On the Transformation
Semitopological Semigroup

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Abstract

In this paper we introduce the notion of weighted (weakly) almost periodic compactification of a semitopological semigroup and generalize this notion to corresponding notion for transformation semigroup. The inclusion relation and equality of some well-known function spaces on a weighted transformation semigroup is also investigated.

Introduction

Let $\omega$ be a weight on a semitopological semigroup $X$ with identity $e$ and $S$ be a semigroup. The pair $(S, X)$ is semitopological transformation semigroup and $AP(X)$ $(WAP(X))$ is denoted almost periodic (weakly almost periodic) function space. For a topological group $G$.

Burckel proved that $G$ is compact if and only if bounded continuous functions on $G$ are weakly almost periodic [2]. Granirer improved the result to $G$ is compact if and only if all the bounded uniformly continuous functions on
$G$ are weakly almost periodic [5]. Dzinotyiweyi proved a same result for very large class of topological semigroup. H. D. Junghenn generalize the notion of (weakly) almost periodic compactification of semitopological semigroup to the corresponding notion for transformation semigroup and show that if $S$ has a (weakly) almost periodic compactification then $(S, X)$ has a (weakly) almost periodic compactification [8].

A. Rejali gave a modification of definition of weighted (weakly) almost periodic function spaces and proved that $AP(G, \omega) = C(G, \omega)$ when $G$ is compact group [12].

The organization of this paper is as follows. In section 1 we introduce preliminaries and some definition. In section 2 we generalize definition of (weakly) almost periodic function space in [12] and introduce weighted transformation functions spaces on weighted transformation semigroup and prove that each function in $C(X, \omega)$ is almost periodic if $S$ and $X$ are compact. Also we show that whenever a member of $C(X, \omega)$ or $WAP(X, \omega)$ belong to $AP(X, \omega)$. We will give examples of the weighted semitopological transformation semigroup and study the equicontinuity of this function spaces.

In section 3 we introduce a notion of weighted transformation compactification and finally we give the main result of the paper which states; each weakly almost periodic function on $X$ with respect to $S$ is left norm continuous.

1 Preliminaries

Let $(S, X, \omega)$ be a weighted transformation semigroup in which $S$ is a semigroup and $\omega$ is a weight on $X$. $(\omega : X \rightarrow \mathbb{R}^+)$ is called a weight on $X$ if $\omega(xy) \leq \omega(x)\omega(y)$, $x, y \in X$ and $\omega, \omega^{-1}$ are bounded on compact subsets of $X$.

A map $(s, x) \rightarrow sx : S \times X \rightarrow X$ is called the action of $S$ on $X$ if $s(tx) = (st)x$. For all $s, t \in S$, $x \in X$.

If $T$ is a subsemigroup of $S$ and $Y$ is a $T$- invariant subset of $X$, i.e. $TY = \{ty : t \in T, y \in Y\} \subseteq Y$. Then $(T, Y, \omega)$ is called a sub-transformation semigroup of $(S, X, \omega)$.

For topological spaces $S$ and $X$ , $(T, Y, \omega)$ is dense in $(S, X, \omega)$ if $T$ and $Y$ are dense in $S$ and $X$ respectively. $C(X)$ is the $C^*$-algebra of bounded continuous real-valued function on $X$ with supnorm.

If $\mathcal{F}$ is a $C^*$-subalgebra of $C(X)$ then $X^\mathcal{F}$ denotes the spectrum of $\mathcal{F}$ (the set of all multiplicative means on $\mathcal{F}$) which is $\omega^*$-compact in $\mathcal{F}^*$

The evaluation map $\epsilon : X \rightarrow X^\mathcal{F}$ with $\omega^*$-dense image is defined by $\epsilon(x)(f) = f(x)$ for $x \in X$, $f \in \mathcal{F}$. 

For $s, t \in S$ and $x \in X$, define $\lambda_s : S \to S$ and $\rho_t : S \to S$ by $\lambda_s(t) = st = \rho_t(s)$.

Also $\hat{\lambda}_s : X \to X$ and $\hat{\rho}_x : S \to X$ by $\hat{\lambda}_s(x) = \hat{\rho}_x(s) = sx$.

These maps are called translation maps.

