy \right\}^{\frac{1}{q}} \\
&= \left\{ \int\limits_0^\infty \omega_p x^{p-\alpha\lambda-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int\limits_0^\infty \omega_q y^{q-\beta\lambda-1} g^q(y)dy \right\}^{\frac{1}{q}} \quad (2.2)
\end{aligned}$$

where $\omega_p = \int\limits_0^\infty \frac{x^{\frac{\alpha\lambda}{p}} y^{\frac{\beta\lambda}{q}-1}}{(x\alpha + y\beta)\,\lambda} dy$, and $\omega_q = \int\limits_0^\infty \frac{x^{\frac{\alpha\lambda}{p}-1} y^{\frac{\beta\lambda}{q}}}{(x\alpha + y\beta)\,\lambda} dx$.

If (2.1) takes the form of an inequality, then there exists real numbers A and B such that they are not all zero and,

$$\frac{Ax^{\frac{\alpha\lambda}{p}}y^{\frac{\beta\lambda}{q}-1}}{(x\alpha+y\beta)\,\lambda}x^{p-\alpha\lambda-1}f^p(x)=\frac{Bx^{\frac{\alpha\lambda}{p}-1}y^{\frac{\beta\lambda}{q}}}{(x\alpha+y\beta)\,\lambda}y^{q-\beta\lambda-1}g^q(y),\ \text{a.e. in }\ (0,\infty)\times(0,\infty)\,.$$

Hence, we find

$$Ax^{p-\alpha\lambda}f^p(x)=By^{q-\beta\lambda}g^q(y),\ \text{a.e. in }\ (0,\infty)\times(0,\infty)\,.$$

It follows that there exists a constant C, such that

$$Ax^{p-\alpha\lambda}f^p(x)=C,\ \text{a.e. in }\ (0,\infty)$$

$$By^{q-\beta\lambda}g^q(y)=C\ \text{a.e. in }\ (0,\infty)\,.$$

Without loss of generality, suppose that $A\neq 0$. Then we have

$$x^{p-\alpha\lambda-1}f^p(x)=\frac{C}{Ax},\ \text{a.e. in }\ (0,\infty)\,,$$

which contradicts the fact that $0<\int_0^\infty x^{p-\alpha\lambda-1}f^p(x)dx<\infty$. Therefore, (2.1) takes the form of strict inequality, and we may rewrite (2.1) as

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{(x\alpha+y\beta)\,\lambda}dxdy<\frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}\left\{\int_0^\infty x^{p-\alpha\lambda-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q-\beta\lambda-1}g^q(x)dx\right\}^{\frac{1}{q}}.$$

We compute the weight function ω_p as follows, let $u=\dfrac{y\beta}{x\alpha}$, then we obtain by (1.3)

$$\omega_p=\frac{1}{\beta}\int_0^\infty\frac{u^{\frac{\lambda}{q}-1}}{(u+1)\,\lambda}du=\frac{1}{\beta}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\tag{2.3}$$

and similarly,

$$\omega_q = \frac{1}{\alpha} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \tag{2.4}$$

From (2.2), (2.3) and (2.4) we get (2.1).

We need to show that the constant factor $\frac{B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}$ contained in (2.1) is the best possible when $\alpha = \beta$. To do that we define two functions

$$f_0(x) = \begin{cases} 0, \, x \in [0,1) \\ x^{\frac{\alpha\lambda - p - \varepsilon}{p}}, \, x \in [1, +\infty) \end{cases}$$

and

$$g_0(y) = \begin{cases} 0, \, y \in [0,1) \\ y^{\frac{\beta\lambda - q - \varepsilon}{q}}, \, y \in [1, +\infty) \end{cases}$$

Assume that $0 < \varepsilon < \alpha\lambda$. Suppose that $\frac{B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}$ is not the best possible, then there exists $0 < K < \frac{B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}$ such that

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f_0(x) g_0(y)}{(x\alpha + y\beta)\,\lambda} dxdy &< K \left\{ \int_0^\infty x^{p-\alpha\lambda-1} f_0^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q-\beta\lambda-1} g_0^q(y) dy \right\}^{\frac{1}{q}} \\ &= K \left\{ \int_1^\infty x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty y^{-1-\varepsilon} dy \right\}^{\frac{1}{q}} = \frac{K}{\varepsilon} \end{aligned} \tag{2.5}$$

