Sensitivity Analysis of Units with Interval Data in DEA

Mohammad Izadikhah ¹

Dept. of Math, Arak branch, Islamic Azad University
Arak, tel: +98-861-3663041-9, P.O. Box 38135/567, Iran

F. Hosseinzadeh Lotfi

Dept. of Math, Science and Research Branch
Islamic Azad University, Tehran 14515-775, Iran

Abstract

Data Envelopment Analysis (DEA) requires that the data for all inputs and outputs are known exactly. When some outputs and inputs are known decision variables, such as interval data and ordinal data, the DEA models become a nonlinear programming problem and is called imprecise DEA (IDEA). When data assume to be interval, decision making units (DMUs) can divide to three classes as follows: (I) efficient in any cases, (II) efficient in maximal sense and inefficient in minimal sense, (III) always inefficient. In this paper, we discuss a new technique for assessing the sensitivity of efficiency and inefficiency classifications in DEA with interval data in above three cases. Also we find radius of stability for all DMUs. Available bank branch data in Iran was used to illustrate the applicability of this approach

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1 Introduction

Data Envelopment Analysis (DEA) is a mathematical programming approach to performance evaluation [1,2,4]. DEA requires that the data for all inputs and outputs are known exactly. Recently Cooper et al. addressed the problem of

¹Corresponding author: Mohammad Izadikhah, E-mail: m_izadikhah@yahoo.com

imprecise data in DEA in its general form [3]. When some outputs and inputs are imprecise data such as interval or ordinal data, the DEA model becomes a nonlinear programming problem and is called imprecise DEA (IDEA). In dealing with this data the obtained models usually are nonlinear. To rectify this drawback Cooper et al. [5] proposed some methods to convert the nonlinear model to a linear one. In addition to it one can read Thompson et al. [11]. Recently Lee et al. [10], Zhu [12], Despotis and Smirlis [6] and Jahanshahloo et al. [7,8,9] investigate IDEA. When evaluate DMUs under interval data, they can classify as follows: (E^{++}) all the DMUs which are efficient with any combination of their inputs and outputs, (E^+) the DMUs which are efficient in their maximal status but for some data levels they loss their efficiency and (E^-) the DMUs that are inefficient in each case.

In this paper, we present a new technique for assessing the sensitivity of efficiency and inefficiency classification in DEA with interval data. The focus is on the stability of DMUs that have interval data. Having identified efficient and inefficient DMUs in a DEA analysis with interval data, one may want to know how sensitive these identification are to possible variations in data. Also, a modified CCR model is suggested to sensitive the DMUs with interval data. We develop several linear programming formulations for investigating radius of stability for all DMUs with interval data. The possible data perturbations for preserving the DMUs classifications are computed from the optimal values.

The structure of the paper is as follows: In section 2, we review DEA models for dealing with interval data. Then on the basis of these models, in section 3, we propose some models for determining radius of stability of DMUs. In section 4, the sensitivity and stability analysis method to several data sets are introduced. A concluding section summarizes our main results.

2 Preliminaries

Suppose we have n DMUs, where each DMU_j , ($j \in J$, $J = 1, \dots, n$) consumes m inputs $X_j = (x_{1j}, \dots, x_{mj})$ to produce s outputs $Y_j = (y_{1j}, \dots, y_{sj})$. We assume $Y_j \geq 0$, $X_j \geq 0$, $Y_j \neq 0$, $X_j \neq 0$. Unlike the original DEA model, we assume further that the levels of inputs and outputs are not known exactly, the true input and output data known to lie within bounded intervals i.e. $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{ij} \in [y_{ij}^L, y_{ij}^U]$ with upper and lower bounds of the intervals given as constants and assumed strictly positive. In this case, the efficiency can be an interval. The upper limit of interval efficiency is obtained from the optimistic viewpoint and the lower limit is obtained from the pessimistic

viewpoint. The following model provides such an upper bound for DMU_o :

$$\begin{aligned}
 h_o^U = \max \quad & \sum_{r=1}^s u_r y_{ro}^U \\
 \text{s.t.} \quad & \sum_{i=1}^m v_i x_{io}^L = 1, \\
 & \sum_{r=1}^s u_r y_{ro}^U - \sum_{i=1}^m v_i x_{io}^L \leq 0, \\
 & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{1}$$

where ε is a non-Archimedean infinitesimal and original variables $u_1, \dots, u_s$ and $v_1, \dots, v_m$ are weights for outputs and inputs, respectively, which to be estimated. We denote by h_o^U the efficiency score attained by DMU_o in (1). The model below provides a lower bound of efficiency score for DMU_o :

$$\begin{aligned}
 h_o^L = \max \quad & \sum_{r=1}^s u_r y_{ro}^L \\
 \text{s.t.} \quad & \sum_{i=1}^m v_i x_{io}^U = 1, \\
 & \sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U \leq 0, \\
 & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{2}$$