$(S, X, \omega)$ is left topological if $S$ and $X$ are topological space and $\lambda_s, \hat{\lambda}_s$ are continuous for all $s \in S$, right topological if $\rho_s$ and $\hat{\rho}_x$ are continuous for all $s \in S, x \in X$, and semitopological if it is both left and right topological.

A semitopological semigroup $S$ is topologically right (left) simple if for each $s \in S$;

$sS : (sS) = X$.

$(S, X, \omega)$ is called topological if the multiplication in $S$ and the action of $S$ on $X$ are continuous.

If there is a separately (jointly) continuous action of a semitopological (topological) group $S$ with identity $e$ on a topological space $X$ and $ex = x$ for all $x \in X$, then $(S, X, \omega)$ is called a semitopological (resp. topological) transformation group.

$(S, X, \omega)$ is said to be compact if so are $S$ and $X$. We put

$L_s = (\lambda_s)^*$ and $R_s = (\rho_s)^*$ on $C(S)$.

$\hat{L}_s = (\hat{\lambda}_s)^*$ and $\hat{R}_x = (\hat{\rho}_x)^*$ on $C(X)$

where $(\lambda_s)^*$ is dual of $(\lambda_s)$ and $(\rho_s)^*$ is dual of $\rho_s$.

Clearly $\hat{L}_s \hat{L}_t = \hat{L}_{st}$ and $\hat{R}_s \hat{R}_x = \hat{R}_{sx}.

A subset $F$ of $C(S)$ is called translation invariant if $L_sF \cup R_sF \subseteq F$ for $s \in S$.

Let $A = \{f_\alpha : \alpha \in I\}$ be a family of real functions on topological space $X$. $A$ is called equicontinuous if for every $\varepsilon > 0$ and $x_0 \in X$ there exit a neighborhood $N$ of $x_0$ such that $|f_\alpha(x) - f_\alpha(x_0)| < \varepsilon$, for every $x \in N$ and $\alpha \in I$.

2 Weighted Function Spaces

**Definition 2.1** Let $(S, X, \omega)$ be a weighted semitopological transformation semigroup, and $e$ be the identity of $X$. Then:

for weight $\omega$, put $\Omega : X \times X \to (0, 1]$ defined $\Omega(x, y) = \frac{\omega(xy)}{\omega(x)\omega(y)}$ and $s\Omega : X \to (0, 1]$ by $s\Omega(x) = \Omega(se_x, x)$ and $\Omega_x : S \to (0, 1]$ by $\Omega_x(s) = \Omega(se_x, x)$.

**Notation:** Always assume that $\Omega$ is separately continuous and we will use from $sf, fx$ instead of $\hat{L}_sf$ and $\hat{R}_x$.

Now we introduce some weight function spaces.

$$C(X, \omega) = \{f \in C(X) : \frac{f}{\omega} \in C(X)\}$$

and for every $f \in C(X, \omega)$ define:

$$\|f\|_\omega = sup\left\{\frac{|f(x)|}{\omega(x)} : x \in X\right\}.$$
\[L_{\omega, s} f = \{s(\frac{f}{\omega}) s \Omega : s \in S\} \quad \text{and} \quad R_{\omega, s} f = \{(\frac{f}{\omega}) x \Omega_x : x \in X\}\]

\[AP(X, \omega) = \{f \in C(X, \omega) : \{L_{\omega, s} (\frac{f}{\omega}) s \Omega : s \in S\} \text{ is relatively compact in norm topology on } C(X) \}\]

\[WAP(X, \omega) = \{f \in C(X, \omega) : \{\hat{L}_{\omega, s} (\frac{f}{\omega}) : s \in S\} \text{ is relatively compact in weak topology on } C(X)\}\]

\[LUC(X, \omega) = \{f \in C(X, \omega) : \text{The map } s \mapsto \hat{L}_{\omega, s} (\frac{f}{\omega}) : S \to C(X) \text{ is norm continuous}\}\]

\[RUC(X, \omega) = \{f \in C(X, \omega) : \text{The map } x \mapsto \hat{R}_{\omega, x} (\frac{f}{\omega}) : X \to C(S) \text{ is norm continuous}\}\]

**Proposition 2.2** Let \(f \in C(X, \omega)\). Then \(f \in AP(X, \omega) \ (WAP(X, \omega)) \) if \(\{\frac{f(s)}{\omega(s)} : s \in S\}\) is relatively (weakly) compact in \(C(X, \omega)\).

