A Simple Proof of Xia's Proposition

Muneo Chō ¹

Department of Mathematics
Kanagawa University
Yokohama 221-8686, Japan
chiyom01@kanagawa-u.ac.jp

Tadasi Huruya

Faculty of Education and Human Sciences
Niigata University, Niigata 950-2181, Japan
huruya@ed.niigata-u.ac.jp

Michio Seto ²

Department of Mathematics
Kanagawa University
Yokohama 221-8686, Japan
chiyom01@kanagawa-u.ac.jp
seto@kanagawa-u.ac.jp

Abstract

This paper studies a spectral property of a k -tuple of hyponormal operators $\mathbf{T} = (T_1, \dots, T_n)$ with compact self commutators.

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INTRODUCTION

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. A k -tuple of operators $\mathbf{T} = (T_1, \dots, T_k)$ on $\mathcal{H}$ is said to be *subnormal* if there exists a k -tuple of commuting normal operators $\mathbf{N} = (N_1, \dots, N_k)$ on $\mathcal{K}$, containing $\mathcal{H}$ as a subspace, satisfying

$$T_j = N_j|_{\mathcal{H}} \quad (j = 1, \dots, k).$$

A k -tuple of operators $\mathbf{T} = (T_1, \dots, T_k)$ is said to be *finite type* if $\text{rank}[T_j^*, T_j] < \infty$ ($j = 1, \dots, k$). For an operator T , the spectrum and the resolvent of T are denoted by $\sigma(T)$ and $\rho(T)$, respectively. For a k -tuple of operators $\mathbf{T} = (T_1, \dots, T_k)$, let

$$\sigma_r(\mathbf{T}) = \{ (z_1, \dots, z_k) \in \mathbf{C}^k : \sum_{j=1}^k (T_j - z_j)(T_j - z_j)^* \text{ is not invertible} \}$$

be the *right spectrum* of $\mathbf{T}$. In [1], D. Xia showed trace formula of subnormal tuples of operators of finite type. Also he showed the following proposition:

Proposition. *Let $\mathbf{T} = (T_1, \dots, T_k)$ be a subnormal k -tuple of operators on $\mathcal{H}$ with minimal normal extension $\mathbf{N} = (N_1, \dots, N_k)$. If $\mathbf{z} = (z_1, \dots, z_k)$ satisfies $z_j \in \sigma(T_j) \cap \rho(N_j)$, $j = 1, \dots, k$ and $\mathbf{z} \in \sigma_r(\mathbf{T})$, then there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $(T_j - z_j)^* x_0 = 0$ for every $j = 1, \dots, k$.*

His proof needs the spectral measure of $\mathbf{N}$. Hence, commutativity of $\mathbf{N}$ is essential. In this paper, we show the following

Theorem. *Let $\mathbf{T} = (T_1, \dots, T_k)$ be a k -tuple of operators on $\mathcal{H}$ with compact self commutators, i.e., $[T_j^*, T_j]$ is compact for every $j = 1, \dots, k$. Assume that $\mathbf{z} = (z_1, \dots, z_k) \in \sigma_r(\mathbf{T})$. If there exist a Hilbert space $\mathcal{K}$, containing $\mathcal{H}$ as a subspace, and a k -tuple of hyponormal operators $\mathbf{S} = (S_1, \dots, S_k)$ on $\mathcal{K}$ such that $T_j = S_j|_{\mathcal{H}}$ and $z_j \in \rho(S_j)$ for every j , then there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $(T_j - z_j)^* x_0 = 0$ for every $j = 1, \dots, k$.*

PROOF

Proof of Theorem. We decompose S_j as follows:

$$S_j = \begin{pmatrix} T_j & A_j \\ 0 & B_j \end{pmatrix} \quad \text{on} \quad \mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp \quad (j = 1, \dots, k).$$

Then we have

$$S_j^* S_j = \begin{pmatrix} T_j^* T_j & T_j^* A_j \\ A_j^* T_j & A_j^* A_j + B_j^* B_j \end{pmatrix} \quad \text{and} \quad S_j S_j^* = \begin{pmatrix} T_j T_j^* + A_j A_j^* & A_j B_j^* \\ B_j A_j^* & B_j B_j^* \end{pmatrix}.$$

Since every S_j is hyponormal, we have $T_j^* T_j - T_j T_j^* \geq A_j A_j^*$. Hence every A_j^* is compact. From the assumption, let $\{x_n\}$ be a sequence of unit vectors of $\mathcal{H}$ such that $\|(T_j - z_j)^* x_n\| \rightarrow 0$ ($j = 1, \dots, k$). Since A_j^* is compact, there exists a subsequence $\{x_{n_m}\} \subset \{x_n\}$ and $y_j \in \mathcal{H}^\perp$ such that $A_j^* x_{n_m} \rightarrow y_j$ ($m \rightarrow \infty$) for all j . Then

$$(S_j - z_j)^*(x_{n_m} \oplus 0) = (T_j - z_j)^* x_{n_m} \oplus A_j^* x_{n_m} \longrightarrow 0 \oplus y_j \quad (m \rightarrow \infty).$$

Since $z_j \in \rho(S_j)$ it holds

$$x_{n_m} \oplus 0 \longrightarrow (S_j - z_j)^{* -1}(0 \oplus y_j) \quad (m \rightarrow \infty).$$

Since $(S_j - z_j)^{* -1}(0 \oplus y_j)$ is the limit of the sequence $\{x_{n_m} \oplus 0\}$, we may write it by $x^{(j)} \oplus 0$. Since $x^{(j)}$ is the limit of x_{n_m} it is independent on j . Hence we may write $x^{(j)}$ as x_0 . From $\|x_{n_m}\| = 1$, $\|x_0\| = 1$. Also we have

$$\begin{aligned} (0 \oplus y_j) &= (S_j - z_j)^*(x_0 \oplus 0) \\ &= \begin{pmatrix} (T_j - z_j)^* & 0 \\ A_j^* & (B_j - z_j)^* \end{pmatrix} (x_0 \oplus 0) = (T_j - z_j)^* x_0 \oplus A_j^* x_0. \end{aligned}$$

Therefore, we have $(T_j - z_j)^*x_0 = 0$ for every $j = 1, \dots, k$. □

Remark. Commutativity of the k -tuple of hyponormal operators $(S_1, \dots, S_k)$ is not needed.

References

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