The efficiency h_o^L attained by DMU_o in (2) serves as a lower bound of its possible efficiency scores. Models (1) and (2) provide each DMU with a bounded interval $[h_o^L, h_o^U]$ in which its possible efficiency scores lie from the worst to the best case. Considering (1) and (2), it is evident that $h_o^L \leq h_o^U$. On the basis of the above efficiency score intervals, DMUs can be classified in three subsets as follows:

$$\begin{aligned}
 E^{++} &= \{j \in J \mid h_j^L = 1\}, \\
 E^+ &= \{j \in J \mid h_j^L < 1 \text{ and } h_j^U = 1\}, \\
 E^- &= \{j \in J \mid h_j^U < 1\}.
 \end{aligned}$$

3 Sensitivity analysis in DEA with interval data

Suppose that DMUs are evaluated by (1) and (2) are classified in E^{++} , E^+ and E^- . Having identified efficient and inefficient DMUs in a DEA analysis, one may want to know how to sensitive these identifications are to possible

variations in the data. We determine "radius of stability" within which data variations will not alter a DMU's classification from efficient to inefficient status or vice versa. We consider the radius of stability of DMU_o in three cases as follow.

3.1 Radius of stability for DMU belonging to E^{++}

In this case, we assume that DMU_o is in E^{++} , that is, $h_o^L = 1$. It is obvious that DMU_o remains in E^{++} if its outputs increase or its inputs decrease. Our aim is to find the scalars α and β such that if we decrease upper bound of outputs and increase lower bound of inputs of DMU_o by α then $h_o^U = 1$, also if we decrease lower bound of outputs and increase upper bound of inputs of DMU_o by β then $h_o^L = 1$, i.e. DMU remains in E^{++} .

We will calculate the upper-bounds of α and β , namely the ranges for α and β , such that the inputs and outputs of DMU_o are perturbed within these ranges and the efficiency of DMU_o is preserved in E^{++} . It has been assumed that α and β are scalars and strictly positive. Here, we consider the following cases:

(1) It is obvious that if $X_o^L - \mathbf{1}_m\alpha$, $Y_o^U + \mathbf{1}_s\alpha$ then $h_o^U = 1$ and if $X_o^U - \mathbf{1}_m\beta$, $Y_o^L + \mathbf{1}_s\beta$ then $h_o^L = 1$; consequently, $DMU_o \in E^{++}$ (Note that $\mathbf{1}$ is a vector with all components equal 1.).

(2) If $X_o^L + \mathbf{1}_m\alpha$, $Y_o^U - \mathbf{1}_s\alpha$ and $X_o^U + \mathbf{1}_m\beta$, $Y_o^L - \mathbf{1}_s\beta$, then it is possible for DMU_o not to be in the E^{++} . We are concerned with finding the largest value for α and β such that $\widehat{DMU}_o \in E^{++}$, where $\widehat{DMU}_o$ is DMU_o with the following changes:

$$(I) : \begin{cases} X_o^L \rightarrow X_o^L + \mathbf{1}_m\alpha, \\ Y_o^U \rightarrow Y_o^U - \mathbf{1}_s\alpha, \end{cases} \text{ and}$$

$$(II): \begin{cases} X_o^U \rightarrow X_o^U + \mathbf{1}_m\beta, \\ Y_o^L \rightarrow Y_o^L - \mathbf{1}_s\beta, \end{cases}$$

For this purpose, we present the following models:

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\ & \sum_{r=1}^s u_r(y_{ro}^L - \beta) - \sum_{i=1}^m v_i(x_{io}^U + \beta) \geq 0, \quad (*) \\ & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o \\ & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \tag{3}$$

and

$$\begin{aligned}
 \max \quad & \alpha \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r (y_{ro}^U - \alpha) - \sum_{i=1}^m v_i (x_{io}^L + \alpha) \geq 0, \quad (**) \\
 & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & x_{io}^L + \alpha \leq x_{io}^U + \beta^*, \quad i = 1, \dots, m, \\
 & y_{ro}^U - \alpha \geq y_{ro}^L - \beta^*, \quad r = 1, \dots, s, \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{4}$$

Where, β^* is optimal value of model (3). In models (3) and (4), inequality in constraints (*) and (**) guarantees that these models are feasible. Here, the constraint $\sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1$ is added for normalization purpose. In the first glance, it seems that (3) and (4) are nonlinear, but the following manipulation transforms the models into linear.

