The Renormalization of Self Intersection Local Times of Fractional Brownian Motion

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Abstract

We present a new approach to treat the problem of self intersection local time of a d-dimensional Fractional Brownian motion based on the property of chaotic representation and the white noise analysis. This approach could be generalized to general Gaussian processes.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space which could be specified later and $\{B_t : t \geq 0\}$ a d-dimensional fractional Brownian Motion (fBm) with Hurst parameter $H = (H_1, \dots, H_d) \in]0, 1[^d$.

It is known that

$$B_t = \int_0^t K_H(t, s) dw_s$$

where $\{w_t : t \geq 0\}$ is a standard Brownian motion and $K_H = (K_{H_1}, \dots, K_{H_d})$ is a kernel, (see for instance [16] for the proof).

We are interested, in this paper, in computing, when it exists, the self-intersection local time of the fBm. More precisely we study the existence of the limit when ϵ goes to zero, of the following sequence of processes

$$L_T^\epsilon = \int_0^T \int_0^t P^\epsilon(B_t - B_s) ds dt$$

where

$$P^\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}^d} e^{-\frac{|x|^2}{2\epsilon}}, \quad x \in \mathbb{R}^d.$$

Many questions arise here

1) Find out a relation between N the number of chaos to be subtracted from L_T^ϵ , d and H such that

$$L_T^\epsilon - \sum_{n=0}^N L_{T,n}^\epsilon$$

admits a limit in an appropriate space when $\epsilon \rightarrow 0$.

2) Under which condition on d and H there is a blow up of the expectation $\mathbb{E}(L_T^\epsilon)$ when $\epsilon \rightarrow 0$?

3) Suppose we are in the blow up case, find out a renormalization factor $r_{d,H}(\epsilon)$ such that

$$r_{d,H}(\epsilon) \left(L_T^\epsilon - \mathbb{E}(L_T^\epsilon) \right)$$

is bounded in $L^2(\Omega)$ when $\epsilon \rightarrow 0$.

4) Find out the limit in $L^2(\Omega)$ or in law, when it exists, of

$$r_{d,H}(\epsilon) \left(L_T^\epsilon - \mathbb{E}(L_T^\epsilon) \right)$$

when $\epsilon \rightarrow 0$.

In view of what has been done in the case of the classical Brownian motion, see for instance [3][8] [9] [11][17] and [18] we should compute explicitly the chaos of the fBm, for this we need some tools from the white noise analysis.

1.1 Tools from White Noise Analysis

We quote some white noise analysis concepts as introduced in [3], referring to [10] for a systematic presentation. Consider a white noise space $(S'(\mathbb{R})^d, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the weak Borel σ -algebra of $S'(\mathbb{R})^d$, and μ is the centered Gaussian measure whose covariance is given by the inner product of $L^2(\mathbb{R})^d$, in the sense that the vector valued white noise has the characteristic function

$$C(\mathbf{f}) = \mathbb{E}(e^{i\langle \omega, \mathbf{f} \rangle}) = \int_{S'(\mathbb{R})^d} d\mu[\omega] e^{i\langle \omega, \mathbf{f} \rangle} = e^{-\frac{1}{2}\langle \mathbf{f}, \mathbf{f} \rangle}, \quad (1)$$

where $\langle \omega, \mathbf{f} \rangle = \sum_{j=1}^d \langle \omega_j, f_j \rangle$ and $f_j \in S(\mathbb{R}, \mathbb{R})$. Then a realization of a vector of independent fractional Brownian motions B_j , $j = 1, \dots, d$, is given by

$$B_j(t) = \langle \omega_j, K_{H_j} \rangle = \int_0^t K_{H_j}(s, t) \omega_j(s) ds. \quad (2)$$

We recall the explicit formula of the kernel of a one dimensional fBm with Hurst parameter $h \in]0, 1[$, see for instance [1]

$h < 1/2$

$$K_h(t, u) = C_h \left\{ \left(\frac{t}{2} \right)^{h-1/2} (t-u)^{h-1/2} - \right. \quad (3)$$

$$(h - 1/2)u^{1/2-h} \int_u^t (r - u)^{h-1/2} r^{h-3/2} dr \Big\} 1_{[0,t]}(u),$$

where C_h is some constant.

$h > 1/2$

$$K_h(t, u) = C_h u^{1/2-h} \int_u^t (r - u)^{h-3/2} r^{h-1/2} dr 1_{[0,t]}(u) \quad (4)$$

where C_h is some constant.

Hence we consider independent d-tuples of Gaussian white noise $\omega = (w_1, \dots, w_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, \dots, f_d) \in S(\mathbb{R}, \mathbb{R}^d)$, and use the following multi-index notation:

$$\vec{n}! = \prod_1^d n_i! \quad (5)$$

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^d \int dt f_i^2(t) \quad (6)$$

$$\langle F_{\vec{n}}, \mathbf{f}^{\otimes \vec{n}} \rangle = \int d^n t F_{\vec{n}}(t_1, \dots, t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, \dots, t_n) \quad (7)$$

and similarly for $\langle : \omega^{\otimes \vec{n}} :, F_{\vec{n}} \rangle$ where for d-tuples of white noise the Wick product $: \dots :$ (see [10]) generalizes to

$$: \omega^{\otimes \vec{n}} := \bigotimes_{i=1}^d : \omega_i^{\otimes n_i} :. \quad (8)$$

The Hilbert space

$$(L^2) = L^2(d\mu) \quad (9)$$

is canonically isomorphic to the d-fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq \left(\bigoplus_{k=0}^{\infty} Sym L^2(\mathbb{R}^k, k! d^k t) \right)^{\otimes d} = \mathcal{F}. \quad (10)$$

