On Chaotic Order and Generalized Heinz-Kato-Furuta-type Inequality

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Abstract

In a Hilbert space the chaotic order $A \gg B$ for positive and invertible operators A and B was characterized in terms of operator inequalities in [3]. In this paper we give different characterizations in terms of both the generalized Heinz-Kato-Furuta-type inequality and the Reid-type inequality.

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1. Notation and Introduction

For the sake of convenience let us recall some standard notations and well-known results first. Let H be a Hilbert space and the capital letters be bounded linear operators on H . A^* denotes the adjoint operator of A . T is positive (written $T \geq O$, the zero operator) in case $(Tx, x) \geq 0$ for all $x \in H$. If S and T are Hermitian, we write $T \geq S$ in case $T - S \geq O$. That $T > O$ means T is positive and invertible, and I denotes the identity operator on H . $T = U |T|$ is the polar decomposition of T with U the partial isometry, $|T|$ the positive square root of the positive operator T^*T satisfying the kernel condition $N(|T|) = N(U)$. It is well-known in this case that $|T^*|^c = U |T|^c U^*$ for $c > 0$ in general. As usual, $\log A = \lim_{\alpha \rightarrow +0} \frac{A\alpha - I}{\alpha}$ for $A > O$. By the Löwner-Heinz inequality, i.e., $S\alpha \geq T\alpha$ if $S \geq T \geq O$ for $\alpha \in [0, 1]$, $\log A \geq \log B$ if $A \geq B > O$. The relation is called $\log A$ majorizes $\log B$, or chaotic order, and denoted by $A \gg B$, which is weaker than the usual operator order $A \geq B$. It

is essential to notice that the Löwner-Heinz inequality does not hold in general if $\alpha > 1$. Let us list four well-known results below.

Theorem A [2, Theorem 1]. *The following statements are equivalent.*

- (i) $\text{range}(B) \subseteq \text{range}(A)$;
 - (ii) A^* majorizes B^* , i.e., $\|B^*x\| \leq \lambda \|A^*x\|$ for some $\lambda \geq 0$ and all $x \in H$, or $BB^* \leq \lambda^2 AA^*$ (majorization);
 - (iii) There exists C such that $B = AC$ (factorization).
- Moreover, $\|C\|^2 = \inf \{\mu \mid BB^* \leq \mu AA^*\}$ (due to the equivalence of (ii) and (iii)).

Theorem B [3]. *For $A, B > O$, the following are equivalent for $\alpha \in [0, 1]$.*

- (1.1) $A \gg B$;
- (1.2) $A^{r\alpha} \geq (A^{r/2} B^p A^{r/2})^{\frac{r\alpha}{p+r}}$ for $p, r \geq 0$;
- (1.3) $(B^{r/2} A^p B^{r/2})^{\frac{r\alpha}{p+r}} \geq B^{r\alpha}$ for $p, r \geq 0$.

In Theorem B we mention that (1.1) $\Rightarrow$ (1.2) is due to the Furuta inequality (Theorem C below), and (1.2) $\Rightarrow$ (1.1) and (1.2) $\Leftrightarrow$ (1.3) could be easily proved. Theorem B is indeed generalizations of Ando's result, which says in a rephrased form that $A \gg B$ if and only if $A^p \geq (A^{p/2} B^p A^{p/2})^{1/2}$ for $p \geq 0$ [1].