The constraint (*) converted to $\sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U \geq \beta$. Similarly, constraint (**) converted to $\sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U \geq \alpha$.

By the above substitutions, the following linear models are obtained:

$$\begin{aligned}
 \max \quad & \beta \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U \geq \beta \\
 & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{5}$$

and

$$\begin{aligned}
 & \max \quad \alpha \\
 & s.t. \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U \geq \alpha \\
 & \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & \quad x_{io}^L + \alpha \leq x_{io}^U + \beta^*, \quad i = 1, \dots, m, \\
 & \quad y_{ro}^U - \alpha \geq y_{ro}^L - \beta^*, \quad r = 1, \dots, s, \\
 & \quad u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{6}$$

Where, β^* is optimal value of model (5).

Theorem 1. After solving models (5) and (6) and applying the changes (I) and (II), the data of DMU_o remain interval.

Proof. With regard to following constrains,

$$\begin{aligned}
 X_{io}^L + \alpha &\leq X_{io}^U + \beta, \quad i = 1, \dots, m \\
 Y_{ro}^U - \alpha &\geq Y_{ro}^L - \beta^*, \quad r = 1, \dots, s.
 \end{aligned}$$

proof is evident.

Theorem 2. Suppose DMU_o is in E^{++} . After solving models (5) and (6) and applying the changes (I) and (II), DMU_o remains a member of E^{++} .

Proof. In theorem 1 we showed that the data of $\widehat{DMU}_o$ are interval. (Where $\widehat{DMU}_o$ is DMU_o after changes (I) and (II)). To showing that $\widehat{DMU}_o^{++}$, it is necessary to prove $h_o^L = 1$. Where

$$\begin{aligned}
 h_o^L &= \max \quad \sum_{r=1}^s u_r (y_{ro}^L - \beta^*) \\
 & s.t. \quad \sum_{i=1}^m v_i (x_{io}^U + \beta^*) = 1, \\
 & \quad \sum_{r=1}^s u_r (y_{ro}^L - \beta^*) - \sum_{i=1}^m v_i (x_{io}^U + \beta^*) \leq 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq 0, \\
 & \quad u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{7}$$

where, β^* is optimal value of model (3). Suppose (u^*, v^*, β^*) be the optimal solution of model (3), then we have:

$$\begin{aligned}
 \sum_{r=1}^s u_r^* (y_{ro}^L - \beta^*) - \sum_{i=1}^m v_i^* (x_{io}^U + \beta^*) &\geq 0, \\
 \sum_{r=1}^s u_r^* y_{rj}^U - \sum_{i=1}^m v_i^* x_{ij}^L &\leq 0, \quad j = 1, \dots, n, \quad j \neq 0.
 \end{aligned}$$

we define γ as follows: $\gamma = \sum_{i=1}^m v_i^*(x_{io}^U + \beta^*) \leq \sum_{r=1}^s u_r^*(y_{ro}^L - \beta^*)$, (clearly, $\gamma > 0$).

And by following substitutions:

$$\begin{aligned}\bar{u}_r &= \frac{u_r^*}{\gamma} > 0, \quad r = 1, \dots, s, \\ \bar{v}_i &= \frac{v_i^*}{\gamma} > 0, \quad i = 1, \dots, m.\end{aligned}$$

We have, $\sum_{i=1}^m \bar{v}_i(x_{io}^U + \beta^*) = 1$ and therefore, $\sum_{r=1}^s \bar{u}_r(y_{ro}^L - \beta^*) = 1$. Therefore $(\bar{u}, \bar{v})$ is a feasible solution for model (7) with objective function $\sum_{r=1}^s \bar{u}_r(y_{ro}^L - \beta^*) = 1$. Then $(\bar{u}, \bar{v})$ is optimal solution of model (7) and we have $h_o^L = 1$.

