On Common Fixed Point and Approximation Results of Gregus Type

S. Al-Mezel and N. Hussain

Department of Mathematics
King Abdul Aziz University
P. O. Box 80203, Jeddah 21589, Saudi Arabia

Abstract

Fixed point theorems of Ciric [3], Fisher and Sessa [4], Gregus [5], Jungck [10] and Mukherjee and Verma [17] are generalized to a locally convex space. As applications, common fixed point and invariant approximation results for subcompatible maps are obtained. Our results unify and generalize various known results to a more general class of noncommuting mappings.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Common fixed points, compatible maps, subcompatible maps, Minkowski functional, invariant approximation

1. Introduction and preliminaries

In the sequel, (E, τ) will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^n U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighborhoods of zero for τ . A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E, τ) , there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [16, page 203]).

The following construction will be crucial. Suppose that M is τ -bounded subset of E . For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly $B = \cap_\alpha \lambda_\alpha U_\alpha$ is τ -bounded, τ -closed absolutely convex and contains M . The linear span E_B of B in E is $\cup_{n=1}^\infty nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B . Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$ (for details see [16,25]).

Let M be a subset of a locally convex space (E, τ) . Let $I : M \rightarrow M$ be a mapping. A mapping $T : M \rightarrow M$ is called I -Lipschitz if there exists $k \geq 0$ such that $p_\alpha(Tx - Ty) \leq kp_\alpha(Ix - Iy)$ for any $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$. If $k < 1$ (respectively, $k = 1$), then T is called an I -contraction (respectively, I -nonexpansive). A point $x \in M$ is a common fixed point of I and T if $x = Ix = Tx$. The set of fixed points of T is denoted by $F(T)$. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$, (2) R -weakly commuting if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rp_\alpha(Ix - Tx)$. If $R = 1$, then the maps are called weakly commuting [20]; (3) compatible [10,11,22] if for all $p_\alpha \in A^*(\tau)$, $\lim_n p_\alpha(TIx_n - ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M . Suppose that M is q -starshaped with $q \in F(I)$ and is both T - and I -invariant. Then T and I are called (4) R -subcommuting on M (see [21]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq \frac{R}{k}p_\alpha(((1-k)q + kTx) - Ix)$ for each $k \in (0, 1)$. If $R = 1$, then the maps are called 1-subcommuting [7]; (5) R -subweakly commuting on M (see [8,9]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rd_{p_\alpha}(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$. It is well known that R -weakly commuting, R -subcommuting and R -subweakly commuting maps are compatible but not conversely in general (see [10-12]).

If $u \in E, M \subseteq E$, then we define the set $P_M(u)$ of best M -approximants to u as $P_M(u) = \{y \in M : p_\alpha(y - u) = d_{p_\alpha}(u, M), \text{ for all } p_\alpha \in A^*(\tau)\}$, where $d_{p_\alpha}(u, M) = \inf\{p_\alpha(x - u) : x \in M\}$. A mapping $T : M \rightarrow E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges to 0, we have $Tx = 0$.

In [4], Fisher and Sessa obtained the following generalization of a theorem of Gregus [5].

Theorem 1.1. Let T and I be two weakly commuting mappings on a closed convex subset C of a Banach space X into itself satisfying the inequality,

$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a) \max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \quad (1.1)$
 for all $x, y \in C$, where $a \in (0, 1)$. If I is linear and nonexpansive on C and $T(C) \subseteq I(C)$, then T and I have a unique common fixed point in C .

In 1988, Mukherjee and Verma [17] replaced linearity of I by affineness in Theorem 1.1. Subsequently, Jungck [12] obtained the following generalization of Theorem 1.1 and the result of Mukherjee and Verma [17].

Theorem 1.2. Let T and I be compatible self maps of a closed convex subset C of a Banach space X . Suppose that I is continuous, linear and that $T(C) \subset I(C)$. If T and I satisfy inequality (1.1), then T and I have a unique common fixed point in C .

In this paper, we first prove that Theorems 1.1-1.2 can appreciably be extended to the setup of Hausdorff locally convex space. As applications, common fixed point and invariant approximation results for a new class of subcompatible maps are derived. Our results extend and unify the work of Al-Thagafi [1], Ćirić [3], Fisher and Sessa [4], Gregus [5], Habiniak [6], Hussain and Khan [7], Hussain et al. [8], Jungck [10], Jungck and Sessa [13], Khan and Hussain [14], Khan et al. [15], Mukherjee and Verma [17], Sahab, Khan and Sessa [18], Singh [23,24] and many others.

