Graphs and Plausibility Logic

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Abstract

The aim of this article is to study some new properties of the plausibility logic. Besides, is analyzed the behavior of the logical operators, in the logic of plausibility and in the formal logic. All the results are showed using the theory of the weight graphs.

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1 Introduction

The theory of plausibility is being successfully applied in different scientific fields such as Economics, Sociology and Computing Sciences. In this latter discipline, it is used in the implementation of computational and/or collaborative tools and in the definition of computational designs (see [1], [2], [3], [4] and [8]). In [6, 7] the Theory of Plausibility is formalized at a wider level so as to explain decision-making processes in mathematical terms. A functional and numerical definition of both evidence and plausibility is given, from which is developed a multivalent (pentavalent) logic of plausibility, which has not been contemplated in the previously mentioned works. On the basis of these mathematical elements, the axiomatization and development of the logic of plausibility is described.

2 Preliminary Notes

In plausibility theory, due to the evolutive nature of decisions that are successively refined or reconsidered, the evidence is subject to change, and, as a result, so are the states of plausibility. In order to resolve this problem, plausibility theory breaks down the decision into stages, in such a way that there is no possibility of changes in the evidence. Therefore, in the process of taking the decision, the evidence depends on the stage and the statement under consideration. This gives rise to functional representations of evidence to the contrary (*ec*), evidence in favour (*ef*), significant evidence to the contrary (*sec*) and significant evidence in favour (*sef*) using the following numerical functions. Let K be the set of stages and F the set of statements, then

Definition 2.1. $ec, ef: K \times F \longrightarrow \{0, 0.5, 1\}$ and $sec, sef: K \times F \longrightarrow \{0, 1\}$ such that,

$$ec(k, A) = \begin{cases} 1 & \text{if it is known that evidence to the contrary exists;} \\ 0.5 & \text{if it is not known whether evidence in contrary exists or not;} \\ 0 & \text{if it is known that no evidence to the contrary exists.} \end{cases}$$

$$ef(k, A) = \begin{cases} 1 & \text{if it is known that evidence in favour exists;} \\ 0.5 & \text{if it is not known whether evidence in favour exists or not;} \\ 0 & \text{if it is known that no evidence in favour exists.} \end{cases}$$

$$sec(k, A) = \begin{cases} 1 & \text{if } ec(k, A) = 1 \text{ and the evidence is known to be significant;} \\ 0 & \text{other cases.} \end{cases}$$

$$sef(k, A) = \begin{cases} 1 & \text{if } ef(k, A) = 1 \text{ and the evidence is known to be significant;} \\ 0 & \text{other cases.} \end{cases}$$

Based on this definition, the plausibility of a statement is expressed through the function qualify, represented by qual, where:

Definition 2.2. $qual: K \times F \longrightarrow P = \{0, 1, 2, 3, 4\}$ such that,

$$qual(k, A) = \begin{cases} 0 & \text{if } ec(k, A) = 1 \text{ and } sec(k, A) = 1; \\ 1 & \text{if } ec(k, A) = 1 \text{ and } sec(k, A) = 0; \\ 2 & \text{if } ec(k, A) = 0.5; \\ 3 & \text{if } ec(k, A) = 0 \text{ and } sef(k, A) = 0; \\ 4 & \text{if } ec(k, A) = 0 \text{ and } sef(k, A) = 1. \end{cases}$$

Evidence to the contrary is determinative for the state of plausibility of a statement, which leads us to classify plausibility by regions. Was show in [7] that R is an equivalence relation on P , it determines a partition on the latter set P , $P/R = \{\{0, 1\}, \{2\}, \{3, 4\}\}$.

Definition 2.3. $R_0 = \{0, 1\}$, $R_1 = \{2\}$ and $R_2 = \{3, 4\}$, in which R_0 is rejection, R_1 indetermination and R_2 acceptance.

From the seven axioms presented A1, A2, A3, A4 and A6 figure among the principles that explicitly or implicitly appear in [1]. In addition, two axioms are included A5 and A7 [6]. The A7 is intuitive, which is to say, if $qual(k, A) = qual(k, B)$ it is to be expected that $sef(k, A) = sef(k, B)$. Although less is demanded by axiom A7, in itself it is a much weaker condition, as it only considers the case of $qual(k, B) = 3$.

