On a More Accurate Hilbert's Type Inequality

Bicheng Yang

Department of Mathematics
Guangdong Institute of Education
Guangzhou, Guangdong 510303, P. R. China
bcyang@pub.guangzhou.gd.cn

Abstract

By using the improved Euler-Maclaurin's summation formula, a more accurate Hilbert's type inequality is built. As applications, the equivalent form is considered. All the lemmas and the theorem provide some new estimates on this type of inequalities.

Mathematics Subject Classification: 26D15

Keywords: Hardy-Hilbert's type inequality, weight coefficient, Cauchy's inequality

1 Introduction

If $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then one has a Hilbert's type inequality as

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m-n} < \pi^2 \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{\frac{1}{2}}, \quad (1)$$

where the constant factor π^2 is the best possible (see [1, Ch.9]). Hilbert's type inequalities are important in analysis and its applications (see[2]). In recent years, Pachpatte et. al [3,4,5,6] gave some new generalizations and improvements of them, and Kuang et. al [7] built a strengthened version of (1) by using the improved Euler-Maclaurin's summation formula. More recently, Yang [8] gave an extension of (1) by introducing a parameter $\lambda \in (0, 2]$ as

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m^{\lambda} - n^{\lambda}} < \left(\frac{\pi}{\lambda} \right)^2 \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 \right\}^{\frac{1}{2}}, \quad (2)$$

where the constant factor $(\pi/\lambda)^2$ is the best possible. Some more accurate Mulholland's inequalities were considered by Yang [9,10].

In this paper, by using the improved Euler-Maclaurin's summation formula and introducing a parameter α , one gives a refinement of the way of weight coefficient, and build a more accurate Hilbert's type inequality, which is a best extension of (1) related to the double series as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{(m+\alpha) - (n+\alpha)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{m-n} \quad (\alpha \geq \frac{3+\sqrt{21}}{24}).$$

As applications, an equivalent form is given. All the lemmas and the theorem provide some new estimates on this type of inequalities.

2 Some lemmas

First, we need the formula as (cf. [1,Ch.9]):

$$\int_0^{\infty} \frac{\ln u}{u-1} u^{-\frac{1}{p}} du = \left[\frac{\pi}{\sin(\pi/p)} \right]^2 \quad (p > 1). \quad (3)$$

LEMMA 2.1(see [7]). If $f \in C^4[0, \infty)$, $(-1)^i f^{(i)}(x) \geq 0$, $f^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3, 4$), then

$$\sum_{m=0}^{\infty} f(m) \leq \int_0^{\infty} f(x) dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0). \quad (4)$$

LEMMA 2.2. Setting $g(u) = \frac{\ln u}{u-1}$, $u \in (0, \infty)$ ($g(1) := 1$), and

$$f(x) = g\left(\frac{x+\alpha}{n+\alpha}\right) \left(\frac{x+\alpha}{n+\alpha}\right)^{-\frac{1}{2}}, x \in (-\alpha, \infty) (\alpha > 0; n \in N_0)$$

(N_0 is the set of non-negative integers), then $f(x)$ possesses the conditions of (4).

Proof. One finds $g \in C^4(0, \infty)$, and

$$\begin{aligned} g'(u) &= \frac{1}{(u-1)^2} \left(1 - \frac{1}{u} - \ln u\right); g''(u) = \frac{1}{(u-1)^3} \left(2 \ln u - 3 + \frac{4}{u} - \frac{1}{u^2}\right); \\ g'''(u) &= \frac{1}{(u-1)^4} \left(-6 \ln u + 11 - \frac{18}{u} + \frac{9}{u^2} - \frac{2}{u^3}\right); \\ g^{(4)}(u) &= \frac{h(u)}{(u-1)^5}, h(u) = 24 \ln u - 50 + \frac{96}{u} - \frac{72}{u^2} + \frac{32}{u^3} - \frac{6}{u^4}. \end{aligned}$$