2. Main Results

We begin with the definition of subcompatible mappings.

Definition 2.1. Let M be a q -starshaped subset of a normed space E . For the selfmaps I and T of M with $q \in F(I)$, we define $S_q(I, T) := \cup\{S(I, T_k) : 0 \leq k \leq 1\}$ where $T_k x = (1 - k)q + kTx$ and $S(I, T_k) = \{\{x_n\} \subset M : \lim_n Ix_n = \lim_n T_k x_n = t \in M \Rightarrow \lim_n \|IT_k x_n - T_k Ix_n\| = 0\}$. Now I and T are subcompatible if $\lim_n \|ITx_n - T Ix_n\| = 0$ for all sequences $\{x_n\} \in S_q(I, T)$. We can extend this definition to locally convex space by replacing norm with a family of seminorms.

Clearly, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.2. Let $X = \mathbb{R}$ with usual norm and $M = [1, \infty)$. Let $I(x) = 2x - 1$ and $T(x) = x^2$, for all $x \in M$. Let $q = 1$. Then M is q -starshaped with $Iq = q$. Note that I and T are compatible. For any sequence $\{x_n\}$ in M with $\lim_n x_n = 2$, we have, $\lim_n Ix_n = \lim_n T_{\frac{2}{3}} x_n = 3 \in M, \Rightarrow \lim_n \|IT_{\frac{2}{3}} x_n - T_{\frac{2}{3}} Ix_n\| = 0$. However, $\lim_n \|ITx_n - T Ix_n\| \neq 0$. Thus I and T are not subcompatible

maps.

Note that R -subweakly commuting and R -subcommuting maps are sub-compatible. The following simple example reveals that the converse is not true, in general.

Example 2.3. Let $X = R$ with usual norm and $M = [0, \infty)$. Let $I(x) = \frac{x}{2}$ if $0 \leq x < 1$ and $Ix = x$ if $x \geq 1$, and $T(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$. Then M is 1-starshaped with $I1 = 1$ and $S_q(I, T) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that I and T are subcompatible but not R -weakly commuting for all $R > 0$. Thus I and T are neither R -subweakly commuting nor R -subcommuting maps.

The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, R -weak commutativity and other variants of commutativity of maps as well.

Example 2.4[22]. Let $X = R$ with usual metric and $M = [0, \infty)$. Let $I(x) = 1 + x$ and $T(x) = 2 + x^2$. Then $|ITx - TIx| = 2x$ and $|Ix - Tx| = |x^2 - x + 1|$. Thus the pair (I, T) is not weakly commuting on M with respect to usual metric. But if X is endowed with the discrete metric d , then $d(ITx, TIx) = 1 = d(Ix, Tx)$ for $x > 1$. Thus the pair (I, T) is weakly commuting on M with respect to discrete metric.

The following lemma gives us affirmative answer in this direction, which improves and extends Lemma 2.1 of [7,14].

Lemma 2.5. Let I and T be compatible selfmaps of a τ -bounded subset M of a Hausdorff locally convex space (E, τ) . Then I and T are compatible on M with respect to $\|\cdot\|_B$.

Proof. By hypothesis, $\lim_{n \rightarrow \infty} p_\alpha(ITx_n - TIx_n) = 0$ for each $p_\alpha \in A^*(\tau)$ whenever $\lim_{n \rightarrow \infty} p_\alpha(Tx_n - t) = 0 = \lim_{n \rightarrow \infty} p_\alpha(Ix_n - t)$ for some $t \in M$.

Taking supremum on both sides, we get

$$\sup_{\alpha} \lim_{n \rightarrow \infty} p_{\alpha} \left(\frac{ITx_n - TIx_n}{\lambda_{\alpha}} \right) = \sup_{\alpha} \left(\frac{0}{\lambda_{\alpha}} \right)$$

whenever

$$\sup_{\alpha} \lim_{n \rightarrow \infty} p_{\alpha} \left(\frac{Tx_n - t}{\lambda_{\alpha}} \right) = \sup_{\alpha} \left(\frac{0}{\lambda_{\alpha}} \right) = \sup_{\alpha} \lim_{n \rightarrow \infty} p_{\alpha} \left(\frac{Ix_n - t}{\lambda_{\alpha}} \right)$$