**Proof.**

Let \(x \in X\). Then:

\[s(\frac{f}{\omega}) s \Omega(x) = \frac{f(sx)}{\omega(sx)} \Omega(se, x) = \frac{f(sx)}{\omega(sx)} \frac{\omega((se)x)}{\omega(se) \omega(x)} = \frac{f(sx)}{\omega(se) \omega(x)} = \frac{sf}{\omega(se) \omega}(x)\]

therefore:

\[s(\frac{f}{\omega}) s \Omega = \frac{sf}{\omega(se) \omega}\]

and so if \(f \in C(X, \omega)\) then:

\(f \in AP(X, \omega) \ (\text{resp.} WAP(X, \omega)) \) if \(\{\frac{f(s)}{\omega(s)} : s \in S\}\) is relatively (weak) compact in \(C(X, \omega)\).

**Example 2.3** If \(S\) be a right zero semigroup and \(S = X\) then \(AP(X, \omega) = C(X, \omega)\).

**Example 2.4** let \((S, X, \omega)\) be a weighted semitopological transformation semigroup and \(\omega\) be a multiplicative weight on \(X\) i.e \(\omega(xy) = \omega(x) \omega(y), x, y \in X\). Then \(AP(X, \omega) = AP(X)\) and \(WAP(X, \omega) = WAP(X)\).

**Remark 2.5** All of these function spaces are translation invariant \(C^*\)-subalgebras of \(C(X, \omega)\) containing the constant function and clearly \(AP(X, \omega) \subseteq WAP(X, \omega)\).
Lemma 2.6 Let \((S, X, \omega)\) be a weighted semitopological transformation semigroup. Then:

1) If \((S, X, \omega)\) is compact, then \(\text{WAP}(X, \omega) = C(X, \omega)\).
2) If \((S, X, \omega)\) is a compact, then \(\text{AP}(X, \omega) = C(X, \omega)\).
3) If \(f \in \text{WAP}(X, \omega)\), then the map \(s \rightarrow_s (\frac{f}{\omega})_s : S \rightarrow \text{WAP}(X, \omega)\) is continuous in the weak topology.
4) If \(f \in \text{WAP}(X, \omega)\), then the map \(s \rightarrow_s (\frac{f}{\omega})_s : S \rightarrow \text{AP}(X, \omega)\) is continuous in the norm topology.

Proof.

1) Let \((S, X, \omega)\) be compact and \(f \in C(X, \omega)\), since the map \(s \rightarrow_s (\frac{f}{\omega})_s \Omega\) is pointwise continuous, \(\{s(\frac{f}{\omega})_s : s \in S\}\) is compact in the pointwise topology of \(C(X)\), and since this topology agrees with the weak topology on norm bounded pointwise compact of \(C(X, \omega)\) ([8]). \(\{s(\frac{f}{\omega})_s : s \in S\}\) is relatively weakly compact. Therefore \(f \in \text{WAP}(X, \omega)\).

3) Let \((s_\alpha)\) a net in \(S\) and converges to \(s\). Since \(f \in \text{WAP}(X, \omega)\) then \(\{s(\frac{f}{\omega})_s : s \in S\}\) is relatively weakly compact therefore \((s_\alpha(\frac{f}{\omega})_{s_\alpha}\Omega)\) has unique weak limit point in \(C(X)\). Therefore \(s_\alpha(\frac{f}{\omega})_{s_\alpha}\Omega \rightarrow (\frac{f}{\omega})_s\Omega\).

Proposition 2.7 Let \((S, X, \omega)\) be a weighted semitopological transformation semigroup. Then:

1) \(\text{AP}(X, \omega) \subseteq LUC(X, \omega) \cap RUC(X, \omega)\)
2) If \((S, X)\) is compact, then \(\text{AP}(X, \omega) = LUC(X, \omega) = RUC(X, \omega)\).
3) If \((S, X, \omega)\) is compact, then \(\text{AP}(X, \omega) = LUC(X, \omega) = RUC(X, \omega)\).
4) If \((S, X, \omega)\) is compact and topological then \(\text{AP}(X, \omega) = \text{LUC}(X, \omega) = \text{RUC}(X, \omega) = \text{WAP}(X, \omega) = C(X, \omega)\).
5) If \((S, X)\) is compact and Hausdorff, then \(C(X, \omega) = \text{RUC}(X, \omega) = (LUC(X, \omega))\)

if and only if the action of \(S\) on \(X\) is (jointly) continuous.