3.2 Radius of stability for DMU belonging to E^+

In this case, we assume that DMU_o is in E^+ , that is, $h_o^L < 0$ and $h_o^U = 1$. Our aim is to find the scalars α and β such that, if we decrease upper bound of outputs and increase lower bound of inputs of DMU_o by α then $h_o^U = 1$, also if we increase lower bound of outputs and decrease upper bound of inputs of DMU_o by β then $h_o^L < 1$, i.e. DMU_o remains in E^+ . We will calculate the upper bounds of α and β , namely the ranges for α and β such that the inputs and outputs of DMU_o perturbations within these ranges preserve the efficiency of DMU_o in E^+ . Here, we consider the following cases:

(1) If $X_o^L \rightarrow X_o^L - \mathbf{1}_m \alpha$, $Y_o^U \rightarrow Y_o^U + \mathbf{1}_s \alpha$ then $h_o^U = 1$ and if X_o^U , Y_o^L are fixed, then $h_o^L < 1$, consequently $DMU_o \in E^+$.

(2) If $X_o^U \rightarrow X_o^U + \mathbf{1}_m \beta$, $Y_o^L \rightarrow Y_o^L - \mathbf{1}_s \alpha$ then $h_o^L < 1$ and if X_o^L , Y_o^U are fixed, then $h_o^U = 1$, consequently $DMU_o \in E^+$.

(3) We are concerned in finding the largest values of α and β such that $\widehat{DMU}_o^+$, where $\widehat{DMU}_o$ is DMU_o with the following changes:

$$(III) : \quad \begin{cases} X_o^L \rightarrow X_o^L + \mathbf{1}_m \alpha, \\ Y_o^U \rightarrow Y_o^U - \mathbf{1}_s \alpha, \end{cases}$$

and

$$(IV) : \quad \begin{cases} X_o^U \rightarrow X_o^U - \mathbf{1}_m \beta, \\ Y_o^L \rightarrow Y_o^L + \mathbf{1}_s \beta, \end{cases}$$

For this purpose, we present the following models:

$$\begin{aligned}
 \max \quad & \alpha \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r (y_{ro}^U - \alpha) - \sum_{i=1}^m v_i (x_{io}^L + \alpha) \geq 0, \quad (*) \\
 & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{8}$$

Assume, α^* is optimal value of objective function. Then we choose $\bar{\alpha}$ such that $0 \leq \bar{\alpha} < \alpha^*$, value of $\bar{\alpha}$ is determined by manager, and then solve the following model:

$$\begin{aligned}
 \max \quad & \beta \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r (y_{ro}^L + \beta) - \sum_{i=1}^m v_i (x_{io}^U - \beta) \leq 0, \quad (**) \\
 & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & x_{io}^U - \beta \geq x_{io}^L + \bar{\alpha}, \quad i = 1, \dots, m, \\
 & y_{ro}^L + \beta \leq y_{ro}^U - \bar{\alpha}, \quad r = 1, \dots, s, \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{9}$$

In models (8) and (9), inequality in constrains (*) and (**) guarantees that these models are feasible. Clearly, models (8) and (9) are nonlinear, but by similar manipulation as mentioned in section (3.1) transform in to following linear models:

$$\begin{aligned}
 \max \quad & \alpha \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r y_{ro}^U - \sum_{i=1}^m v_i x_{io}^L \geq \alpha, \\
 & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{10}$$

and

$$\begin{aligned}
 \max \quad & \beta \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U + \beta \leq 0, \\
 & \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & x_{io}^U - \beta \geq x_{io}^L + \bar{\alpha}, \quad i = 1, \dots, m, \\
 & y_{ro}^L + \beta \leq y_{ro}^U - \bar{\alpha}, \quad r = 1, \dots, s, \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{11}$$

Where $0 \leq \bar{\alpha} < \alpha^*$.

Theorem 3. After solving models (10) and (11) and applying the changes (III) and (IV), the data of DMU_o remain interval.

Proof. proof is evident.

Theorem 4. Suppose DMU_o is in E^+ . Then $\widehat{DMU}_o$ with inputs x_{io} where $x_{io}^L + \bar{\alpha}_{io} \leq x_{io}^U - \beta^*$, and outputs y_{ro} where $y_{ro}^L + \beta_{ro}^* \leq y_{ro}^U - \bar{\alpha}$ be a member of E^+ .

Proof. The proof is similar to that of theorem (2) and is omitted.