For a general element φ of (L^2) this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} \langle : \omega^{\otimes \vec{n}} :, F_{\vec{n}} \rangle, \quad (11)$$

the norm of φ is given by

$$\|\varphi\|_{(L^2)}^2 = \sum_{\vec{n}}^{\infty} \vec{n}! |F_{\vec{n}}|_{2,n}^2 \quad (12)$$

with kernel functions F in $\mathcal{F}$ and where $\|\cdot\|_{2,n}$ is the norm in $L^2(\mathbb{R}^n, dt)$. Given $\xi \in S(\mathbb{R})^d$, let us consider the Wick exponential

$$:\exp\langle\omega, \xi\rangle: \equiv \exp\left(\langle\omega, \xi\rangle - \frac{1}{2}(\xi, \xi)\right) \quad (13)$$

$$= \sum_{\vec{n}} \frac{1}{\vec{n}!} \langle: \omega^{\otimes \vec{n}} :, \xi^{\otimes \vec{n}} \rangle, \quad \omega \in S'(\mathbb{R})^d. \quad (14)$$

The S-transform plays an important role in the study of stochastic processes in particular the computation of their chaos expansion, see for example [?][10]; we define the S-transform of φ in (L^2) as

$$S\varphi(\xi) \equiv \ll \varphi, : \exp\langle \cdot, \xi \rangle : \gg = \sum_{\vec{n}} (\varphi_{\vec{n}}, \xi^{\otimes \vec{n}})_{2,n}. \quad (15)$$

In particular, for Hermitian operators A in $L^2(\mathbb{R})$, we can define the "second quantization" of A as an operator $\Gamma(A)$ in (L^2) given by

$$S\Gamma(A)\Phi(\cdot) = S\Phi(A\cdot) \quad (16)$$

for $\Phi \in (L^2)$.

Generalized functions are obtained via a Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*$$

with (S) the projective limit Hilbert spaces $(S)_k$

$$(S) \equiv \text{projlim}_k (S)_k \quad \text{and} \quad (S)_k \equiv D(\Gamma(A^k))$$

where A is operating on $L^2(\mathbb{R}, du)$

$$Af(u) = (-d^2/du^2 + u^2 + 1)f(u).$$

2 The kernels of the self-intersection local time of the fBm

$$L_T^\epsilon = \int_0^T \int_0^t P^\epsilon(B_t - B_s) ds dt$$

$$P^\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}^d} e^{-\frac{|x|^2}{2\epsilon}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi - \frac{\epsilon \xi^2}{2}} d\xi$$

so

$$L_T^\epsilon = \int_0^T \int_0^t dt ds \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(B_t - B_s) \cdot \xi - \frac{\epsilon \xi^2}{2}} d\xi$$

in view of (2), for $j = 1, \dots, d$, $B_t^j - B_s^j = \langle \Delta K_j, \omega_j \rangle$ with $\Delta K_j = K_{H_j}(u, t) - K_{H_j}(u, s)$

$$\begin{aligned} S\left(e^{i(B_t-B_s)\cdot\xi}\right)(f) &= \prod_{j=1}^d S\left(e^{i\langle\Delta K_j,\omega_j\rangle\xi_j}\right)(f_j) \\ S\left(e^{i\langle\Delta K_j,\omega_j\rangle\xi_j}\right)(f_j) &= e^{-\frac{|f_j|_2^2}{2}} \int_{S'(\mathbb{R})} d\gamma_j(\omega_j) e^{i\langle\Delta K_j,\omega_j\rangle\xi_j + \langle f_j, \omega_j \rangle} \\ &= e^{-\frac{|f_j|_2^2}{2}} \int_{S'(\mathbb{R})} d\gamma_j(\omega_j) e^{i\langle\xi_j\Delta K_j - if_j\rangle} \\ &= e^{-\frac{\xi_j^2|\Delta K_j|_2^2}{2}} e^{i\xi_j\langle\Delta k_j, f_j\rangle} \end{aligned}$$

The last equality was by definition of γ and so

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi e^{-\frac{\epsilon\xi^2}{2}} S\left(e^{i(B_t-B_s)\cdot\xi}\right)(f) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \prod_{j=1}^d e^{-\frac{\xi_j^2}{2}(|\Delta K_j|_2^2 + \epsilon)} e^{i\xi_j\langle\Delta k_j, f_j\rangle} d\xi_j$$

using this elementary equality

$$\int_{\mathbb{R}} e^{-\frac{\alpha x^2}{2} + i\beta x} = \sqrt{\frac{2\pi}{\alpha}} e^{-\frac{\beta^2}{2\alpha}}$$

we obtain

$$\begin{aligned} SP^\epsilon(B_t - B_s)(f) &= \frac{1}{\sqrt{2\pi}^d} \prod_{j=1}^d \frac{1}{\sqrt{|\Delta K_j|_2^2 + \epsilon}} \exp\left\{-\frac{\langle f_j, \Delta K_j \rangle^2}{2(|\Delta K_j|_2^2 + \epsilon)}\right\} \\ &= \frac{1}{\sqrt{2\pi}^d} \prod_{j=1}^d \frac{1}{\sqrt{|t-s|^{2H_j} + \epsilon}} \exp\left\{-\frac{\langle f_j, \Delta K_j \rangle^2}{2(|t-s|^{2H_j} + \epsilon)}\right\} \\ &= \frac{1}{\sqrt{2\pi}^d} \prod_{j=0}^d \sum_{n_j=0}^{\infty} \frac{1}{n_j!} \left(-\frac{1}{2}\right)^{n_j} \frac{1}{(|t-s|^{2H_j} + \epsilon)^{n_j+1/2}} \langle f_j, \Delta K_j \rangle^{2n_j} \\ &= \frac{1}{\sqrt{2\pi}^d} \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \left(-\frac{1}{2}\right)^n \prod_{j=1}^d \left(\frac{1}{(|t-s|^{2H_j} + \epsilon)^{n_j+1/2}}\right) \langle f^{\otimes 2\vec{n}}, \Delta K^{\otimes 2\vec{n}} \rangle. \end{aligned}$$