$$Axiom (A1) \begin{cases} sec(k, \smile A) = sef(k, A); \\ sef(k, \smile A) = sec(k, A); \\ ec(k, \smile A) = 1 - ec(k, A). \end{cases}$$

$$Axiom (A2) \begin{cases} ec(k, A \wedge B) = \max(ec(k, A), ec(k, B)); \\ sec(k, A \wedge B) = \max(sec(k, A), sec(k, B)); \\ sef(k, A \wedge B) = \min(sef(k, A), sef(k, B)). \end{cases}$$

$$Axiom (A3) \begin{cases} ec(k, A \vee B) = ec(k, \smile (\smile A \wedge \smile B)); \\ sec(k, A \vee B) = sec(k, (\smile A \wedge \smile B)); \\ sef(k, A \vee B) = sef(k, \smile (\smile A \wedge \smile B)). \end{cases}$$

$$Axiom (A4) \begin{cases} ec(k, A \implies B) = ec(k, \smile A \vee B); \\ sec(k, A \implies B) = sec(k, \smile A \vee B); \\ sef(k, A \implies B) = sef(k, \smile A \vee B). \end{cases}$$

$$\text{Axiom (A5)} \left\{ \begin{array}{l} ec(k, A \iff B) = ec(k, (A \implies B) \wedge (B \implies A)); \\ sec(k, A \iff B) = sec(k, (A \implies B) \wedge (B \implies A)); \\ sef(k, A \iff B) = sef(k, (A \implies B) \wedge (B \implies A)). \end{array} \right.$$

$$\text{Axiom (A6)} \left\{ \begin{array}{l} ec(k, \forall x Ax) = ec(k, Ax_1 \wedge Ax_2 \cdots \wedge Ax_n); \\ sec(k, \forall x Ax) = sec(k, Ax_1 \wedge Ax_2 \cdots \wedge Ax_n); \\ sef(k, \forall x Ax) = sef(k, Ax_1 \wedge Ax_2 \cdots \wedge Ax_n); \\ ec(k, \exists x Ax) = ec(k, Ax_1 \vee Ax_2 \cdots \vee Ax_n); \\ sec(k, \exists x Ax) = sec(k, Ax_1 \vee Ax_2 \cdots \vee Ax_n); \\ sef(k, \exists x Ax) = sef(k, Ax_1 \vee Ax_2 \cdots \vee Ax_n). \end{array} \right.$$

$$\text{Axiom (A7)} \{ \text{qual}(k, A) < \text{qual}(k, B) = 3 \text{ then } sef(k, A) \leq sef(k, B). \}$$

3 Main Results

Theorem 3.1. $ec(k, \smile A) = 1 - ec(k, A)$ if, and only if, $qual(k, A) \in R_i$ implies $qual(k, \smile A) \in R_{2-i}$.

Proof. $qual(k, A) \in R_i$. $i = 0$. If $qual(k, A) \in R_0$, then $qual(k, A) \in \{0, 1\}$, by Definition 2.3 and $ec(k, A) = 1$ by Definition 2.2. By hypothesis $ec(k, \smile A) = 0$. Therefore, according to Definition 2.2, $qual(k, \smile A) \in \{3, 4\}$, that is, by Definition 2.3 $qual(k, \smile A) \in R_2$. The proofs for $i = 1, 2$ are analogous. Thus, $qual(k, A) \in R_i$ implies $qual(k, \smile A) \in R_{2-i}$.

If $ec(k, A) = 0$, by Definition 2.2, $qual(k, A) \in \{3, 4\}$ and by Definition 2.3, $qual(k, A) \in R_2$, as shown in the latter section $qual(k, \smile A) \in R_0$, by Definition 2.3, $qual(k, \smile A) \in \{0, 1\}$, and by Definition 2.2, $ec(k, \smile A) = 1$. Which is to say: $ec(k, \smile A) = 1 - ec(k, A)$.

If $ec(k, A) = 0.5$, by Definition 2.2, $qual(k, A) \in \{2\}$ and by Definition 2.3, $qual(k, A) \in R_1$, as shown in the latter section $qual(k, \smile A) \in R_1$, by Definition 2.3, $qual(k, \smile A) \in \{2\}$, and by Definition 2.2, $ec(k, \smile A) = 0.5$. Which is to say: $ec(k, \smile A) = 1 - ec(k, A)$. The analogous way we proof the same relation for $ec(k, A) = 1$.

By combining both part we arrive at the general result.

□

Lemma 3.2. [6]

$$a) \text{qual}(k, \smile A) = 4 - \text{qual}(k, A);$$

$$b) \text{qual}(k, A \wedge B) = \min(\text{qual}(k, A), \text{qual}(k, B));$$

- c) $qual(k, A \vee B) = \max(qual(k, A), qual(k, B));$
- d) $qual(k, A \implies B) = \max(4 - qual(k, A), qual(k, B));$
- e) $qual(k, A \iff B) = \min(\max(4 - qual(k, A), qual(k, B)), \max(qual(k, A), 4 - qual(k, B))).$

Corollary 3.3.