This implies that

$$\lim_{n \rightarrow \infty} \sup_{\alpha} p_{\alpha} \left(\frac{ITx_n - TIx_n}{\lambda_{\alpha}} \right) = 0$$

whenever

$$\lim_{n \rightarrow \infty} \sup_{\alpha} p_{\alpha} \left(\frac{Tx_n - t}{\lambda_{\alpha}} \right) = 0 = \lim_{n \rightarrow \infty} \sup_{\alpha} p_{\alpha} \left(\frac{Ix_n - t}{\lambda_{\alpha}} \right)$$

Hence $\lim_{n \rightarrow \infty} \|ITx_n - TIx_n\|_B = 0$

whenever $\lim_{n \rightarrow \infty} \|Tx_n - t\|_B = 0 = \lim_{n \rightarrow \infty} \|Ix_n - t\|_B$ as desired.

The following common fixed point theorem generalizes theorems 1.1-1.2 and the main results of Ćirić [3] and Mukherjee and Verma [17].

Theorem 2.6. Let M be a nonempty τ -bounded, τ -complete, and convex subset of a Hausdorff locally convex space (E, τ) and T and I be compatible selfmaps of M satisfying the inequality

$$p_{\alpha}(Tx - Ty) \leq ap_{\alpha}(Ix - Iy) + (1 - a) \max\{p_{\alpha}(Tx - Ix), p_{\alpha}(Ty - Iy)\} \quad (2.1)$$

for all $x, y \in M$, for all $p_{\alpha} \in A^*(\tau)$ and for some $a \in (0, 1)$. If I is linear and nonexpansive on M and $T(M) \subseteq I(M)$, then T and I have a unique common fixed point.

Proof. Since M is τ -complete, it follows that $(E_B, \|\cdot\|_B)$ is a Banach space and M is complete in it (see [7, 25]). By Lemma 2.5, T and I are $\|\cdot\|_B$ -compatible maps of M . From (2.1) we obtain for any $x, y \in M$,

$$\begin{aligned} & \sup_{\alpha} p_{\alpha} \left(\frac{Tx - Ty}{\lambda_{\alpha}} \right) \\ & \leq a \sup_{\alpha} p_{\alpha} \left(\frac{Ix - Iy}{\lambda_{\alpha}} \right) + (1 - a) \max \left\{ \sup_{\alpha} p_{\alpha} \left(\frac{Tx - Ix}{\lambda_{\alpha}} \right), \sup_{\alpha} p_{\alpha} \left(\frac{Ty - Iy}{\lambda_{\alpha}} \right) \right\}. \end{aligned}$$

Thus $\|Tx - Ty\|_B \leq a\|Ix - Iy\|_B + (1 - a) \max\{\|Tx - Ix\|_B, \|Ty - Iy\|_B\}$. Note that if I is nonexpansive on τ -bounded set M , then I is also nonexpansive with respect to $\|\cdot\|_B$. (cf. [7, 15]). A comparison of our hypothesis with that of Theorem 1.2 tells that we can apply Theorem 1.2 to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exists a unique $a \in M$ such that $a = Ia = Ta$.

The following result generalizes Theorem 2.2 in [1], Theorem 3 in [21] and corresponding results in [8] to a more general class of maps.

Theorem 2.7. Let I and T be selfmaps of a convex subset M of a Hausdorff locally convex space (E, τ) . Suppose that I is nonexpansive and affine on M ,

$q \in F(I)$ and $T(M) \subseteq I(M)$. If the pair $\{I, T\}$ is subcompatible and satisfies, for all $p_\alpha \in A^*(\tau)$, $x, y \in M$, and all $k \in (0, 1)$,

$$p_\alpha(Tx - Ty) \leq p_\alpha(Ix - Iy) + \frac{1-k}{k} \max\{d_{p_\alpha}(Ix, [q, Tx]), d_{p_\alpha}(Iy, [q, Ty])\}, \quad (2.2)$$

then I and T have a common fixed point in M provided one of the following conditions holds:

- (i) M is τ -compact and T is continuous.
- (ii) M is weakly compact in (E, τ) , I is weakly continuous and $I - T$ is demiclosed at 0.