6) If \((S, X, \omega)\) is compact and Hausdorff, then \(LUC(X, \omega) = RUC(X, \omega) = C(X, \omega)\) if and only if the action of \(S\) on \(X\) is continuous.

Proof.

1) If \(f \in \text{AP}(X, \omega)\) then the map \(s \rightarrow_s (\frac{f}{\omega})_s \Omega\) is continuous by [3]. Therefore \(f \in \text{LUC}(X, \omega)\).

Now if \(f \in \text{RUC}(X, \omega)\) the map \(x \rightarrow (\frac{f}{\omega})_x \Omega_x\) is continuous. Let \(U(x) = (\frac{f}{\omega})_x \Omega_x\) and \(V(s) = (\frac{f}{\omega})_s \Omega\) for \(x \in X, s \in S\). Obviously \((V(s))(s)) = (U(x))(s)\). Therefore \(f \in \text{RUC}(X, \omega)\), hence \(\text{AP}(X, \omega) = \text{LUC}(X, \omega) \cap \text{RUC}(X, \omega)\).

2) Let \(S\) be compact semigroup and \(f \in \text{LUC}(X, \omega)\). Then the map \(s \rightarrow_s (\frac{f}{\omega})_s \Omega\) is continuous. Therefore \(\{s(\frac{f}{\omega})_s : s \in S\}\) is compact (since
$S$ is compact) so $f \in AP(X, \omega)$.

Note that if $X$ is compact, then the proof is similar.

3) It follows from (2) and (1 lemme 2.6)

4) It follows from (3) and (2 lemme 2.6)

5) we note that the action of $S$ on $X$ is continuous if and only if for $f \in C(X, \omega)$ the map $(s, x) \rightarrow f(sx) : S \times X \rightarrow \mathbb{R}$ is continuous (App. B 3 of [1]). This equivalent to norm continuity the map $x \rightarrow (\frac{L}{\omega})_x \Omega_x$, when $S$ is compact.

6) If $(S, X, \omega)$ is compact, then by (1) lemma 2.6 and 3 proof is obviously.

**Proposition 2.8** Let $(S, X, \omega)$ be a weighted semitopological transformation semigroup and $f \in C(X, \omega)$. Then:

1) $f \in RUC(X, \omega)$ if and only if $L_{S, f}$ is equicontinuous.

2) $f \in LUC(X, \omega)$ if and only if $R_{S, f}$ is equicontinuous.

**Proof.**

1) Let $f \in RUC(X, \omega)$, Then the map $x \rightarrow (\frac{L}{\omega})_x \Omega_x : X \rightarrow C(S)$ is norm continuous. Hence for every $\epsilon > 0$ and $x_0 \in X$ there is a neighborhood $N$ of $x_0$ such that for every $x \in X$: $\| (\frac{L}{\omega})_x \Omega_x - (\frac{L}{\omega})_{x_0} \Omega_{x_0} \| < \epsilon$. Hence for every $s \in S$ $| \frac{L}{\omega}(sx) \Omega_x(s) - \frac{L}{\omega}(sx_0) \Omega_{x_0}(s) | < \epsilon$ and so $| (\frac{L}{\omega})_s \Omega(s) - (\frac{L}{\omega})_{s_0} \Omega_{x_0} | < \epsilon$. Hence $L_{S, f}$ is equicontinuous. The Converse is clear and the proof of 2 is similar.

**Corollary 2.9**

1) If $S$ is compact then for every $f \in AP(X, \omega)$ the set $R_{S, f}$ is equicontinuous.

2) If $X$ is compact then for every $f \in AP(X, \omega)$, $L_{S, f}$ is equicontinuous.

