On an Extension of Hardy-Hilbert’s
Integral Inequality with Three Parameters

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Abstract. In this paper by introducing three parameters we establish an extension of Hardy-Hilbert’s integral inequality. As an application we give it’s equivalent form.

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1. Introduction

If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(x) > 0, 0 < \int_0^\infty f^p(x)dx < \infty, \) and \( 0 < \int_0^\infty g^q(x)dx < \infty, \) then we have the following two equivalent inequalities as (see[1])

\[
\begin{align*}
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy &= \frac{\pi}{\sin \left( \frac{\pi}{p} \right)} \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x) dx \right\}^{\frac{1}{q}} \\
\end{align*}
(1.1)
\]
\[
\int_0^\infty \left( \int_0^\infty \frac{f(x)}{x+y} \, dx \right)^p \, dy < \left( \frac{\pi}{\sin \left( \frac{\pi}{p} \right)} \right)^p \int_0^\infty f^p(x) \, dx
\]  

(1.2)

where the constant factors \( \frac{\pi}{\sin \left( \frac{\pi}{p} \right)} \) and \( \left( \frac{\pi}{\sin \left( \frac{\pi}{p} \right)} \right)^p \) are the best possible in (1.1) and (1.2) respectively. Inequality (1.1) is called Hardy-Hilbert’s integral inequality, which is important in analysis and its applications (see[2]). Recently, various extensions on inequality (1.1) have appeared in some papers such as [3],[4], [5] and [6].

In 1998, by introducing a parameter \( \lambda \in (0, 1] \) and the Beta function \( B(u, v) \) as

\[
B(u, v) = \int_0^\infty \frac{x^{u-1}}{(x+1)^{u+v}} \, dx = B(v, u) \ (u, v > 0),
\]

(1.3)

Yang [4,5] gave a generalization of (1.1) and (1.2) as: If \( \lambda > 2 - \min \{p, q\} \), \( f, g \) are non-negative functions such that

\[
0 < \int_0^\infty x^{1-\lambda} f^p(x) \, dx < \infty, \text{ and } 0 < \int_0^\infty x^{1-\lambda} g^q(x) \, dx < \infty,
\]

then the following two inequalities are equivalent

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^{\lambda}} \, dx \, dy < k\lambda(p) \left\{ \int_0^\infty x^{1-\lambda} f^p(x) \, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\lambda} g^q(x) \, dx \right\}^{\frac{1}{q}}
\]

and

\[
\int_0^\infty y^{(p-1)(\lambda-1)} \left[ \int_0^{\infty} \frac{f(x)}{(x+y)^{\lambda}} \, dx \right]^p \, dy < [k\lambda(p)]^p \int_0^\infty x^{1-\lambda} f^p(x) \, dx
\]

where the constant factors \( k\lambda(p) = B\left( \frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q} \right) \) and \([k\lambda(p)]^p \) are the best possible.
In 1999, Kuang [3] gave a generalization with a parameter $\lambda$ of (1.1) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\alpha} \, dx \, dy < h_\alpha(p) \left\{ \int_0^\infty x^{1-\alpha} f^p(x) \, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-\alpha} g^q(x) \, dx \right\}^{\frac{1}{q}}$$

where $\max\left\{ \frac{1}{p}, \frac{1}{q} \right\} < \alpha \leq 1$, $h_\alpha(p) = \pi \alpha \sin \left( \frac{\pi}{p} \right) \sin \left( \frac{\pi}{q} \right)^{-1}$. Because of the constant factor $h_\alpha(p)$ being not the best possible, Yang [6] gave a new generalization of (1.1) as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\alpha + y^\alpha} \, dx \, dy < \frac{\pi}{\alpha \sin \left( \frac{\pi}{p} \right)} \left\{ \int_0^\infty x^{(1-\alpha)(p-1)} f^p(x) \, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{(1-\alpha)(q-1)} g^q(x) \, dx \right\}^{\frac{1}{q}}$$