- a) $qual(k, A \implies B) = qual(k, \smile B \implies \smile A);$
- b) $qual(k, A \iff B) = qual(k, \smile B \iff \smile A).$

Theorem 3.4.

- a) $qual(k, \smile (\smile A)) = qual(k, A).$
- b) $qual(k, A) \in R_i$ if and only if $qual(k, \smile A) \in R_{2-i}.$
- c) $qual(k, A) \leq qual(k, B)$ if and only if $qual(k, \smile B) \leq qual(k, \smile A).$

Proof.

- a) According to the Lemma 3.2,
 $qual(k, \smile (\smile A)) = 4 - qual(k, \smile A) = 4 - (4 - qual(k, A)) = qual(k, A).$
Hence, $qual(k, \smile (\smile A)) = qual(k, A).$
- b) 1. Let $qual(k, A) \in R_i$, whence by A1 $ec(k, \smile A) = 1 - ec(k, A)$. By Theorem 3.1 $qual(k, \smile A) \in R_{2-i}$. Hence $qual(k, A) \in R_i$ implies $qual(k, \smile A) \in R_{2-i}$.
2. Let $qual(k, \smile A) \in R_{2-i}$, according with (1) $qual(k, \smile (\smile A)) \in R_i$ and by (a) $qual(k, A) \in R_i$. Hence $qual(k, \smile A) \in R_{2-i}$ implies $qual(k, A) \in R_i$.
By (1) and (2) $qual(k, A) \in R_i$ if and only if $qual(k, \smile A) \in R_{2-i}$.
- c) According to the Lemma 3.2 $qual(k, \smile A) = 4 - qual(k, A) \geq 4 - qual(k, B) = qual(k, \smile B)$ if and only if $qual(k, A) \leq qual(k, B)$. So result follow.

□

Definition 3.5. $A \equiv B$ if and only if $qual(k, A) = qual(k, B).$

Remark 3.6. If $ec(k, A) = ec(k, B)$, $sec(k, A) = sec(k, B)$ and $sef(k, A) = sef(k, B)$, then $A \equiv B$.

Proposition 3.7. $qual(k, A \iff B) = qual(k, (A \wedge B) \vee (\smile A \wedge \smile B)).$

Proof. By items a), b) and c) of Lemma 3.2 and using that if a, b and c are real numbers, then $a - \max(a-b, a-c) = \min(b, c)$. The result follows. $\square$

Theorem 3.8. [7]

- a) $qual(k, A \wedge B) = qual(k, \smile (\smile A \vee \smile B))$;
- b) $qual(k, A \vee B) = qual(k, \smile (\smile A \wedge \smile B))$;
- c) $qual(k, A \implies B) = qual(k, \smile A \vee B)$;
- d) $qual(k, A \iff B) = qual(k, (A \implies B) \wedge (B \implies A))$.

Corollary 3.9.

- a) $A \wedge B \equiv \smile (\smile A \vee \smile B)$.
- b) $A \vee B \equiv \smile (\smile A \wedge \smile B)$.
- c) $A \implies B \equiv \smile A \vee B$.
- d) $A \iff B \equiv (A \implies B) \wedge (B \implies A)$.
- e) $A \implies B \equiv \sim B \implies \sim A$
- f) $(A \iff B) \equiv (\sim B \iff \sim A)$
- g) $A \iff B \equiv (A \wedge B) \vee (\sim A \wedge \sim B)$

Notice that the relation $A \equiv B$ proves the properties of Boolean Algebra. It should be noted that in a statement, if some of the statements that constitute it are substituted by others of equal value, then the latter statement is equivalent to the former one. Logical equivalence has a very significant practical interest with regard to the decision process, especially with respect to any refinement of the restrictions of a decision, since those properties can serve as the frame of reference in which to establish, through the use of connectives, relations between restrictions. For instance, they allow the most appropriate alternatives to be chosen in order to apply the empirical verification method of plausibility.

Furthermore, the following theorem is formulated through equivalence which determines the scope of plausibility logic.

Theorem 3.10.

- a) $\forall x Ax \equiv Ax_1 \wedge Ax_2 \cdots Ax_n$.
- a) $\exists x Ax \equiv Ax_1 \vee Ax_2 \cdots Ax_n$.

Proof.