**Proof.** By Proposition (2.7) and Proposition (2.8) is immediate.

**Definition 2.10** A weighted homomorphism of a weighted semitopological transformation semigroup $(S, X, \omega_1)$ into a weighted semitopological transformation semigroup $(T, Y, \omega_2)$ is a pair $(\sigma, \eta)$, where $\sigma : S \rightarrow T$ is a continuous homomorphism and $\eta : X \rightarrow Y$ is a continuous homomorphism such that $\eta(sx) = \sigma(s) \eta(x)$ for $x \in X$ and $s \in S$ and $\omega_2 \circ \eta = \omega_1$.

**Remark 2.11** Let $(\sigma, \eta)$ be a homomorphism of $(S, X, \omega_1)$ into $(T, Y, \omega_2)$. Let $f \in C(Y, \omega_2), s \in S$ and $x \in X$. Then:

1) $s(\frac{\eta^*(f)}{\omega_1})(x) = (\eta^*(\frac{f}{\omega_2})(sx)) = (\frac{L}{\omega_2}) \sigma(s) \eta(x) = (\frac{L}{\omega_2})(\sigma(s) \eta(x)) = \sigma(s)(\frac{f}{\omega_2}) \eta(x) = \eta^*(\sigma(s)(\frac{f}{\omega_2}))(x)$, and so:

$$s(\frac{\eta^*(f)}{\omega_1}) = \eta^*(\sigma(s)(\frac{f}{\omega_2}))$$

2) $s \Omega_1 = \eta^*(\sigma(s) \Omega_2)$
Lemma 2.12 Let $(\sigma, \eta)$ be a homomorphism of $(S, X, \omega_1)$ into $(T, Y, \omega_2)$. where $\omega_2 \circ \eta = \omega_1$. Then:

$$\eta^*(WAP(Y, \omega_2)) \subseteq WAP(X, \omega_1) \cap \eta^*(C(Y, \omega_2))$$

and equality is hold if $\eta(X)$ is dense in $Y$ and $\sigma(S)$ is dense in $T$.

Proof.
Let $f \in WAP(Y, \omega_2)$ and $A = \{\sigma(s)(\frac{f}{\omega_2}) : s \in S\}$. Then $\eta^*(A)$ is relatively weakly compact, and by remark (2.12) $\eta^*(f) \in WAP(X, \omega_1)$. Since $\eta^*(f) \in \eta^*(C(Y, \omega_2))$. Therefore

$$\eta^*(f) \in WAP(X, \omega_1) \cap \eta^*(C(Y, \omega_2)).$$

Now if $\eta(X)$ and $\sigma(S)$ are dense in $Y$ and $T$, respectively, then $\eta$ is an isometry and by remark (2.12) $$(\eta^*)^{-1}(cl(B)) = cl(A) \quad (1)$$

where $B = \{\sigma(s)(\frac{f}{\omega_1}) : s \in S\}$.

Let $g \in WAP(X, \omega_1) \cap \eta^*(C(Y, \omega_2))$. Then $(\eta^*)^{-1}(g) \in C(Y, \omega_2)$ and by remark (1.7) and (1) $(\eta^*)^{-1}(g) \in WAP(Y, \omega_2)$ and $g = \eta^*(\eta^*)^{-1}(g) \in WAP(Y, \omega_2)$. Hence equality is hold.

3 Weighted Transformation Compactification

Definition 3.1 Let $(S, X, \omega)$ be a weighted transformation semitopological semigroup and $(T, Y, \bar{\omega})$ be a compact Hausdorff right topological weighted semigroup. Further $(\phi, \psi) : (S, X, \omega) \longrightarrow (T, Y, \bar{\omega})$ be a continuous homomorphism such that $\phi(S)$ and $\psi(S)$ are dense semitopological sub transformation semigroup of $T$ and $Y$ respectively. Then $((\phi, \psi), (T, Y, \bar{\omega}))$ is called weighted transformation compactification of $(S, X, \omega)$.

If $((\phi, \psi), (T, Y, \bar{\omega}))$ and $((\phi', \psi'), (T', Y', \bar{\omega}'))$ are two weighted transformation compactification of $(S, X, \omega)$ and $(\pi, \gamma) : (T, Y, \bar{\omega}) \longrightarrow (T', Y', \bar{\omega}')$ is a continuous homomorphism of $(T, Y, \bar{\omega})$ onto $(T', Y', \bar{\omega}')$. we say that $((\phi, \psi), (T, Y, \bar{\omega}))$ is an extension of $((\phi', \psi'), (T', Y', \bar{\omega}'))$.