3.3 Radius of stability for DMU belonging to E^-

In this case, we assume that DMU_o is in E^- . Our aim is to find the scalars α and β such that if we decrease lower bound of inputs and increase upper bound of outputs of DMU_o by α then $h_o^U < 1$, also if we decrease upper bound of inputs and increase lower bound of outputs of DMU_o by β then $h_o^L < 1$, i.e. DMU_o remains in E^- . We will calculate the upper bounds of α and β , namely the ranges for α and β such that the inputs and outputs of DMU_o are perturbed within these ranges and its inefficiency score is not changed. Now we consider following cases:

- (1) If $X_o^L \rightarrow X_o^L + \mathbf{1}_m \alpha$ and $Y_o^U \rightarrow Y_o^U - \mathbf{1}_s \alpha$ then $h_o^U < 1$ and if $X_o^U \rightarrow X_o^U + \mathbf{1}_m \beta$ and $Y_o^L \rightarrow Y_o^L - \mathbf{1}_s \beta$ then $h_o^L < 1$, consequently $DMU_o \in E^-$.
- (2) If $X_o^L \rightarrow X_o^L - \mathbf{1}_m \alpha, Y_o^U \rightarrow Y_o^U + \mathbf{1}_s \alpha$ and if $X_o^U \rightarrow X_o^U - \mathbf{1}_m \beta, Y_o^L \rightarrow$

$Y_o^L + \mathbf{1}_s \beta$ then it is possible for DMU_o not to be in the E^- . We are concerned with finding the largest value for α and β such that $\widehat{DMU}_o^-$, where $\widehat{DMU}_o$ is DMU_o with the following changes:

$$(V) : \quad \begin{cases} X_o^L \rightarrow X_o^L - \mathbf{1}_m \alpha, \\ Y_o^U \rightarrow Y_o^U + \mathbf{1}_s \alpha, \end{cases}$$

and

$$(VI): \quad \begin{cases} X_o^U \rightarrow X_o^U - \mathbf{1}_m \beta, \\ Y_o^L \rightarrow Y_o^L + \mathbf{1}_s \beta, \end{cases}$$

For this purpose, we present the following models:

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\ & \sum_{r=1}^s u_r (y_{ro}^U + \alpha) - \sum_{i=1}^m v_i (x_{io}^L - \alpha) \geq 0, \quad (*) \\ & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\ & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \quad (12)$$

Assume, α^* is optimal value of objective function. Since, we must $h_o^L < 1$, so we choose $\bar{\alpha}$ such that $0 \leq \bar{\alpha} < \alpha^*$, value of $\bar{\alpha}$ is determined by manager, then solve the following model:

$$\begin{aligned} \min \quad & \beta \\ \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\ & \sum_{r=1}^s u_r (y_{ro}^L + \beta) - \sum_{i=1}^m v_i (x_{io}^U - \beta) \geq 0, \quad (**) \\ & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\ & x_{io}^U - \beta \geq x_{io}^L - \bar{\alpha}, \quad i = 1, \dots, m, \\ & y_{ro}^L + \beta \leq y_{ro}^U + \bar{\alpha}, \quad r = 1, \dots, s, \\ & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \quad (13)$$

In models (12) and (13), inequality in constraints (*) and (**) guarantees that these models are feasible. Clearly, models (12) and (13) are nonlinear, so by similar manipulation as mentioned in section (3.1) transform in to following linear models:

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\ & \sum_{r=1}^s u_r y_{ro}^U - \sum_{i=1}^m v_i x_{io}^L + \alpha \geq 0, \\ & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\ & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m. \end{aligned} \quad (14)$$

and

$$\begin{aligned}
 \min \quad & \beta \\
 \text{s.t.} \quad & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \sum_{r=1}^s u_r y_{ro}^L - \sum_{i=1}^m v_i x_{io}^U + \beta \geq 0, \\
 & \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq o, \\
 & x_{io}^U - \beta \geq x_{io}^L - \bar{\alpha}, \quad i = 1, \dots, m, \\
 & y_{ro}^L + \beta \leq y_{ro}^U + \bar{\alpha}, \quad r = 1, \dots, s, \\
 & u_r, v_i \geq \varepsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{15}$$

Theorem 5. After solving models (14) and (15) and applying the changes (V) and (VI), the data of DMU_o remain interval.

Proof. Proof is evident.