Theorem C [4]. *If $A \geq B \geq O$, then, for $r \geq 0$, $p \geq 1$ and $\alpha \in [0, 1]$,*

$$(B^{r/2} A^p B^{r/2})^{\frac{(1+r)\alpha}{p+r}} \geq B^{(1+r)\alpha} \text{ (Furuta inequality).}$$

Theorem D [5, Theorem 1]. *Let $A, B \geq O$ such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in H$. Then, for $r, s \geq 0$, $p, q \geq 1$, and $\alpha, \beta \in [0, 1]$ such that $(1+r)\alpha + (1+s)\beta \geq 1$, and for all $x, y \in H$,*

$$\begin{aligned} & |(T \mid T^{(1+r)\alpha + (1+s)\beta - 1} x, y)|^2 \\ & \leq ([\mid T \mid^r A^{2p} \mid T \mid^r]^{\frac{(1+r)\alpha}{p+r}} x, x) ([\mid T^* \mid^s B^{2q} \mid T^* \mid^s]^{\frac{(1+s)\beta}{q+s}} y, y). \end{aligned}$$

The result above is called the generalized Heinz-Kato-Furuta inequality in the literature, and it is equivalent to the Furuta inequality [5]. In particular, the Heinz-Kato-Furuta inequality is $|(T \mid T^{\alpha + \beta - 1} x, y)| \leq \|A^\alpha x\| \|B^\beta y\|$, and the Heinz-Kato inequality is $|(Tx, x)| \leq \|A^\alpha x\| \|B^{1-\alpha} y\|$. Notice that the condition $(1+r)\alpha + (1+s)\beta \geq 1$ in Theorem D may be omitted if T is invertible.

The Reid's inequality is as follows, which is equivalent to the Löwner-Heinz inequality for $\alpha = 1/2$ [7, Corollary 2].

Theorem E [10]. *Let $S \geq O$, and SK be Hermitian. Then, for all $x \in H$,*

$$|(SKx, x)| \leq \|K\| |(Sx, x)|.$$

The aim of this paper is to characterize the relation $A \gg B$ for $A, B > O$ in terms of both the generalized Heinz-Kato-Furuta-type inequality and the Reid-type inequality by using Theorem A (the Douglas's majorization and factorization theorem) and Theorem B (Fujii's et al. characterization of $A \gg B$ in terms of the Furuta-type inequality).

Incidentally, the notations and the well-known results above, except for Theorem A, D and E, could be found in the recent book [6] with proofs.

2. Main Results

We need the following modified definition: We say that S strictly majorizes T if $\|Tx\| \leq \|Sx\|$ for all $x \in H$. For examples, in Theorem D above we may say that A strictly majorizes T and B strictly majorizes T^* .

In what follows we assume that $A, B > O$ and $T \neq O$. With several different conditions on majorization for operators A, B and T , we have the following four main results characterizing the chaotic order $A \gg B$.

Theorem 2.1. *If A strictly majorizes T , and B majorizes T^* , then the following are equivalent for $\alpha, \beta \in [0, 1]$ and $p, q, r, s \geq 0$ such that $r\alpha + s\beta \geq 2$, some $\mu \geq 0$, and for all $x, y \in H$.*

$$(1.1) \quad A \gg B;$$

$$(2.1) \quad ([A^{r/2} | T |^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} | T^* |^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ \leq \mu^{\frac{s\beta q}{q+s}} (A^{r\alpha} x, x) (B^{s\beta} y, y);$$

$$(2.2) \quad \text{There exists a unique } C \text{ with } \|C\| \leq \mu \text{ such that} \\ ([A^{r/2} | BC |^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} | C^* B |^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ \leq \mu^{\frac{s\beta q}{q+s}} (A^{r\alpha} x, x) (B^{s\beta} y, y);$$

$$(2.3) \quad |(T | T |^{\frac{r\alpha+s\beta}{2}-1} x, y)|^2 \\ \leq \mu^{\frac{s\beta q}{q+s}} ([|T|^{r/2} A^p | T |^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([|T^*|^{s/2} B^q | T^*|^{s/2}]^{\frac{s\beta}{q+s}} y, y);$$

$$(2.4) \quad \text{There exists a unique } C \text{ with } \|C\| \leq \mu \text{ such that} \\ |(BC | BC |^{\frac{r\alpha+s\beta}{2}-1} x, y)|^2 \\ \leq \mu^{\frac{s\beta q}{q+s}} ([|BC|^{r/2} A^p | BC |^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([|C^* B|^{s/2} B^q | C^* B |^{s/2}]^{\frac{s\beta}{q+s}} y, y).$$