Let $(S, X, \omega)$ be a weighted transformation semitopological semigroup and $P$ be a property of compactification of $((\phi, \psi), (T, Y, \bar{\omega}))$. Then a P-compactification of $(S, X, \omega)$ is a compactification of $(S, X, \omega)$ that has the given property $P$.

If a P-compactification of $(S, X, \omega)$ that is extension of each other P-compactification of $(S, X, \omega)$ is called universal P-compactification.
Remark 3.2 If \(((\phi, \psi), (T, Y, \bar{\omega}))\) is a weighted transformation compactification of \((S, X, \omega)\), then \(\psi^*(\bar{\omega}) = \omega\).

Example 3.3 For a weighted transformation semitopological semigroup \((S, X, \omega)\) \(((\epsilon, \delta)(S^{WAP}, X^{WAP}(\bar{\omega}, \bar{\omega})),\) is a universal semitopological compactification of \((S, X, \omega)\), where \(\epsilon\) and \(\delta\) are evaluation map on \(C(S)\) and \(C(X, \omega)\) respectively.

Example 3.4 Let \((S, X, \omega)\) be a semitopological semigroup where \(S = X\) and \(e\) be the identity of \(X\). Further let \(M\) be closure of \(e\) in \(X\), then \(M\) is a closed congruence subsemigroup of \(X\) (5.4 of [6] and (1.24) of [1]). Put \(H = \frac{X}{M}\). Hence \(H\) is a Hausdorff topological semigroup. Suppose that \(\pi : X \to H\) denotes canonical homomorphism. For each \(f \in C(X, \omega)\) define \(g \in C(H, \bar{\omega})\) by \(g(\pi(x)) = f(x)\). Since \(f\) is constant on \(H\) then \(g\) is well-defined and hence \(g \in C(H, \bar{\omega})\). Thus \(\pi^*: C(H, \bar{\omega}) \to C(X, \omega)\) is an isometric isomorphism of \(C(H, \bar{\omega})\) on \(C(X, \omega)\).

Therefore \(\pi^*(R_{\pi(X)}g) = R_Sf\) and it follows \(\pi^*(WAP(H, \bar{\omega})) = WAP(X, \omega)\) and if \(WAP(X, \omega) = C(X, \omega)\), then \(WAP(H, \bar{\omega}) = C(H, \bar{\omega})\)

Lemma 3.5 Let \((S, X, \omega)\) be a compact Hausdorff weighted semitopological transformation semigroup and \(T = \{t \in S : tS = S, tX = X\}\) be a dense subsemigroup of \(S\). Then for each \(f \in C(X, \omega)\) the map \(s \to (\frac{f}{\omega})_s \Omega : S \to C(X)\) is norm continuous at each point of \(T\).

**Proof.**

Let \(f \in C(X, \omega)\). By (App. B1 of [1]); there is \(s_0 \in S\) such that the function \((s, x) \to \frac{f}{\omega}(sx) : S \times X \to \mathbb{R}\) is jointly continuous at each point of \(\{s_0\} \times X\). Further By (App.B3 of [1]) the set

\[
N = \{s \in S : \|s\frac{f}{\omega} - s_0\frac{f}{\omega}\| < \frac{\epsilon}{2}\}
\]

for every \(\epsilon > 0\) is a neighborhood of \(s_0\).

Let \(t_0 \in T\) be arbitrary, then by definition of \(T\) there exist \(u_0 \in S\) such that \(t_0u_0 = s_0\). Since \(T\) is dense in \(S\) and \(s_0S = S\), then \(t_0T\) is dense in \(S\).