Theorem 6. Suppose DMU_o is in E^- . Then $\widehat{DMU}_o$ with inputs x_{io} where $x_{io}^L - \bar{\alpha} \leq x_{io} \leq x_{io}^U - \beta^*$, and outputs y_{ro} where $y_{ro}^L + \beta^* \leq y_{ro} \leq y_{ro}^U + \bar{\alpha}$ is also a member of E^- .

Proof. The proof is similar to that of theorem 2 and is omitted.

4 An application in Bank Branches in Iran

We now apply this approach to some commercial bank branches in Iran. There are 20 branches in this district. Each branch uses 3 inputs to produce 5 outputs. Table 1 shows the kind of these inputs and outputs.

Table 1.

Inputs	Outputs
Payable interest	The total sum of four main deposits
Personnel	Other deposits
Non-Performing loans	Loans granted
	Received interest
	Fee

In Tables 2 and 3 the interval inputs and interval outputs for these DMUs are given.

Table 2. Input-data for the 20 bank branches

DMU_j	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	x_{3j}^L	x_{3j}^U
1	5007.37	9613.37	36.29	36.86	87243	87243
2	2926.81	5961.55	18.8	2016	9945	12120
3	8732.7	17752.5	25.74	27.17	47575	50013
4	945.93	1966.39	20.81	22.54	19292	19753
5	8487.07	17521.66	14.16	14.8	3428	3911
6	13759.35	27359.36	19.46	19.46	13929	15657
7	587.69	1205.47	27.29	27.48	27827	29005
8	4646.39	9559.61	24.52	25.07	9070	9983
9	1554.29	3427.89	20.47	21.59	412036	413902
10	17528.31	36297.54	14.84	15.05	8638	10229
11	2444.34	4955.78	20.42	20.54	500	937
12	7303.27	14178.11	22.87	23.19	16148	21353
13	9852.15	19742.89	18.47	21.83	17163	17290
14	4540.75	9312.24	22.83	23.96	17918	17964
15	3039.58	6304.01	39.32	39.86	51582	55136
16	6585.81	13453.58	25.57	26.52	20975	23992
17	4209.18	8603.79	27.59	27.95	41960	43103
18	1015.52	2037.82	13.63	13.93	18641	19354
19	5800.38	11875.39	27.12	27.26	19500	19569
20	1445.68	2922.15	28.96	28.96	31700	32061

Table 3. Output-data for the 20 bank branches

DMU_j	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	y_{3j}^L	y_{3j}^U	y_{4j}^L	y_{4j}^U	y_{5j}^L	y_{5j}^U
1	2696995	3126798	263643	382545	1675519	1853365	108634.76	125740.28	965.97	6957.33
2	340377	440355	95978	117659	377309	390203	32396.65	37836.56	304.67	749.4
3	1027546	1061260	37911	503089	1233548	1822028	96842.33	108080.01	2285.03	3174
4	1145235	1213541	229646	268460	468520	542101	32362.8	39273.37	207.98	510.93
5	390902	395241	4924	12136	129751	142873	12662.71	14165.44	63.32	92.3
6	988115	1087392	74133	111324	507502	574355	53591.3	72257.28	480.16	869.52
7	144906	165818	180530	180617	288513	323721	40507.97	45847.48	176.58	370.81
8	408163	416416	405396	486431	1044221	1071812	56260.09	73948.09	4654.71	5882.53
9	335070	410427	337971	449336	1584722	1802942	176436.81	189006.12	560.26	2506.67
10	700842	768593	14378	15192	2290745	2573512	662725.21	791463.08	58.89	86.86
11	641680	696338	114183	241081	1579961	2285079	17527.58	20773.91	1070.81	2283.08
12	453170	481943	27196	29553	245726	275717	35757.83	42790.14	375.07	559.85
13	553167	574989	21298	23043	425886	431815	45652.24	50255.75	438.43	836.82
14	309670	342598	20168	26172	124188	126930	8143.79	11948.04	936.62	1468.45
15	286149	317186	149183	270708	787959	810088	106798.63	111962.3	1203.79	4335.24
16	321435	347848	66169	80453	360880	379488	89971.47	165524.22	200.36	399.8
17	618105	835839	244250	404579	9136507	9136507	33036.79	41826.51	2781.24	4555.42
18	248125	320974	3063	6330	26687	29173	9525.6	10877.78	240.04	274.7
19	640890	679916	490508	684372	2946797	3985900	66097.16	95329.87	961.56	1914.25
20	119948	120208	14943	17495	297674	308012	21991.53	27934.19	282.73	471.22