Proof. Since $\|Tx\| \leq \|Ax\|$, and by the Douglas's theorem there exist $\mu > 0$ such that $\|T^*y\| \leq \mu \|By\|$ for all $x, y \in H$, i.e., $|T|^2 \leq A^2$ and $|T^*|^2 \leq (\mu B)^2$. Also, there exists C with $\|C\| \leq \mu$ such that $T = BC$, or $T^* = C^*B$. Clearly C is unique as B is invertible. It follows that $A \gg |T|$ and $\mu B \gg |T^*|$. Now,

(1.1) $\Rightarrow$ (2.1). In (1.1) replace B by $|T|$. By (1.2) of Theorem B we get

$$(A^{r/2} |T|^p A^{r/2})^{\frac{r\alpha}{p+r}} \leq A^{r\alpha}.$$

Similarly, replacing B by $|T^*|$, A by μB , α by β , r by s , and p by q yield

$$(B^{s/2} |T^*|^q B^{s/2})^{\frac{s\beta}{q+s}} \leq \mu^{\frac{s\beta q}{q+s}} B^{s\beta}.$$

The required inequality follows by the two inequalities above.

(2.1) $\Rightarrow$ (2.2). Due to statements at the beginning of the proof.

(2.2) $\Rightarrow$ (1.1). Since $\|T^*y\| \leq \mu \|By\|$, we may put $T = B$ in particular, then $C = I$ and $\mu = 1$. Also, in (2.2) let $s = r$, $\beta = \alpha$, $q = p$ and $y = x$, then

$$([A^{r/2} B^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x)(B^{r\alpha} x, x) \leq (A^{r\alpha} x, x)(B^{r\alpha} x, x), \text{ or } \\ (A^{r/2} B^p A^{r/2})^{\frac{r\alpha}{p+r}} \leq A^{r\alpha}.$$

It follows that $A \gg B$ by Theorem B.

(1.1) $\Rightarrow$ (2.3). The same replacement as in (1.1) $\Rightarrow$ (2.1) and using (1.3) of Theorem B instead yield

$$|T|^{r\alpha} \leq (|T|^{r/2} A^p |T|^{r/2})^{\frac{r\alpha}{p+r}} \text{ and } |T^*|^{s\beta} \leq \mu^{\frac{s\beta q}{q+s}} (|T^*|^{s/2} B^q |T^*|^{s/2})^{\frac{s\beta}{q+s}}.$$

Next, since

$$\begin{aligned} & |(T |T|^{\frac{r\alpha+s\beta}{2}-1} x, y)|^2 \\ &= |(U |T|^{\frac{r\alpha+s\beta}{2}} x, y)|^2 = |(|T|^{r\alpha/2} x, |T|^{s\beta/2} U^* y)|^2 \\ &\leq \| |T|^{r\alpha/2} x \|^2 \| |T|^{s\beta/2} U^* y \|^2 \text{ by the Cauchy-Schwarz inequality} \\ &= (|T|^{r\alpha} x, x)(U |T|^{s\beta} U^* y, y) \\ &= (|T|^{r\alpha} x, x)(|T^*|^{s\beta} y, y) \text{ by a remark in §1} \end{aligned}$$

and the required (2.3) follows.

(2.3) $\Rightarrow$ (2.4). Trivial.

(2.4) $\Rightarrow$ (1.1). Similar to the proof of (2.2) $\Rightarrow$ (1.1), i.e., let $T = B$, etc., then $C = I$ and $\mu = 1$. It follows by (2.4) that

$$B^{r\alpha} \leq (B^{r/2} A^p B^{r/2})^{\frac{r\alpha}{p+r}}.$$

Hence $A \gg B$ by Theorem B, and the proof is completed.