it’s equivalent form is:

$$\int_0^\infty \left[ \int_0^\infty \frac{f(x)}{x^\alpha + y^\alpha} \, dx \right]^{p} \, dy < \left[ \frac{\pi}{\alpha \sin \left( \frac{\pi}{p} \right)} \right]^p \int_0^\infty x^{(1-\alpha)(p-1)} f^p(x) \, dx$$

where the constant factors $\frac{\pi}{\alpha \sin \left( \frac{\pi}{p} \right)}, \left[ \frac{\pi}{\alpha \sin \left( \frac{\pi}{p} \right)} \right]^p$ are the best possible.

In this paper by introducing three parameters, our aim is to estimate the double integral:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^\alpha + y^\beta)^\lambda} \, dx \, dy.$$
2. Main Results

**Theorem 1.** If \( f, g \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha, \beta, \lambda > 0 \) such that \( 0 < \int_0^\infty x^{p-\alpha\lambda-1} f^p(x)dx < \infty \), and \( 0 < \int_0^\infty x^{q-\beta\lambda-1} g^q(x)dx < \infty \), then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x + y)^\lambda} \, dx \, dy \leq \frac{B(\frac{\alpha}{p}, \frac{\lambda}{\beta})}{\alpha \beta \lambda} \left\{ \int_0^\infty x^{p-\alpha\lambda-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-\beta\lambda-1} g^q(x)dx \right\}^{\frac{1}{q}}
\]

where the constant factor \( \frac{B(\frac{\alpha}{p}, \frac{\lambda}{\beta})}{\alpha \beta \lambda} \) is the best possible when \( \alpha = \beta \).

**Proof.** Define two functions,

\[
\mu = \frac{f(x) \left( x^{\frac{\alpha}{p} y^{\frac{\lambda}{\beta} - 1}} \right)^{\frac{1}{p}}}{(x + y)^\lambda} \quad \text{and} \quad \varphi = \frac{g(y) \left( x^{\frac{\alpha}{p} y^{\frac{\lambda}{\beta} - 1}} \right)^{\frac{1}{q}}}{(x + y)^\frac{\lambda}{q}}.
\]

Let us estimate the left-hand side of (2.1) by Holder’s inequality, we obtain

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x + y)^\lambda} \, dx \, dy = \int_0^\infty \int_0^\infty \mu \varphi \, dx \, dy
\]

\[
\leq \left\{ \int_0^\infty \mu^p \, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \varphi^q \, dy \right\}^{\frac{1}{q}}
\]

\[
= \left\{ \int_0^\infty x^{p-\alpha\lambda-1} f^p(x)dx \right\}^{\frac{1}{p}} \times \times \left\{ \int_0^\infty x^{q-\beta\lambda-1} g^q(y)dy \right\}^{\frac{1}{q}}
\]

\[
= \left\{ \int_0^\infty \omega_p x^{p-\alpha\lambda-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega_q y^{q-\beta\lambda-1} g^q(y)dy \right\}^{\frac{1}{q}}
\]

where \( \omega_p = \int_0^\infty \frac{x^{\frac{\alpha}{p} y^{\frac{\lambda}{\beta} - 1}}}{(x + y)^\lambda} \, dy \), and \( \omega_q = \int_0^\infty \frac{x^{\frac{\alpha}{p} y^{\frac{\lambda}{\beta} - 1}}}{(x + y)^\frac{\lambda}{q}} \, dx \).