Table 4. Efficiency units and classification

DMU_j	h_j^U	h_j^L	Classification	DMU_j	h_j^U	h_j^L	Classification
1	1.000	1.000	E^{++}	11	1.000	1.000	E^{++}
2	1.000	0.371	E^+	12	0.496	0.328	E^-
3	1.000	0.523	E^+	13	0.701	0.449	E^-
4	1.000	1.000	E^{++}	14	0.724	0.263	E^-
5	0.763	0.618	E^-	15	1.000	0.413	E^+
6	1.000	0.917	E^+	16	1.000	0.221	E^+
7	1.000	0.728	E^+	17	1.000	1.000	E^{++}
8	1.000	1.000	E^{++}	18	0.952	0.263	E^-
9	1.000	1.000	E^{++}	19	1.000	0.991	E^+
10	1.000	1.000	E^{++}	20	1.000	0.183	E^+

In Table 5 we present the radius of stability for DMUs which is belong to set E^{++} , that is values of β_j^* and α_j^* , $\forall j, j \in E^{++}$.

Table 5. Radius of stability for E^{++}

DMU_j	β_j^*	α_j^*
1	9.733	9.733
4	6.746E+2	6.763E+2
8	13.794	14.344
9	28.107	29.227
10	2.429E+4	2.429E+4
11	7.006E+3	7.006E+3
17	1.129E+3	1.129E+3

In Table 6 we present the radius of stability for DMUs which is belong to sets E^+ and E^- .

Table 6. Radius of stability for DMUs belong to E^+ and E^-

$DMU_j \in E^+$	α_j^*	β_j^*	$\bar{\alpha}$	$DMU_j \in E^-$	α_j^*	β_j^*	$\bar{\alpha}$
2	29.27	4.3E+2	14.63	5	3.95	2.61	1.97
3	2.22E+2	-1.09E+2	1.11E+2	12	14.35	7.49	7.17
6	6.50	-3.25	3.25	13	7.45	7.08	3.72
7	1.74E+3	-8.70E+2	8.70E+2	14	12.28	7.27	6.14
15	2.81E+3	-1.40E+3	1.40E+3	18	6.60	3.60	3.30
16	1.74E+3	-8.73E+2	8.74E+2				
19	1.01E+4	-5.04E+3	5.04E+3				

5 Discussion

From results of tables (5,6) manager can decided that which data perturbations for preserving the DMUs classifications are suitable. For instance, we can see that, by applying the changes (I),(II),(III),(IV),(V) and (VI) inputs and outputs of $\widehat{DMU}_9 \in E^{++}$ and $\widehat{DMU}_6 \in E^+$ and $\widehat{DMU}_{14} \in E^-$ are as tables 7 and 8:

Table 7. Input-data

DMU_j	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	x_{3j}^L	x_{3j}^U
9	1583.51	3455.99	49.69	49.69	412065.22	413930.10
6	13762.60	27362.61	22.71	22.71	13932.25	15660.25
14	4534.61	9304.96	16.69	16.69	17911.86	17956.72

Table 8. Output-data

DMU_j	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	y_{3j}^L	y_{3j}^U
9	335041.89	410397.77	337942.42	449306.77	1584693.89	1802912.77
6	988111.74	1087388.74	74129.74	111320.74	507498.74	574351.74
14	309677.27	342604.14	20175.27	26178.14	124195.27	126936.14

DMU_j	y_{4j}^L	y_{4j}^U	y_{5j}^L	y_{5j}^U
9	176408.70	188976.89	532.15	2477.44
6	53588.04	72254.02	476.90	856.26
14	8151.06	11954.18	943.79	1474.59

Where, we consider that, in models (10),(11),(14) and (15) manager chooses $\bar{\alpha} = \frac{\alpha^*}{2}$. Clearly, by Tables 7 and 8 $\widehat{DMU}_9, \widehat{DMU}_6$ and $\widehat{DMU}_{14}$ have interval data.

6 Conclusion

The current paper develops a new approach for the sensitivity analysis of DMUs with interval data in three following cases: (I) efficient in any cases, (II) efficient in maximal sense and inefficient in minimal sense, (III) always inefficient. And also, finds the radius of stability and largest region for preserving classification of each DMU. Available bank branch data in Iran was used to illustrate the applicability of this approach.

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