Remark that the proof of each of the following three theorems are similar to the proof of Theorem 2.1, and we shall give the outlines only.

Theorem 2.2. *If A strictly majorizes T , and B majorizes T , then the following are equivalent for $\alpha, \beta \in [0, 1]$ and $p, q, r, s \geq 0$ such that $r\alpha + s\beta \geq 0$, some $\delta \geq 0$, and for all $x, y \in H$.*

$$(1.1) \quad A \gg B;$$

$$(2.5) \quad ([A^{r/2} | T |^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} | T |^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ \leq \delta^{\frac{s\beta q}{q+s}} (A^{r\alpha} x, x) (B^{s\beta} y, y);$$

$$(2.6) \quad \text{There exists a unique } D \text{ with } \|D\| \leq \delta \text{ such that} \\ ([A^{r/2} | DB |^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} | DB |^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ \leq \delta^{\frac{s\beta q}{q+s}} (A^{r\alpha} x, x) (B^{s\beta} y, y);$$

$$(2.7) \quad |(|T|^{\frac{r\alpha+s\beta}{2}} x, y)|^2 \\ \leq \delta^{\frac{s\beta q}{q+s}} (|T|^{r/2} A^p |T|^{r/2}]^{\frac{r\alpha}{p+r}} x, x) (|T|^{s/2} B^q |T|^{s/2}]^{\frac{s\beta}{q+s}} y, y);$$

$$(2.8) \quad \text{There exists a unique } D \text{ with } \|D\| \leq \delta \text{ such that} \\ |(|DB|^{\frac{r\alpha+s\beta}{2}} x, y)|^2 \\ \leq \delta^{\frac{s\beta q}{q+s}} (|DB|^{r/2} A^p |DB|^{r/2}]^{\frac{r\alpha}{p+r}} x, x) (|DB|^{s/2} B^q |DB|^{s/2}]^{\frac{s\beta}{q+s}} y, y).$$

Proof. By assumption $\|Tx\| \leq \|Ax\|$ and there exist $\delta \geq 0$ such that $\|Ty\| \leq \delta \|By\|$ for all $x, y \in H$. Thus, $|T|^2 \leq A^2$ and $|T|^2 \leq (\delta B)^2$. It follows that $A \gg |T|$ and $\delta B \gg |T|$, and there exists a unique D with $\|D\| \leq \delta$ such that $T^* = BD^*$, or $T = DB$.

In the proof of (1.1) $\Rightarrow$ (2.7) we need to know that $|(|T|^{\frac{r\alpha+s\beta}{2}} x, y)|^2 \leq (|T|^{r\alpha} x, x) (|T|^{s\beta} y, y)$, which is obtained by a straightforward computation.

It should be noted that the conditions $r\alpha + s\beta \geq 2$ and $r\alpha + s\beta \geq 0$ in Theorem 2.1 and 2.2, respectively, may be omitted if T is invertible.

Theorem 2.3. *If T majorizes A , and T^* strictly majorizes B , then the following are equivalent for $\alpha, \beta \in [0, 1]$ and $p, q, r, s \geq 0$, some $\eta > 0$, and for all $x, y \in H$.*

$$(1.1) \quad A \gg B;$$

$$(2.9) \quad (|T|^{r/2} A^p |T|^{r/2}]^{\frac{r\alpha}{p+r}} x, x) (|T^*|^{s/2} B^q |T^*|^{s/2}]^{\frac{s\beta}{q+s}} y, y)$$

$$\leq \eta^{\frac{r\alpha p}{p+r}}(|T|^{r\alpha} x, x)(|T^*|^{s\beta} y, y);$$

(2.10) *There exists E (E is unique if T is invertible) with $\|E\| \leq \eta$ such that*

$$\begin{aligned} & ([|T|^{r/2} (ET)^p | T|^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([|T^*|^{s/2} B^q | T^*|^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ & \leq \eta^{\frac{r\alpha p}{p+r}}(|T|^{r\alpha} x, x)(|T^*|^{s\beta} y, y); \end{aligned}$$

$$\begin{aligned} & (2.11) \quad (A^{r\alpha} x, x)(B^{s\beta} y, y) \\ & \leq \eta^{\frac{r\alpha p}{p+r}}([A^{r/2} | T|^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([B^{s/2} | T^*|^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y); \end{aligned}$$