If (2.1) takes the form of an inequality, then there exists real numbers \( A \) and \( B \) such that they are not all zero and,
Extension of Hardy-Hilbert’s integral inequality

\[
\frac{Ax^{\frac{\alpha}{p} y^{\frac{\lambda}{q}}}}{(x \alpha + y \beta)^{\lambda}} x^{p-\alpha \lambda-1} f^p(x) = \frac{B x^{\frac{\alpha}{p} - 1} y^{\frac{\lambda}{q}}}{(x \alpha + y \beta)^{\lambda}} y^{q-\beta \lambda-1} g^q(y), \text{ a.e. in } (0, \infty) \times (0, \infty).
\]

Hence, we find

\[
Ax^{p-\alpha \lambda} f^p(x) = By^{q-\beta \lambda} g^q(y), \text{ a.e. in } (0, \infty) \times (0, \infty).
\]

It follows that there exists a constant \(C\), such that

\[
Ax^{p-\alpha \lambda} f^p(x) = C, \text{ a.e. in } (0, \infty)
\]

\[
By^{q-\beta \lambda} g^q(y) = C \text{ a.e. in } (0, \infty).
\]

Without loss of generality, suppose that \(A \neq 0\). Then we have

\[
x^{p-\alpha \lambda-1} f^p(x) = \frac{C}{Ax}, \text{ a.e. in } (0, \infty),
\]

which contradicts the fact that \(0 < \int_0^\infty x^{p-\alpha \lambda-1} f^p(x) dx < \infty\). Therefore, (2.1) takes the form of strict inequality, and we may rewrite (2.1) as

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x \alpha + y \beta)^{\lambda}} dx dy < \frac{B \left( \lambda, \frac{\lambda}{p}, \frac{\lambda}{q} \right)}{\alpha^{\frac{1}{p}} / \beta^{\frac{1}{q}}} \left\{ \int_0^\infty x^{p-\alpha \lambda-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{q-\beta \lambda-1} g^q(x) dx \right\}^{\frac{1}{q}}.
\]

We compute the weight function \(\omega_p\) as follows, let \(u = \frac{y \beta}{x \alpha}\), then we obtain by (1.3)

\[
\omega_p = \frac{1}{\beta} \int_0^\infty \frac{u^{\frac{\lambda}{p} - 1}}{(u + 1)^{\lambda}} du = \frac{1}{\beta} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right)
\]

and similarly,
\[\omega_q = \frac{1}{\alpha} B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \quad (2.4)\]

From (2.2), (2.3) and (2.4) we get (2.1).

We need to show that the constant factor \( B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right) \) contained in (2.1) is the best possible when \( \alpha = \beta \). To do that we define two functions

\[f_0(x) = \begin{cases} \frac{0}{x}, & x \in [0, 1) \\ \frac{x^{\alpha \lambda - p - \varepsilon}}{x}, & x \in [1, +\infty) \end{cases}\]

and

\[g_0(y) = \begin{cases} \frac{0}{y}, & y \in [0, 1) \\ \frac{y^{\beta \lambda - q - \varepsilon}}{y}, & y \in [1, +\infty) \end{cases}\]

Assume that \( 0 < \varepsilon < \alpha \lambda \). Suppose that \( \frac{B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right)}{\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}}} \) is not the best possible, then there exists \( 0 < K < \frac{B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right)}{\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}}} \) such that

\[
\begin{align*}
\int_0^\infty \int_0^\infty f_0(x)g_0(y) \frac{dxdy}{(x\alpha + y\beta)^{\lambda}} &< K \left\{ \int_0^\infty x^{p-\alpha \lambda - 1} f_0^p(x) dx \right\} \left\{ \int_0^\infty y^{q-\beta \lambda - 1} g_0^q(y) dy \right\}^{\frac{1}{q}} \\
&= K \left\{ \int_1^\infty x^{-1-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_1^\infty y^{-1-\varepsilon} dy \right\}^{\frac{1}{q}} = K \varepsilon \quad (2.5)
\end{align*}
\]