Finally we obtain

Proposition 1 Let $\vec{n} \in \mathbb{N}^d$ and $d \geq 1$,

$$L_T^\epsilon = \sum_{\vec{n}=0}^{\infty} L_{T,2\vec{n}}^\epsilon = \frac{1}{\sqrt{2\pi}^d} \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \left(-\frac{1}{2}\right)^n \langle \omega^{\otimes 2\vec{n}} : , l_{T,2\vec{n}}^\epsilon \rangle$$

then

$$l_{T,2\vec{n}}^\epsilon = \int_0^T dt \int_0^t ds \prod_{j=1}^d \left(\frac{1}{(|t-s|^{2H_j} + \epsilon)^{n_j+1/2}}\right) \Delta K^{\otimes 2\vec{n}}$$

2.1 The expectation of the self-intersection local time of the fBm

Let us now compute the expectation of the self-intersection local time of the fBm, $\mathbb{E}(L_T^\epsilon)$, it is just the first chaos, so in view of the last proposition

$$\begin{aligned}\mathbb{E}(L_T^\epsilon) &= \frac{1}{\sqrt{2\pi}^d} \int_0^T dt \int_0^t ds \prod_{j=1}^d \left(\frac{1}{(|t-s|^{2H_j} + \epsilon)^{1/2}} \right) \\ &= \frac{1}{\sqrt{2\pi}^d} \int_0^T ds \frac{T-s}{\prod_{j=1}^d (s^{2H_j} + \epsilon)^{1/2}}\end{aligned}$$

we use the change of variables $s = \epsilon^{\frac{d}{2H^*}} z = \alpha(\epsilon)z$ with $H^* = \sum_{j=1}^d H_j$

$$\mathbb{E}(L_T^\epsilon) = \frac{\alpha(\epsilon)\epsilon^{-\frac{d}{2}}}{\sqrt{2\pi}^d} \int_0^{\frac{T}{\alpha(\epsilon)}} \frac{T - \alpha(\epsilon)z}{\prod_{j=1}^d (\epsilon^{\frac{dH_j}{H^*}-1} z^{2H_j} + 1)^{1/2}} dz$$

we divide the integral in two parts

$$\begin{aligned}\mathbb{E}(L_T^\epsilon) &= \frac{\alpha(\epsilon)\epsilon^{-\frac{d}{2}}}{\sqrt{2\pi}^d} \left\{ \int_0^1 \frac{T - \alpha(\epsilon)z}{\prod_{j=1}^d (\epsilon^{\frac{dH_j}{H^*}-1} z^{2H_j} + 1)^{1/2}} dz \right. \\ &\quad \left. + \int_1^{\frac{T}{\alpha(\epsilon)}} \frac{T - \alpha(\epsilon)z}{\prod_{j=1}^d (\epsilon^{\frac{dH_j}{H^*}-1} z^{2H_j} + 1)^{1/2}} dz \right\}\end{aligned}$$

the first integral in braces is bounded.

Set π the second integral in braces, so

$$\pi \leq T \int_1^{\frac{T}{\alpha(\epsilon)}} \frac{dz}{z^{H^*}} = \begin{cases} T^{1-H^*} \epsilon^{\frac{d}{2}-\frac{d}{2H^*}} - 1 & H^* \neq 1 \\ \log T - \frac{d}{2H^*} \log \epsilon & H^* = 1 \end{cases}$$

Proposition 2 Let $T > 0$ and $d \geq 1$.

If $H^* \geq 1$

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}(L_T^\epsilon) = \infty.$$

Moreover if $H^* = 1$

$$\mathbb{E}(L_T^\epsilon) \leq \text{const.} \times |\log \epsilon|,$$

and if $H^* > 1$

$$\mathbb{E}(L_T^\epsilon) \leq \text{const.} \times \epsilon^{\frac{d}{2}-\frac{d}{2H^*}}$$

where the constants depend on T , H^* and d .

If $H^* < 1$ there is no blow up

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}(L_T^\epsilon) = \frac{1}{\sqrt{2\pi}^d} \int_0^T ds \frac{T-s}{s^{H^*}}.$$

Proposition 3 Suppose all $H_j = H$. Let $T > 0$ and $d \geq 1$.
If $H^* = dH = 1$

$$\mathbb{E}(L_T^\epsilon) = \frac{1}{2H} |\log \epsilon| \left(1 + o(1)\right).$$

If $dH > 1$

$$\mathbb{E}(L_T^\epsilon) = C_H \epsilon^{\frac{1}{2H} - \frac{d}{2}} \left(1 + o(1)\right),$$

$$\text{where } C_H = \int_0^\infty \frac{ds}{(s^{2H} + 1)^{d/2}}.$$

Remark

If all $H_j = \frac{1}{2}$ and $d = 2$ we are in the case $H^* = 1$ and we obtain the Varadhan renormalization term [23].

3 The self-intersection local time of the fBm as a generalized function

First of all let us recall

Theorem 1. [9]

Let $(\Omega, \mathcal{F}, m)$ be a measure space, and Φ_λ a mapping defined on Ω with values in $(S)^*$. We assume that the S -transform of Φ_λ

(1) is an m -measurable function of λ for any test function $f \in S(\mathbb{R})^d$

(2) obeys to the following estimate

$$|S\Phi_\lambda(f)| \leq C_1(\lambda) \exp\{C_2(\lambda) |A^p f|_2^2\}$$

for some fixed p and for $C_1 \in L^1(m)$, $C_2 \in L^\infty(m)$.