On the other hand, setting $u = \frac{x\alpha}{y\beta}$,we have

$$
\begin{aligned}
\int_0^\infty \int_0^\infty \frac{f_0(x)g_0(y)}{(x\alpha+y\beta)\,\lambda}dxdy &= \int_1^\infty \int_1^\infty \frac{x^{\frac{\alpha\lambda-p-\varepsilon}{p}}y^{\frac{\beta\lambda-q-\varepsilon}{q}}}{(x\alpha+y\beta)\,\lambda}dxdy \\
&= \frac{1}{\alpha}\int_1^\infty y^{-\frac{\varepsilon}{q}-\frac{\varepsilon\beta}{\alpha p}-1}\left\{\int_{\frac{1}{y\beta}}^\infty \frac{u^{\frac{\lambda}{p}-1}u^{-\frac{\varepsilon}{\alpha p}}}{(u+1)\,\lambda}du\right\}dy \\
&= \frac{1}{\alpha}\int_1^\infty y^{-\frac{\varepsilon}{q}-\frac{\varepsilon\beta}{\alpha p}-1}\left\{\int_0^\infty \frac{u^{\frac{\lambda}{p}-1}u^{-\frac{\varepsilon}{\alpha p}}}{(u+1)\,\lambda}du-\int_0^{\frac{1}{y\beta}} \frac{u^{\frac{\lambda}{p}-1}u^{-\frac{\varepsilon}{\alpha p}}}{(u+1)\,\lambda}du\right\}dy \\
&= \frac{1}{\alpha}\int_1^\infty y^{-\frac{\varepsilon}{q}-\frac{\varepsilon\beta}{\alpha p}-1}\left\{B(\frac{\lambda}{p},\frac{\lambda}{q})+o(1)-\int_0^{\frac{1}{y\beta}} \frac{u^{\frac{\lambda}{p}-\frac{\varepsilon}{\alpha p}-1}}{(u+1)\,\lambda}du\right\}dy \\
&= \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})+o(1)}{\varepsilon\left(\frac{\alpha}{q}+\frac{\beta}{p}\right)}-\frac{1}{\alpha}\int_1^\infty y^{-\frac{\varepsilon}{q}-\frac{\varepsilon\beta}{\alpha p}-1}\left[\int_0^{\frac{1}{y\beta}} \frac{u^{\frac{\lambda}{p}-\frac{\varepsilon}{\alpha p}-1}}{(u+1)\,\lambda}du\right]dy \\
&> \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})+o(1)}{\varepsilon\left(\frac{\alpha}{q}+\frac{\beta}{p}\right)}-\frac{1}{\alpha}\int_1^\infty y^{-1}\left[\int_0^{\frac{1}{y\beta}} u^{\frac{\lambda}{p}-\frac{\varepsilon}{\alpha p}-1}du\right]dy \\
&= \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})+o(1)}{\varepsilon\left(\frac{\alpha}{q}+\frac{\beta}{p}\right)}-\frac{\alpha p^2}{\beta\left(\alpha\lambda-\varepsilon\right)^2} \\
&= \frac{B(\frac{\lambda}{p},\frac{\lambda}{q})+o(1)}{\varepsilon\left(\frac{\alpha}{q}+\frac{\beta}{p}\right)}-O(1) \\
&= \left[\frac{B(\frac{\lambda}{p},\frac{\lambda}{q})+o(1)}{\varepsilon\left(\frac{\alpha}{q}+\frac{\beta}{p}\right)}\right]\quad\left(\varepsilon\to 0^+\right) \qquad (2.6)
\end{aligned}
$$

By Young's inequality, we have $\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}} \leq \dfrac{\alpha}{q}+\dfrac{\beta}{p}$ (i.e. $\frac{1}{\frac{\alpha}{q}+\frac{\beta}{p}} \leq \frac{1}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}$). Consider the form of equality, we get $\alpha=\beta$. Then

$$\frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{\left(\frac{\alpha}{q}+\frac{\beta}{p}\right)}+o(1)=\frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}+o(1)<k$$

Clearly, when $\varepsilon \to 0^{+}$, the inequality (2.5) is in contradiction with (2.6). Thus the constant factor $\frac{B(\frac{\lambda}{p},\frac{\lambda}{q})}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}$ is the best possible for $\alpha=\beta$, and the proof of the theorem is completed.

Remark. For $\alpha=\beta$, inequality (2.1) becomes

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x\alpha+y\alpha)\,\lambda}dxdy<\frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha}\left\{\int_0^\infty x^{p-\alpha\lambda-1}f^{p}(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q-\alpha\lambda-1}g^{q}(x)dx\right\}^{\frac{1}{q}},$$

where the constant factor $\frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha}$ is the best possible.