(2.12) *There exists E (E is unique if T is invertible) with $\|E\| \leq \eta$ such that*

$$\begin{aligned} & ((ET)^{r\alpha} x, x)(B^{s\beta} y, y) \\ & \leq \eta^{\frac{r\alpha p}{p+r}}([(ET)^{r/2} | T|^p (ET)^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([B^{s/2} | T^*|^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y). \end{aligned}$$

Proof. Note first by assumption there exists $\eta > 0$ such that $\|Ax\| \leq \eta \|Tx\|$, and $\|By\| \leq \|T^*y\|$ for all $x, y \in H$. In other words, $A^2 \leq \eta T^2$ and $B^2 \leq |T^*|^2$. So, $\eta T \gg A$ and $|T^*| \gg B$, and there exists E such that $A = T^*E^* = (T^*E^*)^* = ET$. Clearly E is unique if T is invertible.

Theorem 2.4. *If T majorizes A , and T strictly majorizes B , then the following are equivalent for $\alpha, \beta \in [0, 1]$, $p, q, r, s \geq 0$, some $\xi > 0$, and for all $x, y \in H$.*

$$(1.1) \quad A \gg B;$$

$$\begin{aligned} & (2.13) \quad ([|T|^{r/2} A^p | T|^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([|T|^{s/2} B^q | T|^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ & \leq \xi^{\frac{r\alpha p}{p+r}}(|T|^{r\alpha} x, x)(|T|^{s\beta} y, y); \end{aligned}$$

(2.14) *There exists F (F is unique if T is invertible) with $\|F\| \leq \xi$ such that*

$$\begin{aligned} & ([|T|^{r/2} (FT)^p | T|^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([|T|^{s/2} B^q | T|^{s/2}]^{\frac{s\beta}{q+s}} y, y) \\ & \leq \xi^{\frac{r\alpha p}{p+r}}(|T|^{r\alpha} x, x)(|T|^{s\beta} y, y); \end{aligned}$$

$$\begin{aligned} & (2.15) \quad (A^{r\alpha} x, x)(B^{s\beta} y, y) \\ & \leq \xi^{\frac{r\alpha p}{p+r}}([A^{r/2} | T|^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([B^{s/2} | T^*|^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y); \end{aligned}$$

(2.16) *There exists F (F is unique if T is invertible) with $\|F\| \leq \xi$ such that*

$$\begin{aligned} & ((FT)^{r\alpha} x, x)(B^{s\beta} y, y) \\ & \leq \xi^{\frac{r\alpha p}{p+r}}([(FT)^{r/2} | T|^p (FT)^{r/2}]^{\frac{r\alpha}{p+r}} x, x)([B^{s/2} | T^*|^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y). \end{aligned}$$

Proof. As in the proof before, there exists $\xi > 0$ such that $\|Ax\| \leq \xi \|Tx\|$, and $\|By\| \leq \|Ty\|$ for all $x, y \in H$. In other words, $A^2 \leq \xi T^2$ and $B^2 \leq T^2$. So, $\xi T \gg A$ and $T \gg B$, and there exists F such that $A = T^*F^* = (T^*F^*)^* = FT$ and F is unique if T is invertible.

The next result is generalizations and characterizations of Reid-type inequality in terms of the chaotic order, and the proof is similar to that of Theorem 2.1 and 2.2.