Then Φ_λ is Bochner-integrable in the Hilbert space $(S)_{-q}$ for q large enough,

$$\int_\Omega dm(\lambda) \Phi_\lambda \in (S)^*$$

and

$$S\left(\int_\Omega dm(\lambda) \Phi_\lambda\right)(f) = \int_\Omega dm(\lambda) S\Phi_\lambda(f).$$

Using the last theorem and the following "formula"

$$\delta'' = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} d\xi$$

we could define $\delta(B_t - B_s)$ as a Bochner integral in $(S)^*$ and

$$S(\delta(B_t - B_s))(f) = \frac{1}{\sqrt{2\pi}^d} \frac{1}{|t-s|^{H^*}} \prod_{j=1}^d \exp\left\{-\frac{\langle f_j, \Delta K_j \rangle^2}{|t-s|^{2H_j}}\right\}$$

And so, again by the last theorem, if $H^* < 1$

$$L_T = \int_0^T \int_0^t ds dt \delta(B_t - B_s)$$

is well defined in $(S)^*$.

Suppose now that $H^* \geq 1$.

The idea is that if we subtract some of the first terms in the expansion of the exponential function in the expression of the S-transform of $\delta(B_t - B_s)$, we could obtain an integrable function in factor of the remaining part, then the second condition of Theorem 1 will be satisfied.

And so we could define a renormalization of the self-intersection local time in $(S)^*$.

Let $N \in \mathbb{N}$ and define $\delta^{(2N)}(B_t - B_s)$ by its S-transform

$$S\delta^{(2N)}(B_t - B_s)(f) = \frac{1}{\sqrt{2\pi}^d} \frac{1}{|t-s|^{H^*}} \sum_{\vec{n}, n \geq N} \left(\frac{-1}{2}\right)^n \prod_{j=1}^d \frac{\langle f_j, \Delta K_j \rangle_2^{2n_j}}{|t-s|^{2H_j n_j}}$$

so

$$|S\delta^{(2N)}(B_t - B_s)(f)| \leq \frac{1}{\sqrt{2\pi}^d} \frac{1}{|t-s|^{H^*}} \sum_{\vec{n}, n \geq N} \frac{1}{2^n} \prod_{j=1}^d \frac{|f_j|_\infty^{2n_j} |\Delta K_j|_1^{2n_j}}{|t-s|^{2H_j n_j}}$$

we need to estimate the L^1 -norm of ΔK_j , $|\Delta K_j|_1$ for fixed j .

1) We treat first the case when all $H_j > 1/2$.

Let $h > 1/2$, in view of (4)

$$\begin{aligned} \Delta K_h &= K_h(t, u) - K_h(s, u) = C_h u^{1/2-h} \left\{ 1_{[0,s]}(u) \int_s^t (r-u)^{h-3/2} r^{h-1/2} dr \right. \\ &\quad \left. + 1_{[s,t]}(u) \int_u^t (r-u)^{h-3/2} r^{h-1/2} dr \right\} \end{aligned}$$

we obtain

$$|\Delta K_h|_1 \leq C_h \{c_1 T^{h-1/2} |t-s| + c_2 |t-s|^{h+1/2}\}$$

suppose that $|t-s|$ is small enough, we get

$$\begin{aligned} |S\delta^{(2N)}(B_t - B_s)(f)| &\leq \frac{1}{\sqrt{2\pi}^d} \frac{1}{|t-s|^{H^*}} \sum_{\vec{n}, n \geq N} \frac{1}{2^n} |t-s|^{2n} \prod_{j=1}^d \frac{1}{|t-s|^{2H_j n_j}} \prod_{j=1}^d |f_j|_\infty^{2n_j} \\ &\leq \frac{1}{\sqrt{2\pi}^d} \frac{1}{|t-s|^{H^*+2N(\bar{H}-1)}} \exp\left\{\frac{1}{2} \sum_{j=1}^d |f_j|_\infty^2\right\} \end{aligned}$$

where $\bar{H} = \max_j H_j$.

Theorem 2. *Let $T > 0$ and $N \in \mathbb{N}$, suppose that*

$$2N > \frac{H^* - 1}{1 - \bar{H}}$$

then

$$L_T^{(2N)} \equiv \int_0^T \int_0^t ds dt \delta^{(2N)}(B_t - B_s)$$

is well defined as an element of $(S)^$ and*

$$\lim_{\epsilon \rightarrow 0} L_\epsilon^{(2N)} = L^{(2N)} \quad \text{in } (S)^*$$

2) When all $H_j < 1/2$ we obtain a bad estimation of $|\Delta K_j|_1$ and so we don't have a result.

Remark

When all $H_j = 1/2$, the condition under which $L^{(2N)}$ is well defined in $(S)^*$ is that $2N > d - 2$, this correspond to a result obtained in [9].