Theorem 2 If $f>0$, $p>1$, and $\alpha,\beta,\lambda>0$ such that $0<\int_0^\infty x^{p-\alpha\lambda-1}f^{p}(x)dx<\infty$, $0<\int_0^\infty y^{q-\beta\lambda-1}g^{q}(y)dy<\infty$ then:

$$\int_0^\infty y^{\beta\lambda(p-1)-1}\left[\int_0^\infty \frac{f(x)}{(x\alpha+y\beta)\,\lambda}dx\right]^{p}dy<\left[\frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha}\right]^{p}\int_0^\infty x^{p-\alpha\lambda-1}f^{p}(x)dx \tag{2.7}$$

where the constant factor $\left[\frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha}\right]^{p}$ is the best possible when $\alpha=\beta$. Inequality (2.7) is equivalent to (2.1).

Proof. Since $0<\int_0^\infty y^{q-\beta\lambda-1}g^{q}(y)dy<\infty$, there exists $T_0>0$, such that for any $T>T_0$, one has $0<\int_0^T y^{q-\beta\lambda-1}g^{q}(y)dy<\infty$. We set

$$g(y,T)=y^{\beta\lambda(p-1)-1}\left[\int_0^T \frac{f(x)}{(x\alpha+y\beta)\,\lambda}dx\right]^{p-1}.$$

Then, by (2.1), we have

$$\begin{aligned} 0 &< \int_0^T y^{q-\beta\lambda-1} g^q(y,T)dy = \int_0^T y^{\beta\lambda(p-1)-1}\left[\int_0^T \frac{f(x)}{(x\alpha+y\beta)\lambda}dx\right]^p dy \\ &= \int_0^T\int_0^T \frac{f(x)g(y,T)}{(x\alpha+y\beta)\lambda}dxdy \\ &< \frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}\left\{\int_0^T x^{p-\alpha\lambda-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^T y^{q-\beta\lambda-1}g^q(y)dy\right\}^{\frac{1}{q}} \qquad (2.8)\end{aligned}$$

Hence, we get

$$\begin{aligned} 0 &< \left\{\int_0^T y^{q-\beta\lambda-1} g^q(y,T)dy\right\}^{1-\frac{1}{q}} = \left\{\int_0^T y^{\beta\lambda(p-1)-1}\left[\int_0^T \frac{f(x)}{(x\alpha+y\beta)\lambda}dx\right]^p dy\right\}^{\frac{1}{p}} \\ &< \frac{B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)}{\alpha^{\frac{1}{q}}\beta^{\frac{1}{p}}}\left\{\int_0^T x^{p-\alpha\lambda-1}f^p(x)dx\right\}^{\frac{1}{p}} \qquad (2.9)\end{aligned}$$

It follows that $0 < \int_0^\infty y^{q-\beta\lambda-1}g^q(y,\infty)dy < \infty$. Hence (2.8) and (2.9) are strict inequalities as $T \to \infty$. Therefore, inequality (2.7) holds.

On the other hand assume that (2.7) is valid. By Holder's inequality, we obtain

$$\begin{aligned} \int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x\alpha+y\beta)\lambda}dxdy &= \int_0^\infty\left[y^{\frac{\beta\lambda+1-q}{q}}\int_0^\infty \frac{f(x)}{(x\alpha+y\beta)\lambda}dx\right]\left[y^{-\frac{\beta\lambda+1-q}{q}}g(y)dy\right] \\ &\le \left\{\int_0^\infty y^{\beta\lambda(p-1)}\left[\int_0^\infty \frac{f(x)}{(x\alpha+y\beta)\lambda}dx\right]^p\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q-\beta\lambda-1}g^q(y)dy\right\}^{\frac{1}{q}}\end{aligned}$$

Then by (2.7) we obtain (2.1). Therefore, (2.1) and (2.7) are equivalent. If the constant factor $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$ in (2.7) is not the best possible when $\alpha = \beta$, using (2.10) we may get a contradiction that the constant factor in (2.1) is not the best possible. Thus the theorem is proved.

References

[1] Hardy G.H., Littlewood J.E., and Polya G.,"*Inequalities*", Cambridge Univ. Press, London, 1952.

[2] Mitrinovic, D.S., Pecaric, J.E., Fink, A.M., *Inequalities Involving Functions and their Integrals and Derivatives,* Kluwer Academic Publishers,1991.

[3] Jichang Kuang, On new extensions of Hilbert's integral inequality, *J. Math.Anal.Appl.*,235(2)(1999),608-614.

[4] Bicheng Yang, On a general Hardy-Hilbert's inequality with a best value, *Chinese Ann. Math.(Ser. A),* 21(4)(2000), 401-408.

[5] Bicheng Yang, On Hardy-Hilbert's Integral Inequality, *J. Math. of Anal. Appl.* **261**, 295-306(2001).

[6] Bicheng Yang, On an extension of Hardy-Hilbert's inequality, *Chinese Ann. of Math.,* 23A(2)(2002), 247-254.

[7] Bicheng Yang L. Debnath, On a generalization of Hilberts integral inequality and its applications, Research Report Collection, 2:5.

Received: February 8, 2007