Theorem 2.5. *Let BK be Hermitian for some $K \neq O$. If $p, q, r, s \geq 0$, and $\alpha, \beta \in [0, 1]$ such that $r\alpha + s\beta \geq 2$, then the following are equivalent for all $x, y \in H$, and $a = \frac{r\alpha p}{p+r} + \frac{s\beta q}{q+s}$.*

$$(1.1) \quad A \gg B;$$

$$(2.17) \quad ([A^{r/2} | BK|^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} | BK|^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \leq \|K\|^a (A^{r\alpha} x, x) (B^{s\beta} y, y);$$

$$(2.18) \quad |(BK | BK|^{\frac{r\alpha+s\beta}{2}-1} x, y)|^2 \leq \|K\|^a ([BK | BK|^{\frac{r\alpha}{p+r}} A^p | BK|^{\frac{r\alpha}{p+r}} x, x) ([BK | BK|^{\frac{s\beta}{q+s}} B^q | BK|^{\frac{s\beta}{q+s}} y, y);$$

$$(2.19) \quad |(|BK|^{\frac{r\alpha+s\beta}{2}} x, y)|^2 \leq \|K\|^a ([BK | BK|^{\frac{r\alpha}{p+r}} A^p | BK|^{\frac{r\alpha}{p+r}} x, x) ([BK | BK|^{\frac{s\beta}{q+s}} B^q | BK|^{\frac{s\beta}{q+s}} y, y).$$

Proof. We first note that the following is well-known [7]: If SK is Hermitian for $S \geq O$, then $|SK|^2 \leq (\|K\| \|S\|)^2$. Let us prove it for the completeness sake. Since $KK^* \leq \|K\|^2$ as $K/\|K\|$ is a contraction operator. Hence, the required relation follows as

$$O \leq \frac{|SK|^2}{\|K\|^2} = \frac{|(SK)^*|^2}{\|K\|^2} = \frac{SKK^*S}{\|K\|^2} \leq S^2.$$

The relation means that S majorizes SK with $\lambda = \|K\|$ in the sense of the Douglas's theorem. So, there exists G such that $SK = SG$, or $G = K$ as S is invertible. In our case $|BK|^2 \leq (\|K\| \|B\|)^2$, and hence $|BK| \ll \|K\| B$. Notice that if $A \gg B$, then $\|K\| A \gg \|K\| B \gg |BK|$, i.e., A majorizes $|BK|$ and B majorizes $|BK|$. Now,

(1.1) $\Rightarrow$ (2.17). Replace A by $\|K\| A$ and B by $|BK|$ in (1.1), then (1.2) in Theorem B yields

$$(A^{r/2} | BK|^p A^{r/2})^{\frac{r\alpha}{p+r}} \leq \|K\|^b A^{r\alpha} \text{ with } b = \frac{r\alpha p}{p+r}.$$

Similarly, replacing A by $\|K\| B$, B by $|BK|$, r by s , α by β , and p by q yield

$$(B^{s/2} \mid BK \mid^q B^{s/2})^{\frac{s\beta}{q+s}} \leq \|K\|^c B^{s\beta} \text{ with } c = \frac{s\beta q}{q+s}.$$

Clearly $a = b + c = \frac{r\alpha p}{p+r} + \frac{s\beta q}{q+s}$.

(2.17) $\Rightarrow$ (1.1). We may let $BK = B$ in particular, then $K = I$ and $\|K\| = 1$. In (2.17) let $s = r$, $\beta = \alpha$, $q = p$ and $y = x$. Then

$$([A^{r/2} B^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x)(B^{r\alpha} x, x) \leq (A^{r\alpha} x, x)(B^{r\alpha} x, x),$$

so, $(A^{r/2} B^p A^{r/2})^{\frac{r\alpha}{p+r}} \leq A^{r\alpha}$ and (1.1) follows Theorem B.

That (1.1) $\Leftrightarrow$ (2.18) and (1.1) $\Leftrightarrow$ (2.19) could be similarly proved, and we leave it to the reader.

