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Essential Ideals and Finsler Modules

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Abstract

In this paper, the notion of associated (essential) ideal submodule in Finsler module over C^*-algebras is introduced. Moreover, it is shown that if essential ideal submodule V_I is a Hilbert I-module, then V is itself a Hilbert A-module.

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1 Introduction

A (right) Hilbert C^*-module over a C^*-algebra A is a right A-module V equipped with A-valued inner product $\langle .,. \rangle$ which is a $A-$linear in the second and conjugate linear in the first variable such that V is a Banach space with the norm $\| x \| = \| \langle x, x \rangle \|^{\frac{1}{2}}$ (see [3]).

Finsler modules over C^*-algebras are generalization of Hilbert C^*-modules that first investigated in [5]. Let A be a C^*-algebra and A^+ denote the set of positive elements of a C*-algebra A. Let V be a right module over C^*-algebra A and the map $\rho : V \to A^+$ satisfies the following condition.

(i) the map $\|.\|_V : x \to \|\rho(x)\|$ is a Banach space norm on V; and

(ii) $\rho(xa)^2 = a^*\rho(x)^2 a$ for all $a \in A$ and $x \in V$.

Then V is called a Finsler module over C^*-algebra A.

A Finsler module over C^*-algebra A is said to be full if the linear span $\{\rho(x)^2; x \in V\}$ denoted by $\mathcal{F}(V)$ is dense in A.

Ideal submodules in Hilbert C*-modules are investigated in [2] and [6]. In [5] Phillips and Weaver proved that if (V, ρ) is a Finsler module over C^*-algebra A such that ρ satisfies the parallelogram law

$$\rho(v + w)^2 + \rho(v - w)^2 = 2\rho(v)^2 + 2\rho(w)^2,$$

for $v, w \in V$, then V is a Hilbert A-module. In this paper, the notion of associated (essential) ideal submodule in Finsler modules over C^*-algebras is introduced and it is shown that if ρ satisfies the parallelogram law only on a essential ideal submodule V_I then V is a Hilbert A-module, that is, if essential ideal submodule V_I is a Hilbert I-module, then V is itself a Hilbert A-module.

2 Preliminaries

Definition 2.1. *let V be a Finsler modules over $C^\star-$algebra A, and let I be an ideal in A. The associated ideal submodule V_I is defined by*

$$V_I = [VI]^- = [\{vb : v \in V, b \in I\}]^-$$

(the closed linear span of the action of I on V).

Clearly, V_I is a closed submodule of V. It can be also regarded as a Finsler module over I.

In general, there exist closed submodules which are not ideal submodule. For instance, if a C^*-algebra A is regarded as a Hilbert A-module (with the inner product $\langle a, b\rangle = a^*b$), then ideal submodules of A are precisely ideals in A, while closed submodules of A are closed right ideals in A.

We arise some properties of ideal submodules. Following results are already known of ([2],[1]). let V be a Finsler module over C*-algebra A, and I be an ideal of A. By application of Hewitt-Cohen factorization theorem ([4], Theorem 4.1,[6], proposition 2.31) it is easy to that $V_I = VI = \{vb : v \in V, b \in I\}$. If V be a full Finsler module over A, V_I will be full over I [1, Lemma 2.2].

Remark 2.2. let V be a Finsler module and I be an ideal of A, and V_I be associated ideal submodule. Define by $q : V \to \frac{V}{V_I}$ and $\pi : A \to \frac{A}{I}$ the quotient maps. By definition right action of $\frac{A}{I}$ on linear space $\frac{V}{V_I}$ with $q(v)\pi(a) = q(va)$, $\frac{V}{V_I}$ will be a $\frac{A}{I}$-module and by [5 Lemma 12], $\frac{V}{V_I}$ is a Finsler $\frac{A}{I}$-module with norm Finsler $\rho_{\frac{A}{I}}(q(v)) = \pi(\rho_A(v))$. Then $\rho_{\frac{A}{I}}(q(V)) = \pi(\rho_A(V))$, so $[\rho_{\frac{A}{I}}(q(V)] = \pi([\rho_A(V)])$.

In addition, $\frac{V}{V_I}$ is a full Finsler $\frac{A}{I}$-module if and only if V is full. This follows at once from the evident equality $[\rho_{\frac{A}{I}}(q(V)] = \pi([\rho_A(V)])$.