Proof. Define $T_n : M \rightarrow M$ by

$$T_n x = (1 - k_n)q + k_n T x$$

for some q and all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, for each n , $T_n(M) \subseteq I(M)$ as M is convex, I is linear, $Iq = q$ and $T(M) \subseteq I(M)$. Further, since the pair $\{I, T\}$ is subcompatible and I is linear with $Iq = q$ so, for any $\{x_m\} \subset M$ with $\lim_m Ix_m = \lim_m T_n x_m = t \in M$, we have

$$\begin{aligned} \lim_m p_\alpha(T_n Ix_m - IT_n x_m) &= k_n \lim_m p_\alpha(TIx_m - ITx_m) \\ &= 0. \end{aligned}$$

Thus the pair $\{I, T_n\}$ is compatible on M for each n . Also, we obtain from (2.2),

$$\begin{aligned} p_\alpha(T_n x - T_n y) &= k_n p_\alpha(Tx - Ty) \\ &\leq k_n \left\{ p_\alpha(Ix - Iy) + \frac{1-k_n}{k_n} \max\{p_\alpha(Ix - T_n x), p_\alpha(Iy - T_n y)\} \right\} \\ &= k_n p_\alpha(Ix - Iy) + (1 - k_n) \max\{p_\alpha(Ix - T_n x), p_\alpha(Iy - T_n y)\}, \end{aligned}$$

for each $x, y \in M$, $p_\alpha \in A^*(\tau)$ and $0 < k_n < 1$.

(i) M being τ -compact is τ -bounded and τ -complete. Thus by Theorem 2.6, for each $n \geq 1$, there exists an $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Now the τ -compactness of M ensures that $\{x_n\}$ has a convergent subsequence $\{x_j\}$ which converges to a point $x_0 \in M$. Since $x_j = T_j x_j = k_j T x_j + (1 - k_j)q$ and T is continuous, so we have, as $j \rightarrow \infty$, $T x_0 = x_0$. The continuity of I implies that

$$Ix_0 = I(\lim_j x_j) = \lim_j I(x_j) = \lim_j x_j = x_0.$$

(ii) Weakly compact sets in (E, τ) are τ -bounded and τ -complete so again by Theorem 2.6, T_n and I have a common fixed point x_n in M for each n . The set M is weakly compact so there is a subsequence $\{x_j\}$ of $\{x_n\}$ converging weakly to some $y \in M$. The map I being weakly continuous gives that $Iy = y$. Now

$$x_j = I(x_j) = T_j(x_j) = k_j T x_j + (1 - k_j)q$$

implies that $Ix_j - Tx_j = (1 - k_j)[q - Tx_j] \rightarrow 0$ as $j \rightarrow \infty$. The demiclosedness of $I - T$ at 0 implies that $(I - T)(y) = 0$. Hence $Iy = Ty = y$.

An application of Theorem 2.7 establishes the following result in best approximation theory.

Theorem 2.8. Let T and I be selfmaps of Hausdorff locally convex space (E, τ) and M a subset of E such that $T(\partial M) \subseteq M$, where ∂M denotes boundary of M and $u \in F(T) \cap F(I)$. Suppose that $P_M(u)$ is nonempty convex containing q , $q \in F(I)$, I is nonexpansive and linear on $P_M(u)$ and $I(P_M(u)) = P_M(u)$. If the pair $\{I, T\}$ is subcompatible on $P_M(u)$ and satisfies, for all $x \in P_M(u) \cup \{u\}$, $p_\alpha \in A^*(\tau)$ and $k \in (0, 1)$,

$$p_\alpha(Tx - Ty) \leq \begin{cases} p_\alpha(Ix - Iu) & \text{if } y = u, \\ p_\alpha(Ix - Iy) + \frac{1-k}{k} \max\{d_{p_\alpha}(Ix, [q, Tx]), d_{p_\alpha}(Iy, [q, Ty])\}, & \text{if } y \in P_M(u), \end{cases}$$

then $P_M(u) \cap F(I) \cap F(T) \neq \emptyset$, provided one of the following conditions is satisfied:

- (i) $P_M(u)$ is τ -compact and T is continuous on $P_M(u)$.
- (ii) $P_M(u)$ is weakly compact in (E, τ) , I is weakly continuous and $I - T$ is demiclosed at 0.

Proof. Let $y \in P_M(u)$. Then as in the proof of Theorem 2.6 of [15] (see also [9,12]) $Ty \in P_M(u)$ which implies that T maps $P_M(u)$ into itself and the conclusion follows from Theorem 2.7.