Incidentally, several characterizations of the Reid's inequality apperaed in [7].

3. Consequences

In this final section we start with a generalization of the Ando's condition mentioned in § 1 in terms of generalized Heinz-Kato-type inequality.

Corollary 3.1. *Let A strictly majorize T , and B strictly majorize T^* . then the following are equivalent for $p \geq 1$.*

$$(3.1) \quad A^p \geq (A^{p/2} B^p A^{p/2})^{1/2} \text{ (Ando's condition);}$$

$$(3.2) \quad ([A^{p/2} \mid T \mid^p A^{p/2}]^{1/2} x, x)([B^{p/2} \mid T^* \mid^p B^{p/2}]^{1/2} y, y) \leq (A^p x, x)(B^p y, y);$$

$$(3.3) \quad \text{There exists a unique } C \text{ with } \|C\| \leq 1 \text{ such that} \\ ([A^{p/2} \mid BC \mid^p A^{p/2}]^{1/2} x, x)([B^{p/2} \mid C^* B \mid^p B^{p/2}]^{1/2} y, y) \leq (A^p x, x)(B^p y, y);$$

$$(3.4) \quad |(T \mid T \mid^{p-1} x, y)|^2 \\ \leq ([T \mid^{p/2} A^p \mid T \mid^{p/2}]^{1/2} x, x)([T^* \mid^{p/2} B^p \mid T^* \mid^{p/2}]^{1/2} y, y);$$

$$(3.5) \quad \text{There exists a unique } C \text{ with } \|C\| \leq 1 \text{ such that} \\ |(BC \mid BC \mid^{p-1} x, y)|^2 \\ \leq ([BC \mid^{p/2} A^p \mid BC \mid^{p/2}]^{1/2} x, x)([C^* B \mid^{p/2} B^p \mid C^* B \mid^{p/2}]^{1/2} y, y).$$

Proof. In Theorem 2.1 let $r = s = q = p$ and $\alpha = \beta = 1$.

The following two results are about mutually reversed inequalities in § 2. Yet, they are all equivalent to the chaotic order $A \gg B$.

Corollary 3.2. *For $\alpha, \beta \in [0, 1]$ and $p, q, r, s \geq 0$, and for all $x, y \in H$ the following are equivalent.*

$$(1.1) \quad A \gg B;$$

$$(3.6) \quad ([A^{r/2} \mid T \mid^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} \mid T^* \mid^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \leq (A^{r\alpha} x, x) (B^{s\beta} y, y),$$

if A strictly majorizes T , and B strictly majorizes T^* ;

$$(3.7) \quad (A^{r\alpha} x, x) (B^{s\beta} y, y) \leq ([A^{r/2} \mid T \mid^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} \mid T^* \mid^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y),$$

if T strictly majorizes A , and T^* strictly majorizes B .

Proof. Obviously by (2.1) and (2.11) in Theorem 2.1 and 2.3, respectively.

Corollary 3.3. For $\alpha, \beta \in [0, 1]$ and $p, q, r, s \geq 0$, and for all $x, y \in H$ the following are equivalent.

$$(1.1) \quad A \gg B;$$

$$(3.8) \quad ([A^{r/2} \mid T \mid^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} \mid T \mid^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y) \leq (A^{r\alpha} x, x) (B^{s\beta} y, y),$$

if A strictly majorizes T , and B strictly majorizes T ;

$$(3.9) \quad (A^{r\alpha} x, x) (B^{s\beta} y, y) \leq ([A^{r/2} \mid T \mid^p A^{r/2}]^{\frac{r\alpha}{p+r}} x, x) ([B^{s/2} \mid T \mid^q B^{s/2}]^{\frac{s\beta}{q+s}} y, y),$$

if T strictly majorizes A , and T strictly majorizes B .

Proof. By (2.5) and (2.15) in Theorem 2.2 and 2.4, respectively.

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