It is obvious that $g^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3, 4$). Since $h'(u) = \frac{24}{u}(1 - \frac{1}{u})^4 \geq 0$, then $h(u)$ is increasing in $(0, \infty)$. In view of $h(1) = 0$, one has $h(u) \leq 0$, $u \in (0, 1)$; $h(u) \geq 0$, $u \in (1, \infty)$, and then $g^{(4)}(u) \geq 0$, for $u \in (0, \infty)$. Hence $g'''(u)$

is increasing and $g'''(u) \leq 0$, since $g'''(\infty) = 0$. By the same way, it follows that $g''(u)$ is decreasing and $g''(u) \geq 0$ since $g''(\infty) = 0$, and $g'(u) \leq 0$ since $g'(\infty) = 0$. Hence one can conclude that $(-1)^i [g(u)u^{-\frac{1}{r}}]^{(i)} \geq 0$ ($i = 0, 1, 2, 3, 4$), and then one has

$$(-1)^i f^{(i)}(x) = (-1)^i \left[g(u)u^{-\frac{1}{r}} \right]_{u=\frac{x+\alpha}{n+\alpha}}^{(i)} \frac{1}{(n+\alpha)^i} \geq 0 (x \in [0, \infty)),$$

and $f^{(i)}(\infty) = 0$ ($i = 0, 1, 2, 3, 4$). The lemma is proved.

Note. In virtue of Lemma 2.2, by (4), one has

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{n+\alpha}{m+\alpha} \right)^{\frac{1}{2}} &= \frac{1}{n+\alpha} \sum_{m=0}^{\infty} f(m) \\ &\leq \frac{1}{n+\alpha} \left[\int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0) \right] = \frac{1}{n+\alpha} \left[\int_{-\alpha}^{\infty} f(x)dx - R_{\alpha}(n) \right], \\ R_{\alpha}(n) &:= \int_{-\alpha}^0 f(x)dx - \frac{1}{2}f(0) + \frac{1}{12}f'(0). \end{aligned} \quad (5)$$

Since $g''(u) \geq 0$, one obtains

$$\begin{aligned} \int_{-\alpha}^0 f(x)dx &= 2(n+\alpha) \int_{-\alpha}^0 g\left(\frac{x+\alpha}{n+\alpha}\right) d\left(\frac{x+\alpha}{n+\alpha}\right)^{\frac{1}{2}} \\ &= 2(n+\alpha)g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{\frac{1}{2}} - 2 \int_{-\alpha}^0 g'\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{\frac{1}{2}} dx \\ &= 2(n+\alpha)g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{\frac{1}{2}} - \frac{4(n+\alpha)}{3}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{\frac{3}{2}} \\ &\quad + \frac{4}{3} \int_{-\alpha}^0 g''\left(\frac{x+\alpha}{n+\alpha}\right)\left(\frac{x+\alpha}{n+\alpha}\right)^{\frac{3}{2}} dx \\ &> 2(n+\alpha)g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{\frac{1}{2}} - \frac{4(n+\alpha)}{3}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{\frac{3}{2}}; \end{aligned}$$

$$f'(0) = \frac{1}{(n+\alpha)}g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{2}} - \frac{1}{2(n+\alpha)}g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{3}{2}},$$

and $f(0) = g(\frac{\alpha}{n+\alpha})(\frac{\alpha}{n+\alpha})^{-1/2}$. Hence one finds from (5) and the above results that

$$\begin{aligned} R_{\alpha}(r, n) &> g\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{-\frac{1}{2}} \left(2\alpha - \frac{1}{2} - \frac{1}{24\alpha}\right) \\ &\quad - g'\left(\frac{\alpha}{n+\alpha}\right)\left(\frac{\alpha}{n+\alpha}\right)^{\frac{1}{2}} \left(\frac{4\alpha}{3} - \frac{1}{12\alpha}\right). \end{aligned} \quad (6)$$