On the other hand, setting \( u = \frac{x\alpha}{y\beta} \), we have
\[
\int_0^\infty \int_0^\infty \frac{f_0(x)g_0(y)}{(x\alpha + y\beta)\lambda} \, dx dy = \int_1^1 \int_1^1 \frac{x^{\lambda - p - e} y^{\beta\lambda - q - e}}{(x\alpha + y\beta)\lambda} \, dx dy
\]

\[
= \frac{1}{\alpha} \int_1^1 y^{-\frac{e}{q} - \frac{e}{\alpha p} - 1} \left\{ \int_0^1 u^{\frac{1}{p} - 1} u^{-\frac{e}{p q}} (u + 1) \lambda \, du \right\} \, dy
\]

\[
= \frac{1}{\alpha} \int_1^1 y^{-\frac{e}{q} - \frac{e}{\alpha p} - 1} \left\{ \int_0^1 u^{\frac{1}{p} - 1} u^{-\frac{e}{p q}} (u + 1) \lambda \, du \right\} \, dy
\]

\[
= \frac{B(\lambda, \frac{1}{q}) + o(1)}{\varepsilon \left( \frac{\alpha}{q} + \frac{\beta}{p} \right)} - \frac{1}{\alpha} \int_1^1 y^{-\frac{e}{q} - \frac{e}{\alpha p} - 1} \left[ \int_0^1 u^{\frac{1}{p} - 1} u^{-\frac{e}{p q}} \, du \right] \, dy
\]

\[
> \frac{B(\lambda, \frac{1}{q}) + o(1)}{\varepsilon \left( \frac{\alpha}{q} + \frac{\beta}{p} \right)} - \frac{1}{\alpha} \int_1^1 y^{-1} \left[ \int_0^1 u^{\frac{1}{p} - \frac{e}{p q}} \, du \right] \, dy
\]

\[
= \frac{B(\lambda, \frac{1}{q}) + o(1)}{\varepsilon \left( \frac{\alpha}{q} + \frac{\beta}{p} \right)} - \frac{\alpha^p}{\beta (\alpha \lambda - e)^2}
\]

\[
= \frac{B(\lambda, \frac{1}{q}) + o(1)}{\varepsilon \left( \frac{\alpha}{q} + \frac{\beta}{p} \right)} - O(1)
\]

\[
= \left[ \frac{B(\lambda, \frac{1}{q}) + o(1)}{\varepsilon \left( \frac{\alpha}{q} + \frac{\beta}{p} \right)} \right] (\varepsilon \to 0^+)
\] (2.6)

By Young’s inequality, we have \( \alpha \frac{1}{p} + \beta \frac{1}{q} \leq \frac{\alpha}{q} + \frac{\beta}{p} \) (i.e. \( \frac{1}{q} + \frac{1}{p} \leq \frac{1}{\alpha \lambda - e} \)). Consider the form of equality, we get \( \alpha = \beta \). Then
\[
\frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{(\frac{1}{\alpha} + \frac{1}{\beta})} + o(1) = \frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{\alpha^{-\frac{1}{p}} \beta^{\frac{1}{q}}} + o(1) < k
\]

Clearly, when \( \varepsilon \to 0^+ \), the inequality (2.5) is in contradiction with (2.6). Thus the constant factor \( \frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{\alpha^{-\frac{1}{p}} \beta^{\frac{1}{q}}} \) is the best possible for \( \alpha = \beta \), and the proof of the theorem is completed.

**Remark.** For \( \alpha = \beta \), inequality (2.1) becomes

\[
\int_0^\infty \frac{f(x)g(y)}{(x\alpha + y\beta)\lambda} \, dx \, dy < \frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{\alpha} \left\{ \int_0^\infty x^{p-\alpha\lambda-1} f^p(x) \, dx \right\} \frac{1}{\alpha} \left\{ \int_0^\infty x^{q-\alpha\lambda-1} g^q(x) \, dx \right\}^{\frac{1}{q}},
\]

where the constant factor \( \frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{\alpha} \) is the best possible.