With similar argument of [2 p. 4], if X be a closed submodule of V, J be an ideal of A such that $\rho(V) \subseteq J$, then $\frac{V}{X}$ with module action $q(x)\pi(a) = q(xa)$ is a $\frac{A}{J}$-module iff $X = V_J$. Note that smallest of such ideals is $A-$linear hull $(\rho(V)^2)$.

3 main results

Definition 3.1. Let I be an ideal of C^*-algebra A, define $I^{\perp} = \{a \in A : aI = 0\}$ (that is ideal of A). I is essential if $I^{\perp} = \{0\}$, that is equivalent $I \bigcap J \neq \{0\}$ for all closed ideal J of A.

The following lemma is a much similar relative of Lemma 1.10 of [2]. (Enough that replacing I with I^+ denoted positive elements of I).

Lemma 3.2. *Let I be an ideal in a C^*-algebra A. The following condition are mutually equivalent:*
(a) I is an essential ideal in A
(b) $\|a\| = sup_{b\in I^+, \|b\|\leq 1}(\|ab\|), \forall a \in A$
(c) $\|a\| = sup_{b\in I^+, \|b\|\leq 1}(\|ba\|), \forall a \in A$
(d) $\|a\| = sup_{b\in I^+, \|b\|\leq 1}(\|bab\|), \forall a \in A^+$.

Theorem 3.3. *let V be a Finsler module and I be an essential ideal of A, and V_I be (associated) ideal submodule. Then $\|v\| = sup_{b\in I^+, \|b\|\leq 1}(\|vb\|), \forall v \in V$. Conversely, if V is a full Finsler module over C^*-algebra A and $\|v\| = sup_{b\in I^+, \|b\|\leq 1}(\|vb\|), \forall v \in V$ respect to some ideal I of A, then I is an essential ideal in A.*

Proof . Let I be an essential ideal in A, then $\forall v \in V$

$$\begin{aligned} \|v\|^2 &= \|\rho(v)\|^2 = sup_{b\in I^+, \|b\|\leq 1}(\|b\rho(v)^2 b\|) \\ &= sup_{b\in I^+, \|b\|\leq 1}(\|\rho(vb)^2\|) \\ &= sup_{b\in I^+, \|b\|\leq 1}(\|vb\|)^2, \end{aligned}$$

by definition of norm Finsler and Lemma 3.2.
To prove the converse, suppose that V is a full Finsler module and I is not essential so that $I^{\perp} \neq \{0\}$. Take any $c \in I^{\perp}, c \neq 0$. By [1 proof Theorem 3.2(iii)], there exists $v \in V$ such that $vc \neq 0$. Now by hypothesis of theorem we have

$$\begin{aligned} \|vc\| &= sup_{b\in I^+, \|b\|\leq 1}(\|(vc)b\|) \\ &= sup_{b\in I^+, \|b\|\leq 1}(\|v(cb)\|) = 0. \end{aligned}$$

Means $vc = 0$, that is a contradiction. So I is an essential ideal of A. □

Recall that a Finsler A-module V with map ρ is a Hilbert module if ρ produce a A-valued inner product $< .,. >$ on V such that V is a Hilbert A-module and $\rho(x) =< x, x >^{1/2}$ for all $x \in V$.
Theorem 3.4. *Let V be a Finsler module of commutative C^*-algebra A and I be an essential ideal of A. If essential ideal submodule V_I is a Hilbert*

I-module, then V is itself a Hilbert A-module.

Proof . Let ρ be norm Finsler map over V. Note that essential ideal submodule V_I is a Finsler I-module with map $\rho|_{V_I}$. Hence ρ satisfies in parallelogram law on V_I. Therefore, for all $v, w \in V$ and $a \in I$, we have

$$\begin{array}{ll} \rho & (va+wa)^2+\rho(va-wa)^2-2\rho(va)^2-2\rho(wa)^2=0 \\ \Leftrightarrow & a^*(\rho(v+w)^2+\rho(v-w)^2)-2\rho(v)^2-2\rho(w)^2)a=0 \\ \Leftrightarrow & (\rho(v+w)^2+\rho(v-w)^2-2\rho(v)^2-2\rho(w)^2)a^*a=0. \end{array}$$