4 Estimation of the L^2 -norms of the chaos of the fBm

Suppose, in this section that we are in the blow up case i.e $H^* \geq 1$ and that $H < \frac{1}{2}$. Now we state our main result

Theorem 3. *Let $T > 0$, $\vec{n} \neq 0$ and $d \geq 2$. Then, if $dH = 1$*

$\frac{1}{\sqrt{|\log \epsilon|}} |L_{T, 2\vec{n}}^\epsilon|_2$ has a finite non trivial limit when ϵ goes to zero and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{|\log \epsilon|}} |L_{T, 2\vec{n}}^\epsilon|_2 \leq \frac{1}{\sqrt{2\pi}^d} \frac{\sqrt{2}T \sqrt{(2\vec{n})!}}{(\vec{n}!)} b(H, d, n)^{1/2}.$$

If $dH \in]1, 3/2[$,

$$\lim_{\epsilon \rightarrow 0} |L_{2\vec{n}}^\epsilon|_2^{\leq} \frac{1}{\sqrt{2\pi}^d} \frac{\sqrt{(2\vec{n})!}}{(\vec{n}!)} \frac{c(H, d, n)^{1/2}}{\sqrt{(3 - 2Hd)(2 - Hd)}} T^{2-Hd}.$$

If $dH = 3/2$

$\frac{1}{\sqrt{|\log \epsilon|}} |L_{T,2\vec{n}}^\epsilon|_2$ has a finite non trivial limit when ϵ goes to zero and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{|\log \epsilon|}} |L_{T,2\vec{n}}^\epsilon|_2 \leq \frac{1}{\sqrt{2\pi}^d} \sqrt{\frac{T}{H}} \frac{\sqrt{(2\vec{n})!}}{(\vec{n}!)} c(H, d, n)^{1/2}.$$

If $dH > 3/2$

$$\lim_{\epsilon \rightarrow 0} \epsilon^{\frac{d}{2} - \frac{3}{4H}} |L_{2\vec{n}}^\epsilon|_2 = \frac{1}{\sqrt{2\pi}^d} \frac{\sqrt{2T} \sqrt{2\vec{n}!}}{\vec{n}! 2^n} \left(f(H, d, n) + g(H, d, n) \right)^{1/2},$$

where the constants b, c, f and g are given in (21), (23), (19) and (22)

Remark

1) The results of the last theorem generalizes the case of the classical Brownian motion i.e $H = 1/2$, in fact it was shown in [3] that the renormalization factor was $|\log \epsilon|^{-1/2}$ when $d = 3$ and $\epsilon^{\frac{d-3}{2}}$ when $d \geq 4$.

2) It was shown in [21] that when $d = 2$ and $1/d < H < \frac{3}{2d}$ there is no need to renormalize by multiplication, the result in our theorem could be seen as a generalization for $H < 1/2$ and $d \geq 2$.

3) The case $H > 1/2$ is more complicated because we obtain a specific singularity in ϵ for each chaos. Which is not so surprising because when $H > 1/2$ the fBm is smoother and, intuitively, its self intersection local time is worse.

Proof of Theorem 3

For $\vec{n} \in \mathbb{N}^d$ the $2\vec{n}$ th chaos is given by (see proposition 1)

$$L_{T,2\vec{n}}^\epsilon = \frac{1}{\sqrt{2\pi}^d} \frac{1}{\vec{n}!} \left(-\frac{1}{2}\right)^n \langle \omega^{\otimes 2\vec{n}} :, l_{T,2\vec{n}}^\epsilon \rangle$$

$$l_{T,2\vec{n}}^\epsilon = \int_0^T dt \int_0^t ds \prod_{j=1}^d \left(\frac{1}{(|t-s|^{2H_j} + \epsilon)^{n_j+1/2}} \right) \Delta K^{\otimes 2\vec{n}}$$

so

$$\mathbb{E}\{(L_{T,2\vec{n}}^\epsilon)^2\} = \frac{1}{(2\pi)^d} \frac{1}{(\vec{n}!)^2} \left(\frac{1}{2}\right)^{2n} (2\vec{n})! |l_{T,2\vec{n}}^\epsilon|_{L^2(\mathbb{R}^{2n})}^2 \quad (17)$$

In view of (17), we need to estimate $|l_{T,2\vec{n}}^\epsilon|_{\mathbb{R}^{2n}}^2 = |l_{T,2\vec{n}}^\epsilon|_{2,2n}^2$,

$$|l_{T,2\vec{n}}^\epsilon|_{2,2n}^2 =$$

$$\int_{\mathbb{R}^{2n}} d^{2n}u \int_0^T \int_0^t \int_0^T \int_0^{t'} dt ds dt' ds' \prod_{j=1}^d \frac{\Delta K_j(t, s)^{\otimes 2n_j}(t, s) \Delta K_j(t, s)^{\otimes 2n_j}(t', s')}{\left((|t-s|^{2H_j} + \epsilon)(|t'-s'|^{2H_j} + \epsilon)\right)^{n_j+1/2}}$$

by using Fubini theorem we first get

$$\begin{aligned}
 & \int_{\mathbb{R}^{2n}} d^{2n}u \prod_{j=1}^d \Delta K_j(t, s)^{\otimes 2n_j} \Delta K_j(t, s)^{\otimes 2n_j}(t', s') = \\
 &= \prod_{j=1}^d \int_{\mathbb{R}^{2n_j}} d^{2n_j}u \Delta K_j(t, s)^{\otimes 2n_j} \Delta K_j(t, s)^{\otimes 2n_j}(t', s') \\
 &= \prod_{j=1}^d \prod_{i=1}^{2n_j} \int_{\mathbb{R}} du_i^j \Delta K_j(t, s)(t, s) \Delta K_j(t, s)(t', s') \\
 &= \prod_{j=1}^d \prod_{i=1}^{2n_j} \mathbb{E} \left(\Delta B_j(t, s) \Delta B_j(t', s') \right) \\
 &= \prod_{j=1}^d \left(|t - t'|^{2H_j} + |s - s'|^{2H_j} - |t - s'|^{2H_j} - |s - t'|^{2H_j} \right)^{2n_j}
 \end{aligned}$$

so

$$\begin{aligned}
 |l_{T, 2\vec{n}}^\epsilon|_{2, 2n}^2 &= \int_0^T \int_0^t \int_0^T \int_0^{t'} dt ds dt' ds' \times \\
 & \prod_{j=1}^d \frac{\left(|t - t'|^{2H_j} + |s - s'|^{2H_j} - |t - s'|^{2H_j} - |s - t'|^{2H_j} \right)^{2n_j}}{\left((|t - s|^{2H_j} + \epsilon)(|t' - s'|^{2H_j} + \epsilon) \right)^{n_j + 1/2}} \quad (18)
 \end{aligned}$$

in view of the symmetry of the domain and the integrand function it suffices to integrate only on $T_1 \cup T_2$ where $T_1 = \{0 < s' < t' < s < t < T\}$ and $T_2 = \{0 < s' < s < t' < t < T\}$.