Remark 2.9. (i) 1-subcommuting maps are subcompatible, consequently, Theorem 2.2-Theorem 3.3 due to Hussain and Khan [7] and Theorem 2.3 of Khan and Hussain [14] are improved and extended.

(ii) Commuting maps are subcompatible so Theorems 2.7-2.8 are proper generalization of the main results of Brosowski [2], Habiniak [6], Sahab et al. [18], Sahney et al. [19], Singh [23,24], Tarafdar [25], Theorems 6-7 due to Jungck and Sessa [13] and Theorem 2.6 due to Khan et al. [15].

References

- [1] M. A. Al-Thagafi, Common fixed points and best approximation, *J. Approx. Theory* 85(1996), 318-323.
- [2] B. Brosowski, Fixpunktsatze in der Approximationstheorie, *Mathematica (Cluj)* 11(1969), 195-220.
- [3] Lj. B. Ćirić, On a common fixed point theorem of a Gregus type, *Publ. Inst. Math.*, 49(1991), 174-178.
- [4] B. Fisher and S. Sessa, On a fixed point theorem of Gregus, *Int. J. Math. Math. Sci.*, 9(1986), 23-28.
- [5] M. Gregus, Jr., A fixed point theorem in Banach space, *Boll. Un. Mat. Ital.*, 5(1980), 193-198.
- [6] L. Habiniak, Fixed point theorems and invariant approximation, *J. Approx. Theory*, 56(1989), 241-244.
- [7] N. Hussain and A. R. Khan, Common fixed point results in best approximation theory, *Applied Math. Lett.*, 16(2003), 575-580.
- [8] N. Hussain, D. O'Regan and R. P. Agarwal, Common fixed point and invariant approximation results on non-starshaped domains, *Georgian Math. J.* 12(2005), 659-669.
- [9] N. Hussain and B. E. Rhoades, C_q -commuting maps and invariant approximations, *Fixed Point Theory and Appl.*, 2006(2006), 9 pages.
- [10] G. Jungck, On a fixed point theorem of Fisher and Sessa, *Internat. J. Math. Math. Sci.*, 13(1990), 497-500.
- [11] G. Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, 103(1988), 977-983.
- [12] G. Jungck and N. Hussain, Compatible maps and invariant approximations, *J. Math. Anal. Appl.*, 325(2007), 1003-1012.
- [13] G. Jungck and S. Sessa, Fixed point theorems in best approximation theory, *Math. Japon.*, 42(1995), 249-252.
- [14] A. R. Khan and N. Hussain, An extension of a theorem of Sahab, Khan and Sessa, *Int. J. Math. Math. Sci.*, 27(2001), 701-706.
- [15] A. R. Khan, N. Hussain, and L. A. Khan, A note on Kakutani type fixed point theorems, *Int. J. Math. Math. Sci.*, 24 (2000), 231-235.

- [16] G. Kothe, Topological Vector Spaces I, Springer-verlag, New York, 1969.
- [17] R. N. Mukherjee and V. Verma, A note on fixed point theorem of Gregus, Math. Japon., 33 (1988), 745-749.
- [18] S. A. Sahab, M. S. Khan, and S. Sessa, A result in best approximation theory, J. Approx. Theory 55 (1998), 349-351.
- [19] B. N. Sahney, K. L. Singh and J.H.M. Whitfield, Best approximation in locally convex spaces, J. Approx. Theory, 38(1983), 182-187.
- [20] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math., (beograd) 32 (1982), 149-153.
- [21] N. Shahzad, On R -subcommuting maps and best approximations in Banach spaces, Tamkang J. Math., 32(2001), 51-53.
- [22] S. L. Singh and A. Tomar, Weaker forms of commuting maps and existence of fixed points, J. Korea Soc. Math. Educ. Ser.B:Pure Appl. Math., 10(2003), 145-161.
- [23] S. P. Singh, An application of a fixed-point theorem to approximation theory, J. Approx. Theory 25 (1979), 89-90.
- [24] S. P. Singh, Some results on best approximation theory in locally convex spaces, J. Approx. Theory, 28(1980), 329-333.
- [25] E. Tarafdar, Some fixed-point theorems on locally convex linear topological spaces, Bull. Austral. Math. Soc., 13(1975), 241-254.

Received: November 21, 2006