LEMMA 2.3. For $\alpha \geq (3 + \sqrt{21})/24 = 0.31594^+$, one has

$$\omega_\alpha(n) := \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{n+\alpha}{m+\alpha} \right)^{\frac{1}{2}} < \pi^2 \quad (n \in N_0). \quad (7)$$

Proof. For $\alpha \geq (3 + \sqrt{21})/24$, one can obtain

$$\begin{aligned} 2\alpha - \frac{1}{2} - \frac{1}{24\alpha} &= \frac{48\alpha^2 - 12\alpha - 1}{24\alpha} \geq 0; \\ \frac{4\alpha}{3} - \frac{1}{12\alpha} &= \frac{16\alpha^2 - 1}{12\alpha} > 0. \end{aligned} \quad (8)$$

Since $g(u) \geq 0$ and $g'(u) \leq 0$, by (6), one has $R_\alpha(n) \geq 0$. Setting $u = (x + \alpha)/(n + \alpha)$, one finds from (3) that

$$\frac{1}{n + \alpha} \int_{-\alpha}^{\infty} f(x) dx = \int_0^{\infty} \frac{\ln u}{u - 1} u^{-\frac{1}{2}} du = \pi^2.$$

In view of (5), one has (7). The lemma is proved.

Note. If $\alpha < (3 + \sqrt{21})/24$, one can't conform that $R_\alpha(n) \geq 0$ from (8).

LEMMA 2.4. If $(3 + \sqrt{21})/24 \leq \alpha \leq 1, 0 < \varepsilon < 1$, one has

$$I := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{1}{m+\alpha} \right)^{\frac{1+\varepsilon}{2}} \left(\frac{1}{n+\alpha} \right)^{\frac{1+\varepsilon}{2}} \geq \frac{1}{\varepsilon} \{ \pi^2 + o(1) \} (\varepsilon \rightarrow 0^+). \quad (9)$$

Proof. For fixed y , setting $u = (x + \alpha)/(y + \alpha)$, one finds

$$\begin{aligned} I &\geq \int_{1-\alpha}^{\infty} \left(\frac{1}{y+\alpha} \right)^{\frac{1+\varepsilon}{2}} \left[\int_{1-\alpha}^{\infty} \frac{\ln(\frac{x+\alpha}{y+\alpha})}{x-y} \left(\frac{1}{x+\alpha} \right)^{\frac{1+\varepsilon}{2}} dx \right] dy \\ &= \int_{1-\alpha}^{\infty} \left(\frac{1}{y+\alpha} \right)^{1+\varepsilon} \left[\int_{\frac{1}{y+\alpha}}^{\infty} \frac{\ln u}{u-1} u^{-\frac{1+\varepsilon}{2}} du \right] dy \\ &= \int_{1-\alpha}^{\infty} \left(\frac{1}{y+\alpha} \right)^{1+\varepsilon} \left[\int_0^{\infty} \frac{\ln u}{u-1} u^{-\frac{1+\varepsilon}{2}} du - \int_0^{\frac{1}{y+\alpha}} \frac{\ln u}{u-1} u^{-\frac{1+\varepsilon}{2}} du \right] dy \\ &\geq \frac{1}{\varepsilon} \int_0^{\infty} \frac{\ln u}{u-1} u^{-\frac{1+\varepsilon}{2}} du - \int_{1-\alpha}^{\infty} \frac{1}{y+\alpha} \left[\int_0^{\frac{1}{y+\alpha}} \frac{\ln u}{u-1} u^{-\frac{1+\varepsilon}{2}} du \right] dy \\ &= \frac{1}{\varepsilon} \int_0^{\infty} \frac{\ln u}{u-1} u^{-\frac{1+\varepsilon}{2}} du - \sum_{n=0}^{\infty} \int_{1-\alpha}^{\infty} \frac{1}{y+\alpha} \left[\int_0^{\frac{1}{y+\alpha}} (-\ln u) u^{n-\frac{1+\varepsilon}{2}} du \right] dy \\ &= \frac{1}{\varepsilon} \{ \pi^2 + o(1) \} + \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1-\varepsilon}{2})^2} \int_{1-\alpha}^{\infty} \left[\ln(y+\alpha) + \frac{1}{n + \frac{1-\varepsilon}{2}} \right] d(y+\alpha)^{-n-\frac{1-\varepsilon}{2}} \\ &= \frac{1}{\varepsilon} \left\{ \pi^2 + o(1) - 2\varepsilon \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1-\varepsilon}{2})^3} \right\} \dots \end{aligned}$$