Now chooses \(t \in T\) such that \(t_0t \in N\) and put \(V = \rho_t^{-1}(N)\). Then \(V\) is a neighborhood of \(t_0\). For every \(s \in V\) we have :

\[
\|s(\frac{f}{\omega}) - u_0(\frac{f}{\omega})\| = \sup_{x \in X}|\frac{f}{\omega}(sx) - \frac{f}{\omega}(t_0x)| = \sup_{x \in X}|\frac{f}{\omega}(stx) - \frac{f}{\omega}(t_0tx)| \leq \|st(\frac{f}{\omega}) - u_0(\frac{f}{\omega})\| + \|t_0(\frac{f}{\omega}) - t_0(\frac{f}{\omega})\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Since \(s_0 = t_0u_0\) and \(st, t_0t \in N\). Therefore the map \(s \to (\frac{f}{\omega})_s \Omega : S \to C(X)\) is norm continuous at \(t_0\).

Corollary 3.6 If \((S, X, \omega)\) is a compact Hausdorff weighted semitopological transformation semigroup and \(T = \{t \in S : tS = S, tX = X\}\) be a dense subsemigroup of \(S\), then \((T, X, \omega)\) is a topological transformation subsemigroup of \((S, X, \omega)\).
**Definition 3.7** Let \((S, X, \omega)\) be weighted transformation semigroup. Then \(S\) acts surjectively (resp. topologically surjectively) on \(X\) if for every \(s \in S\) have \(sX = X\). (resp. \(sX\) is dense in \(X\).)

We now state the main result of this paper.

**Theorem 3.8** For weight transformation semigroup \((S, X, \omega)\); if \(S\) is topologically right simple and acts topologically surjective on \(X\), then \(\text{WAP}(X, \omega) \subseteq \text{LUC}(X, \omega)\).

**Proof.**

Let \(((\epsilon, \delta), (T, Y, \overline{\omega}))\) denotes the universal semitopological compactification of \((S, X, \omega)\); where \(\epsilon, \delta\) are evaluation map on \(S\) and \(X\) respectively.

Put: \(T_1 = \{t \in T : tT = T, tY = Y\}\). Then \(T_1\) is a subsemigroup of \(T\) contains \(\epsilon(S)\) and by lemma [3.6]; for each \(f \in C(Y, \omega)\) the map \(t(\frac{f}{\omega})_{\epsilon(s)} : T \longrightarrow C(Y)\) is norm continuous at each point of \(\epsilon(S)\).

For \(s \in S\) and \(f \in C(Y, \overline{\omega})\) we have:

\[
\delta^*(\epsilon(s)(\frac{f}{\omega})_{\epsilon(s)}(\overline{\omega})) = \hat{L}_s \delta^*(\frac{f}{\omega})_{\epsilon(s)}(\hat{\omega})
\]

where \(\delta^* : C(Y, \overline{\omega}) \longrightarrow \text{WAP}(X, \omega)\) is the dual of evaluation map \(\delta : X \longrightarrow X^{WAP}\). Therefore

\[
\text{WAP}(X, \omega) = \delta^*(C(Y, \overline{\omega})) \subseteq \text{LUC}(X, \omega).
\]

**Corollary 3.9** If \((S, X, \omega)\) is a weighted transformation group, then:

\[\text{WAP}(X, \omega) \subseteq \text{LUC}(X, \omega)\]

**Proof.**

Since \((S, X, \omega)\) is weighted transformation group by 1.1.17 of [1]; \(S\) is left simple and right simple.

Let \(e\) be the identity of \(S\). For \(s \in S\) and \(x \in X\) we have: \(x = ex = ss^{-1}x \in SX\) and so \(sX = X\). By theorem 3.8 the proof is obviously.

The following example show that the condition of above theorem is necessary.

**Example 3.10** If \(S = (\mathbb{N}, +), X = (Q, +), \omega(x) = e^x\) and suppose that action of \(S\) on \(X\) is \((s, x) \mapsto s + x\). Then \(\text{WAP}(X, \omega) \nsubseteq \text{LUC}(X, \omega)\). Since by 4.19 of [1], there exist \(f_0 \in \text{WAP}(X)\) such that \(f_0 \notin \text{LUC}(X)\). Now assume that \(f = f_0\omega\). Therefore \(f \in \text{WAP}(X, \omega)\) but \(f \notin \text{LUC}(X, \omega)\).

**Acknowledgements** This research was supported by center of excellence for mathematics at Isfahan university and first author would like to thank from Razi University for his support.
References


Received: January 15, 2007