Let us first integrate over T_1 , we make the following change of variables

$$\begin{cases} x = t - s \\ y = t' - s' \\ z = s - t' \end{cases}$$

where t is considered as a parameter. Set $|l_{T, 2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2$ (resp. $|l_{T, 2\vec{n}}^{\epsilon, 2}|_{2, 2n}^2$) the integral over T_1 (resp. T_2), we obtain

$$\begin{aligned}
 |l_{T, 2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2 &= \int_0^T dt \int_{0 \leq x+y+z \leq t} dx dy dz \times \\
 & \prod_{j=1}^d \frac{\left((x+z)^{2H_j} + (y+z)^{2H_j} - (x+y+z)^{2H_j} - z^{2H_j} \right)^{2n_j}}{\left((x^{2H_j} + \epsilon)(y^{2H_j} + \epsilon) \right)^{n_j + 1/2}}
 \end{aligned}$$

it is almost impossible to compute this integral (at least for us) when all H_j are different, so let us suppose that all H_j are equal to some H .

Denote by $\theta_t(\epsilon) = \frac{t}{\epsilon^{2H}}$, and make the following change of variables, $(x, y, z) = \epsilon^{\frac{1}{2H}}(x', y', z')$, we get

$$|l_{T,2n}^{\epsilon,1}|_{2,2n}^2 = \epsilon^{\frac{3}{2H}-d} \int_0^T \int_{0 \leq x+y+z \leq \theta_t(\epsilon)} dx dy dz \times \frac{\left((x+z)^{2H} + (y+z)^{2H} - (x+y+z)^{2H} - z^{2H}\right)^{2n}}{\left((x^{2H}+1)(y^{2H}+1)\right)^{n+d/2}},$$

by a symmetry argument in (x, y)

$$|l_{T,2n}^{\epsilon,1}|_{2,2n}^2 = 2\epsilon^{\frac{3}{2H}-d} \int_0^T \int_{\substack{0 \leq x+y+z \leq \theta_t(\epsilon) \\ 0 \leq x \leq y \leq \theta_t(\epsilon)}} dx dy dz \times \frac{\left((x+z)^{2H} + (y+z)^{2H} - (x+y+z)^{2H} - z^{2H}\right)^{2n}}{\left((x^{2H}+1)(y^{2H}+1)\right)^{n+d/2}},$$

denote by

$$f_H(x, y, z) = (x+z)^{2H} + (y+z)^{2H} - (x+y+z)^{2H} - z^{2H}$$

and by

$$f(t, H, d, n, \epsilon) = \int_{\substack{0 \leq x+y+z \leq \theta_t(\epsilon) \\ 0 \leq x \leq y \leq \theta_t(\epsilon)}} dx dy dz \frac{\left(f_H(x, y, z)\right)^{2n}}{\left((x^{2H}+1)(y^{2H}+1)\right)^{n+d/2}}.$$

First, note that $|f_H(x, y, z)| \leq x^{2H}$ and for $dH > 1$

$$\int_{0 \leq x \leq y} dx dy \frac{x^{4Hn}}{\left((x^{2H}+1)(y^{2H}+1)\right)^{n+d/2}} < \infty,$$

this implies that for every $z \in \mathbb{R}_+$

$$\int_{0 \leq x \leq y} dx dy \frac{f_H(x, y, z)^{2n}}{\left((x^{2H}+1)(y^{2H}+1)\right)^{n+d/2}}$$

exists. On the other hand for $\alpha \in]1, 2 - 2H[$, $z^\alpha f_H(x, y, z)^{2n}$ decreases to zero when z tends to infinity, then

$$f(H, d, n) = \int_{\mathbb{R}_+} dz \int_{0 \leq x \leq y} dx dy \frac{f_H(x, y, z)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}} < \infty$$

and so that

$$\lim_{\epsilon \rightarrow 0} f(t, H, d, n, \epsilon) = f(H, d, n). \quad (19)$$

Therefore we obtain, if $dH \in]1, 3/2[$

$$\lim_{\epsilon \rightarrow 0} |l_{2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2 = 0$$

and, if $dH \geq 3/2$

$$\lim_{\epsilon \rightarrow 0} \epsilon^{d - \frac{3}{2H}} |l_{2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2 = 2T f(H, d, n).$$

Suppose now $dH = 1$

$$|l_{T, 2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2 = 2\epsilon^{\frac{1}{2H}} \int_0^T dt f(t, H, d, n, \epsilon)$$

$$f(t, H, d, n, \epsilon) \leq \theta_t(\epsilon) \int_{0 \leq x \leq y \leq \theta_t(\epsilon)} dx dy \frac{x^{4Hn}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}}$$

and then

$$\frac{f(t, H, d, n, \epsilon)}{|\log \epsilon|} \leq \theta_t(\epsilon) \left\{ \frac{a(H, d, n)}{|\log \epsilon|} + b(H, d, n) \right\},$$

where

$$a(H, d, n) = \int_{\substack{0 \leq x \leq y \\ x \leq 1}} dx dy \frac{x^{4Hn}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}} \quad (20)$$

and

$$b(H, d, n) = \sup_{0 \leq t \leq T} \sup_{\epsilon > 0} \frac{1}{|\log \epsilon|} \int_{1 \leq x \leq y \leq \theta_t(\epsilon)} dx dy \frac{x^{4Hn}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}}. \quad (21)$$