- a) By A6, $ec(k, \forall xAx) = ec(k, Ax_1 \wedge Ax_2 \cdots Ax_n)$, $ecs(k, \forall xAx) = ecs(k, Ax_1 \wedge Ax_2 \cdots Ax_n)$ and $efs(k, \forall xAx) = efs(k, Ax_1 \wedge Ax_2 \cdots Ax_n)$. By Lemma 3.2 and Definition 3.5 the result follows.
- a) By A6, $ec(k, \exists xAx) = ec(k, Ax_1 \vee Ax_2 \cdots Ax_n)$, $ecs(k, \exists xAx) = ecs(k, Ax_1 \vee Ax_2 \cdots Ax_n)$ and $efs(k, \exists xAx) = efs(k, Ax_1 \vee Ax_2 \cdots Ax_n)$. Notice that if $ec(k, A) = ec(k, B)$, $sec(k, A) = sec(k, B)$, $sef(k, A) = sef(k, B)$ then $qual(k, A) = qual(k, B)$. So Definition 3.5 the result follows. $\square$

The theorem proves that the logic of plausibility is a propositional logic. From the perspective of its application, it is of great importance that the logic of plausibility be reduced to a propositional logic, given that this is the least complex level of logic.

Theorem 3.11. *Plausibility logic is not monotonic.*

Proof. The theory of plausibility, due to the evolutive nature of decisions that are successively refined or reconsidered, the evidence is subject to change.

It is sufficient to consider: $k, m \in K$ with $k < m$ such that $ec(k, A) = 1$, $sec(k, A) = 1$, $ec(m, A) = 1$ and $sec(m, A) = 0$. By definition 2.2, $qual(k, A) = 0$ and $qual(m, A) = 1$. $\square$

Graphs and Plausibility

Figure 1:

Let $P(k, A, i)$ the number of time that the proposition A take the plausibility value i for $i \in \{0, 1, 2, 3, 4\}$. For any proposition, A , there exist total uncertainty if $P(k, A, i) = P(k, A, j)$, $i \neq j$ with $i, j \in \{0, 1, 2, 3, 4\}$.

When A is a simple proposition there exist total certainty if $\exists! i$ with $0 \leq i \leq 4$ so that $P(k, A, i) = 5$ and $P(k, A, j) = 0$ for $j \neq i$ and when the proposition A is molecular $A = A_1 * A_2 * \cdots A_n$ ($*$ denote a logic operator) there exist total certainty if $\exists! i$ with $0 \leq i \leq 4$ so that $P(k, A, i) = 5^n$ and $P(k, A, j) = 0$ for $j \neq i$.

For example, if A is a simple proposition, then $P(k, A, i) = 1$ for all i . Hence, there exist total uncertainty. For $A \vee B$ we have: $P(k, A \vee B, i) = 2i + 1$ for $i = \{0, 1, 2, 3, 4\}$; its graph of certainty is showed in the Figure 1. Notice that

$$P(k, A \wedge B, i) + P(k, A \vee B, i) = 10$$

for $i = \{0, 1, 2, 3, 4\}$. Hence, if we know the certainty of $A \wedge B$ we can know the certainty of $A \vee B$ and conversely.

Figure 2:

For $A \Rightarrow B$ we have: $P(k, A \Rightarrow B, i) = 2i + 1$ for $i = \{0, 1, 2, 3, 4\}$. The proposition $A \Rightarrow B$ has the graph of certainty showed in the left side of Figure 2.

For $A \Leftrightarrow B$ we have: $P(k, A \Leftrightarrow B, i) = P(k, A \Leftrightarrow B, 4 - i)$.

$$P(k, A \Leftrightarrow B, i) = \begin{cases} 2 & \text{if } i = 0, \\ 6 & \text{if } i = 1, \\ 9 & \text{if } i = 2. \end{cases}$$

The proposition $A \Leftrightarrow B$ has the graph of certainty showed in the right side of Figure 2. Notice that in this case no exist total uncertainty like in the formal logic. The proposition $A \wedge (\sim A)$ is not a imposible proposition,

Figure 3:

like in formal logic. Notice that this proposition takes the values with menor

plausibility (and minor probability for the more plausible, that is to say $i = 2$) and for $A \vee (\sim A)$ we do not obtain a tautological proposition like in formal logic: the most plausible values are the largest with minor probability the minor of them (that is $i = 2$), so there is not total uncertainty like in formal logic.

The inference laws of the formal logic (see the graph of certainty for Modus ponens showed in the right side of Figure 3) do not generate tautological propositions. The behavior of low Modus tollens is similar. The reader interested in to study a relation for the ponens modus and the alliances in tree (see [5]).

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