Hence one has (9). The lemma is proved.

3 Main results and applications

THEOREM 3.1. If $(3 + \sqrt{21})/24 \leq \alpha \leq 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then one has the following equivalent inequalities:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha}) a_m b_n}{m-n} < \pi^2 \left\{ \sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right\}^{\frac{1}{2}}; \quad (10)$$

$$\sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha}) a_m}{m-n} \right]^2 < \pi^4 \sum_{n=0}^{\infty} a_n^2, \quad (11)$$

where the constant factors π^2 and π^4 are all the best possible.

Proof. By Cauchy's inequality with weight and (7)(see [11]), one has

$$\begin{aligned} H_{\alpha}(a_m, b_n) &: = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha}) a_m b_n}{m-n} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left[\left(\frac{m+\alpha}{n+\alpha} \right)^{\frac{1}{4}} a_m \right] \left[\left(\frac{n+\alpha}{m+\alpha} \right)^{\frac{1}{4}} b_n \right] \\ &\leq \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{m+\alpha}{n+\alpha} \right)^{\frac{1}{2}} a_m^2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})}{m-n} \left(\frac{n+\alpha}{m+\alpha} \right)^{\frac{1}{2}} b_n^2 \right\}^{\frac{1}{2}} \\ &= \left\{ \sum_{m=0}^{\infty} \omega_{\alpha}(m) a_m^2 \sum_{n=0}^{\infty} \omega_{\alpha}(n) b_n^2 \right\}^{\frac{1}{2}} < \pi^2 \left\{ \sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Hence, one has (10).

For $0 < \varepsilon < 1$, setting $\tilde{a}_m, \tilde{b}_n$ as

$$\tilde{a}_m = \left(\frac{1}{m+\alpha} \right)^{\frac{1+\varepsilon}{2}}, \tilde{b}_n = \left(\frac{1}{n+\alpha} \right)^{\frac{1+\varepsilon}{2}}, m, n \in N_0,$$

one has

$$\begin{aligned} \left\{ \sum_{n=0}^{\infty} \tilde{a}_n^2 \sum_{n=0}^{\infty} \tilde{b}_n^2 \right\}^{\frac{1}{2}} &= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{n+\alpha} \right)^{1+\varepsilon} \\ &< 1 + \int_0^{\infty} \left(\frac{1}{x+\alpha} \right)^{1+\varepsilon} dx = 1 + \frac{1}{\varepsilon \alpha^{\varepsilon}}. \end{aligned} \quad (12)$$

If the constant factor π^2 in (10) is not the best possible, then, there exists a positive number $k < \pi^2$, such that (10) is still valid if one replaces π^2 by k . In particular, by (9), one has

$$\pi^2 + o(1) \leq \varepsilon I = \varepsilon H_{\alpha}(\tilde{a}_m, \tilde{b}_n) < \varepsilon k \left\{ \sum_{n=0}^{\infty} \tilde{a}_n^2 \sum_{n=0}^{\infty} \tilde{b}_n^2 \right\}^{\frac{1}{2}} < k \left(\varepsilon + \frac{1}{\alpha^{\varepsilon}} \right),$$

and then $\pi^2 \leq k$ ($\varepsilon \rightarrow 0^+$). This contradicts the fact that $k < \pi^2$. Hence the constant factor π^2 in (10) is the best possible.