Now Lemma 3.2(b), show that $\|\rho(v+w)^2+\rho(v-w)^2-2\rho(v)^2-2\rho(w)^2\|=0$. Hence, $\rho(v+w)^2+\rho(v-w)^2=2\rho(v)^2+2\rho(w)^2$ means norm Finsler ρ satisfies in parallelogram law on V, so by [5 lemma 12] V is a Hilbert A-module. □

Recall that if A, B, and D are C^*-algebra , and if homomorphisms $\phi : A \to D$ and $\psi : B \to D$ are given , then the C^*-algebra $A \oplus_D B$ is defined as

$$A \oplus_D B = \{(a, b) \in A \oplus B \ : \phi(a) = \psi(b)\}.$$

We use the same notation for modules, Banach spaces, etc.

Let A be a C^*-algebra. By [5 lemmas 10 and 11] A has a unique maximal commutative ideal I_0 and a closed ideal J such that $I_0 \bigcap J = \{0\}$ and $\frac{A}{J}$ is commutative, moreover, $A \cong \frac{A}{J} \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}$ by *-isomorphism $\varphi : A \to \frac{A}{J} \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}$ such that $\varphi(a) = (a + J, a + I_0)$.

Lemma 3.5. *Let A be a C^*-algebra, I_0, J and φ be as in above argument, I be an essential ideal in C^*-algebra A if and only if $\frac{I}{J}$ and $\frac{I}{I_0}$ are essential ideal in C^*-algebras $\frac{A}{J}$ and $\frac{A}{I_0}$ respectively.*

Proof . It is clear that quotient map $\pi : A \to \frac{A}{J}$ is a *-surjective homomorphism. It is enough to show that $\pi(K) \bigcap \frac{I}{J} \neq \{0\}$ for arbitrary closed ideal K of A such that $K \bigcap J \neq K$. Suppose that $\pi(K)$ is a non zero closed ideal of $\frac{A}{J}$. There exists $k \in K - J$, so we have

$$0 \neq \varphi(k) = (k + J, k + I_0) \in \pi(K) \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}.$$

So $\pi(K) \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}$ is a non zero closed ideal of $\frac{A}{J} \oplus_{\frac{A}{I_0+J}} \frac{A}{I_0}$. Since $I \cong \frac{I}{J} \oplus_{\frac{A}{I_0+J}} \frac{I}{I_0}$ is essential ideal in C^*-algebra A. Now we have $\{0\} \neq \pi(K) \bigcap \frac{I}{J}$. Hence $\frac{I}{J}$ is a essential ideal of $\frac{A}{J}$. Similar statement is true for $\frac{I}{I_0}$.

Conversely, let $a \in I^\perp$. Then $ac = o$ for every $c \in I$. Thus $ac \in J$ and $ac \in I_0$. It implies that $a + J \in (\frac{I}{J})^\perp$ and $a + I_0 \in (\frac{I}{I_0})^\perp$. Therefore $a \in I_0 \bigcap J = \{0\}$ because $\frac{I}{J}$ and $\frac{I}{I_0}$ are essential ideal. Hence in view of the mention of above this Lemma $a = 0$. Consequently, $I^{\perp} = \{0\}$, that means I is an essential ideal. □

Theorem 3.6. Let V be a Finsler module over C^*-algebra A and I be an essential ideal of A. If essential ideal submodule V_I is a Hilbert I-module, then V is itself a Hilbert A-module.

Proof . By Theorem 17 of [5], we can write $V \cong V_1 \oplus_{V_0} V_2$, where V_2 and V_0 are Hilbert $\frac{A}{I_0}$ and $\frac{A}{I_0+J}$ modules resp., and V_1 is a Finsler module over commutative C*-algebra $\frac{A}{J}$. Also, we note that $\rho_V(x_1, x_2) = (\rho_{V_1}(x_1), \rho_{V_2}(x_2))$, for every $x_1 \in V_1$ and $x_2 \in V_2$ [5, Lemma 16]. Now by Lemma 3.5 and Theorem 3.4, V_1 is a Hilbert module over C^*-algebra $\frac{A}{J}$. Consequently, V is a Hilbert module over C^*-algebra A. □

Remark 3.7. Essentiality can not be dropped. For instance, let V be a non- Hilbert Finsler module and $I = 0$ so is not essential. $V_I = 0$ is a Hilbert I-module but V is not.

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