So we obtain

$$\frac{1}{|\log \epsilon|} |l_{T, 2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2 \leq T^2 \left\{ \frac{a(H, d, n)}{|\log \epsilon|} + b(H, d, n) \right\}.$$

Let us now treat $|l_{2\vec{n}}^{\epsilon, 2}|_{2, 2n}^2$,

$$|l_{2\vec{n}}^{\epsilon, 2}|_{2, 2n}^2 = 2\epsilon^{\frac{3}{2H} - d} \int_0^T dt g(t, H, d, n, \epsilon)$$

where

$$g(t, H, d, n, \epsilon) = \int_{\substack{0 \leq x+y-z \leq \theta_t(\epsilon) \\ 0 \leq z \leq x \leq y \leq \theta_t(\epsilon)}} dx dy dz \frac{\left(g_H(x, y, z)\right)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}}$$

and

$$g_H(x, y, z) = (x - z)^{2H} + (y - z)^{2H} - (x + y - z)^{2H} - z^{2H}$$

we have $|g_H(x, y, z)| \leq 2x^{2H}$ and so

$$\begin{aligned} z \longmapsto & \int_{0 \leq z \leq x \leq y} dx dy \frac{\left(g_H(x, y, z)\right)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}} = \\ & z^{4Hn+2} \int_{1 \leq x \leq y} dx dy \frac{\left(g_H(x, y, 1)\right)^{2n}}{\left((z^{2H}x^{2H} + 1)(z^{2H}y^{2H} + 1)\right)^{n+d/2}} \end{aligned}$$

is well defined on $\mathbb{R}_+$ and for $dH > 3/2$, one can choose $\alpha \in]1, 2dH - 2[$ such that when $z \rightarrow +\infty$

$$z^\alpha \int_{0 \leq z \leq x \leq y} dx dy \frac{\left(g_H(x, y, z)\right)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}} \longrightarrow 0$$

and then

$$\int_{\mathbb{R}_+} dz \int_{0 \leq z \leq x \leq y} dx dy \frac{\left(g_H(x, y, z)\right)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1)\right)^{n+d/2}} = g(H, d, n) < \infty \quad (22)$$

so that

$$\lim_{\epsilon \rightarrow 0} g(t, H, d, n, \epsilon) = g(H, d, n).$$

Suppose $dH = 1$, the same computation as in the case of $|l_{2\vec{n}}^{\epsilon,1}|_{2,2n}^2$ leads to

$$\frac{1}{|\log \epsilon|} |l_{T,2\vec{n}}^{\epsilon,2}|_{2,2n}^2 \leq 2^{2n} T^2 \left\{ \frac{a(H, d, n)}{|\log \epsilon|} + b(H, d, n) \right\}$$

and finally

$$\frac{1}{|\log \epsilon|} |l_{T,2\vec{n}}^{\epsilon}|_{2,2n}^2 \leq 22^{2n} T^2 \left\{ \frac{a(H, d, n)}{|\log \epsilon|} + b(H, d, n) \right\}.$$

Suppose now $1 < dH < 3/2$, we have

$$\begin{aligned} g(t, H, d, n, \epsilon) &\leq \int_0^{\theta_t(\epsilon)} dz z^{4Hn+2} \int_{1 \leq x \leq y \leq \frac{\theta_t(\epsilon)}{z}} dx dy \frac{g_H(x, y, 1)}{\left(((zx)^{2H} + 1)((zy)^{2H} + 1) \right)^{n+d/2}} \\ &\leq \int_0^{\theta_t(\epsilon)} dz z^{2-2Hd} \int_{1 \leq x \leq y} dx dy \frac{2^{2n} x^{4Hn}}{(x^{2H} y^{2H})^{n+d/2}} \\ &= 2^{2n} c(H, d, n). \end{aligned}$$

So that

$$g(t, H, d, n, \epsilon) \leq 2^{2n} c(H, d, n). \quad (23)$$

then

$$|l_{2\vec{n}}^{\epsilon, 2}|_{2, 2n}^2 \leq \frac{2^{2n} c(H, d, n)}{(3 - 2Hd)(2 - Hd)} T^{4-2Hd}$$

we know that $\lim_{\epsilon \rightarrow 0} |l_{2\vec{n}}^{\epsilon, 1}|_{2, 2n}^2 = 0$, then $\lim_{\epsilon \rightarrow 0} |l_{2\vec{n}}^{\epsilon}|_{2, 2n}$ exists.

Finally let us suppose that $dH = 3/2$

$$|l_{2\vec{n}}^{\epsilon, 2}|_{2, 2n}^2 = 2 \int_0^T dt g(t, H, d, n, \epsilon)$$

$$\begin{aligned} g(t, H, d, n, \epsilon) &\leq \int_0^1 dz \int_{z \leq x \leq y \leq \theta_t(\epsilon)} dx dy \frac{\left(g_H(x, y, z) \right)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1) \right)^{n+d/2}} \\ &\quad + \int_1^{\theta_t(\epsilon)} dz \int_{z \leq x \leq y \leq \theta_t(\epsilon)} dx dy \frac{\left(g_H(x, y, z) \right)^{2n}}{\left((x^{2H} + 1)(y^{2H} + 1) \right)^{n+d/2}}. \end{aligned}$$