Setting b_n as $b_n := \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n}$ ($n \in N_0$), and use (10) to obtain

$$\begin{aligned} \left\{ \sum_{n=0}^{\infty} b_n^2 \right\}^2 &= \left\{ \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n} \right]^2 \right\}^2 \\ &= \left\{ \sum_{n=0}^{\infty} H_{\alpha}(a_m, b_n) \right\}^2 \leq \pi^4 \sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2; \end{aligned} \quad (13)$$

$$0 < \sum_{n=0}^{\infty} b_n^2 = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n} \right]^2 \leq \pi^4 \sum_{n=0}^{\infty} a_n^2 < \infty. \quad (14)$$

It follows that (13) takes the form of strict inequality by using (10); so does (14). Hence (11) holds.

On the other hand, if (11) holds, by Cauchy's inequality, one has

$$\begin{aligned} H_{\alpha}(a_m, b_n) &= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n} \right] [b_n] \\ &\leq \left\{ \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m}{m-n} \right]^2 \sum_{n=0}^{\infty} b_n^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (15)$$

In view of (11), one has (10). It follows that (10) and (11) are equivalent.

If the constant factor π^4 in (11) is not the best possible, then by (15), one can conclude that the constant factor π^2 in (10) is not the best possible. The theorem is proved.

REMARK 3.2. (i) For $\alpha = 1/3$ in (10) and (11), one has the following new equivalent inequalities:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{3m+1}{3n+1})a_m b_n}{m-n} < \pi^2 \left\{ \sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right\}^{\frac{1}{2}}; \quad (16)$$

$$\sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} \frac{\ln(\frac{3m+1}{3n+1})a_m}{m-n} \right]^2 < \pi^4 \sum_{n=0}^{\infty} a_n^2. \quad (17)$$

(ii) For $\alpha = 1$, (10) reduces to (1). It follows that (10) is a best extension of (1). Since for $(3 + \sqrt{21})/24 \leq \alpha < 1$ and $a_n, b_n > 0$ in Theorem 3.1, one has

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+1}{n+1})a_m b_n}{m-n} < \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(\frac{m+\alpha}{n+\alpha})a_m b_n}{m-n},$$

it follows that inequality (10) is an improvement of (1) for any $(3 + \sqrt{21})/24 \leq \alpha < 1$. It is obvious that (10) is valid for $\alpha > 1$, but in this case, it is not more accurate than (1).

References

- [1] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1952.
- [2] D. S. Mintrinic, J. E. Pecaric and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Boston, 1991.
- [3] B. G. Pachpatte, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.*, 226(1998), 166-179.
- [4] Gao Mingzhe and Yang Bicheng, On the extended Hilbert's inequality, *Proc. Amer. Math. Soc.*, 126:3(1998), 751-759.
- [5] Zhao Changjian, Generalization on two new Hilbert type inequalities, *J. Math.*, 20:4(2000), 413-416.
- [6] Bicheng Yang and T. M. Rassias, On the way of weight coefficient and research for the Hilbert-type inequalities, *Math. Ineq. Appl.*, 6: 4(2003), 625-658.
- [7] Kuang Jichang and L. Debnath, On new generalizations of Hilbert's inequality and their applications, *J. Math. Anal. Appl.*, 245(2000), 248-265.
- [8] Yang Bicheng, On a generalization of a Hilbert's type inequality, *Acta Mathematica Scientia*, 25 (2005)(in appear).
- [9] Bicheng Yang, A new inequality similar to Hilbert's inequality, *Journal of Inequalities in Pure and Applied Mathematics*, 3:5(2002), Article 75.
- [10] Bicheng Yang, A new Hardy-Hilbert's type inequality and applications, *Acta Mathematica Sinica*, 46:6(2003), 1079-1086.
- [11] Kuang Jichang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, 2004.

Received: December 21, 2006