**Theorem 2** If \( f > 0, p > 1, \) and \( \alpha, \beta, \lambda > 0 \) such that \( 0 < \int_0^\infty x^{p-\alpha\lambda-1} f^p(x) \, dx < \infty \), \( 0 < \int_0^\infty y^{q-\beta\lambda-1} g^q(y) \, dy < \infty \) then:

\[
\int_0^\infty y^{\beta(p-1)-1} \left[ \int_0^\infty \frac{f(x)}{(x\alpha + y\beta)\lambda} \, dx \right]^p \, dy < \frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{\alpha} \int_0^\infty x^{p-\alpha\lambda-1} f^p(x) \, dx
\]

(2.7)

where the constant factor \( \frac{B(\frac{p}{\alpha}, \frac{q}{\beta})}{\alpha} \) is the best possible when \( \alpha = \beta \). Inequality (2.7) is equivalent to (2.1).

**Proof.** Since \( 0 < \int_0^\infty y^{q-\beta\lambda-1} g^q(y) \, dy < \infty \), there exists \( T_0 > 0 \), such that for any \( T > T_0 \), one has \( 0 < \int_0^T y^{q-\beta\lambda-1} g^q(y) \, dy < \infty \). We set

\[
g(y, T) = y^{\beta(p-1)-1} \left[ \int_0^T \frac{f(x)}{(x\alpha + y\beta)\lambda} \, dx \right]^{p-1}.
\]

Then, by (2.1), we have
\[
0 < \int_0^T y^{q-\beta \lambda -1} g^q(y, T) dy = \int_0^T y^{\beta \lambda (p-1)-1} \left[ \int_0^T \frac{f(x)}{(x \alpha + y \beta) \lambda} dx \right]^p dy
= \int_0^T \int_0^T \frac{f(x)g(y, T)}{(x \alpha + y \beta) \lambda} dxdy
\]
\[
< \frac{B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right)}{\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}}} \left\{ \int_0^T x^{p-\alpha \lambda -1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^T y^{q-\beta \lambda -1} g^q(y) dy \right\}^{\frac{1}{q}} (2.8)
\]

Hence, we get
\[
0 < \left\{ \int_0^T y^{q-\beta \lambda -1} g^q(y, T) dy \right\}^{1-\frac{1}{q}} = \left\{ \int_0^T y^{\beta \lambda (p-1)-1} \left[ \int_0^T \frac{f(x)}{(x \alpha + y \beta) \lambda} dx \right]^p dy \right\}^{\frac{1}{p}}
\]
\[
< \frac{B \left( \frac{\lambda}{p}, \frac{\lambda}{q} \right)}{\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}}} \left\{ \int_0^T x^{p-\alpha \lambda -1} f^p(x) dx \right\}^{\frac{1}{p}} (2.9)
\]

It follows that \(0 < \int_0^\infty y^{q-\beta \lambda -1} g^q(y) dy < \infty\). Hence (2.8) and (2.9) are strict inequalities as \(T \to \infty\). Therefore, inequality (2.7) holds.

On the other hand assume that (2.7) is valid. By Holder’s inequality, we obtain
\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x \alpha + y \beta) \lambda} dxdy = \int_0^\infty \left[ \int_0^\infty \frac{f(x)}{(x \alpha + y \beta) \lambda} dx \right] \left[ y^{-\frac{\beta \lambda + 1}{\lambda}} g(y) dy \right]
\]
\[
\leq \left\{ \int_0^\infty y^{\beta \lambda (p-1)} \left[ \int_0^\infty \frac{f(x)}{(x \alpha + y \beta) \lambda} dx \right]^p \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q-\beta \lambda -1} g^q(y) dy \right\}^{\frac{1}{q}}
\]

Then by (2.7) we obtain (2.1). Therefore, (2.1) and (2.7) are equivalent. If the constant factor \(\left( \frac{\pi \sin \pi p}{\sin \frac{\pi}{p}} \right)^p\) in (2.7) is not the best possible when \(\alpha = \beta\), using (2.10) we may get a contradiction that the constant factor in (2.1) is not the best possible. Thus the theorem is proved.
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