The first term is bounded by

$$2^{2n} \int_{0 \leq x \leq y} dx dy \frac{x^{4Hn}}{\left((x^{2H} + 1)(y^{2H} + 1) \right)^{n+d/2}} = 2^{2n} e(H, d, n),$$

the second term is bounded by

$$\int_1^{\theta_t(\epsilon)} \frac{dz}{z} \int_{1 \leq x \leq y} \frac{2^{2n} x^{4Hn}}{\left(x^{2H} y^{2H} \right)^{n+d/2}} = 2^{2n} c(H, d, n) \log \theta_t(\epsilon),$$

so

$$\frac{1}{|\log \epsilon|} |l_{2\vec{n}}^{\epsilon, 2}|_{2, 2n}^2 \leq \frac{2}{|\log \epsilon|} 2^{2n} e(H, d, n) T + \frac{2^{2n}}{H} c(H, d, n) T + \frac{2 \cdot 2^{2n}}{|\log \epsilon|} \int_0^T \log t dt.$$

Therefore

$$\frac{1}{|\log \epsilon|} |l_{2\vec{n}}^\epsilon|_{2,2n}^2$$

has a finite non trivial limit and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{|\log \epsilon|} |l_{2\vec{n}}^\epsilon|_{2,2n}^2 \leq \frac{2^{2n}}{H} c(H, d, n) T.$$

References

- [1] **Als E. and Nualart N.:** *Stochastic integration with respect to the fractional Brownian motion. Stochastics and Stochastics Reports* 75, 129-152, 2003.
- [2] **Bolthausen E.:** *On the construction of the three dimensional polymer measure. Prob. Theory Rel. Fields* 97 (1993) 81-101.
- [3] **Drumond C., de Faria D. and Streit L.:** *The renormalization of self intersection local times I: The chaos expansion. IDAQP* 3 (2000) 223-236.
- [4] **Drumond C., de Faria D. and Streit L.:** *The square of self intersection local time of Brownian motion. CMS Conf. Proc.* 28 (2000) 115-122.
- [5] **Dynkin E.B.:** *Polynomials of the occupation field and related random fields. J. Func. Anal.* 58 (1984) 20-52.
- [6] **Dynkin E.B.:** *Regularized self-intersection local times of planar Brownian motion. Ann. Prob.* 16 (1988) 58-74.
- [7] **Edwards S.:** *The statistical mechanics of polymers with excluded volume. Proc. Phys. Sci.* 85 (1965) 613-624.
- [8] **Eddahbi M., Lacayo R., Solé J.L., Vives J. and Tudor C.A.:** *Regularity and asymptotic behaviour of the local time for the d-dimensional fractional Brownian motion with N-parameters. Preprint.*
- [9] **de Faria M., Hida T., Streit L. and Watanabe H.:** *Intersection local times as generalized white noise functionals. Acta Appl. Math.* 46 (1997) 351-362.
- [10] **Hida T., Kuo H-H., Potthoff J. and Streit L.:** *White Noise. An Infinite-Dimensional Calculus. Mathematics and its Applications* 253. Kluwer-Academic, Dordrecht (1993).

- [11] **Imkeller P., Perez-Abreu V. and Vives J.:** *Chaos expansions of double intersection local time of brownian motion in $\mathbb{R}^d$ and renormalization. Stoch. Proc. App. 56 (1995) 1-34.*
- [12] **Imkeller P. and Yan J-A.:** *Multiple intersection local time of planar Brownian motion as a particular Hida distribution. J. Func. Anal. 140 (1996) 256-273.*
- [13] **Kuo H-H.:** *Donker's delta function as a generalized Brownian functional and its application. Lecture Notes in Control and Information Sciences, 49 (1983) 167-178.*
- [14] **Le Gall J.F.:** *Sur le temps local d'intersection du mouvement Brownian plan et la méthode de renormalization de Varadhan. L.N.M 1123 314-331. Springer, Berlin (1985).*
- [15] **Lévy P.:** *Le mouvement brownien plan. Amer. J. Math. 62 (1940) 440-487.*
- [16] **Nualart D.:** *Stochastic integration with respect to the fractional Brownian motion and applications. Preprint.*
- [17] **Nualart D. and Vives J.:** *Chaos expansion and local times. Publ. Math. 36(2) (1992) 827-836.*
- [18] **Rezgui A. and Streit L.:** *The renormalization of self intersection local times of Brownian motion. Preprint.*
- [19] **Rosen J.:** *A local time approach to the self-intersections of Brownian paths in space. Comm. Math. Phys. 88 (1983) 327-338.*
- [20] **Rosen J.:** *Tanaka's formula and renormalization for intersections of planar Brownian motion. Ann. Prob. 14 (1986) 1245-1251.*
- [21] **Rosen J.:** *The intersection local time of fractional Brownian motion in the plane. J. Multi. Anal. 23 (1987) 37-46.*
- [22] **Symanzik K.:** *Euclidean Quantum Field Theory, in R. Jost ed., Local Quantum Theory. Academic, New York. (1969).*
- [23] **Varadhan S.R.S.:** *Appendix to "Euclidean Quantum Field theory" by Szymanzik K., in: R. Jost ed., local Quantum Theory. Academic, New York (1969).*

- [24] **Watanabe H.:** *The local time of self-intersections of Brownian motions as a generalized Brownian functionals. Lett. Math. Phys. 23 (1991) 1-9.*
- [25] **Westwater M. J.:** *On Edwards' model for long polymer chains. Commun. Math. Phys. 72 (1980) 131-174.*
- [26] **Wolpert R.:** *Weiner path intersections and local time. J. Func. Anal. 30 (1978) 329-340.*
- [27] **Wolpert R.:** *Local time and a particle picture for Euclidean field theory. J. Func. Anal. 30 (1978) 341-357.*
- [28] **Yor M.:** *Renormalization et convergence en loi pour les temps locaux d'intersections du mouvement brownien dans $\mathbb{R}^3$. L.N.M 1123 350-365. Springer, Berlin (1985).*

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