On Common Fixed Point and Approximation Results of Gregus Type

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Abstract

Fixed point theorems of Ciric [3], Fisher and Sessa [4], Gregus [5], Jungck [10] and Mukherjee and Verma [17] are generalized to a locally convex space. As applications, common fixed point and invariant approximation results for subcompatible maps are obtained. Our results unify and generalize various known results to a more general class of noncommuting mappings.

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1. Introduction and preliminaries

In the sequel, $(E, \tau)$ will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ is said to be an associated family of seminorms for $\tau$ if the family $\{\gamma U : \gamma > 0\}$, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighborhoods of zero for $\tau$. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ is called an augmented associated family for $\tau$ if $\{p_\alpha : \alpha \in I\}$ is an associated family with property that the seminorm $max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space $(E, \tau)$, there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on $E$ such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see[16, page 203]).
The following construction will be crucial. Suppose that $M$ is $\tau$-bounded subset of $E$. For this set $M$ we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly $B = \cap_{\alpha} \lambda_\alpha U_\alpha$ is $\tau$-bounded, $\tau$-closed absolutely convex and contains $M$. The linear span $E_B$ of $B$ in $E$ is $\bigcup_{n=1}^\infty nB$. The Minkowski functional of $B$ is a norm $\|\cdot\|_B$ on $E_B$. Thus $(E_B, \|\cdot\|_B)$ is a normed space with $B$ as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$ (for details see [16,25]).

Let $M$ be a subset of a locally convex space $(E, \tau)$. Let $I : M \rightarrow M$ be a mapping. A mapping $T : M \rightarrow M$ is called $I$-Lipschitz if there exists $k \geq 0$ such that $p_\alpha(Tx - Ty) \leq k p_\alpha(Ix - Iy)$ for any $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$. If $k < 1$ (respectively, $k = 1$), then $T$ is called an $I$-contraction (respectively, $I$-nonexpansive). A point $x \in M$ is a common fixed point of $I$ and $T$ if $x = Ix = Tx$. The set of fixed points of $T$ is denoted by $F(T)$. The pair $(I,T)$ is called (1) commuting if $T Ix = ITx$ for all $x \in M$, (2) $R$-weakly commuting if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rp_\alpha(Ix - Tx)$. If $R = 1$, then the maps are called weakly commuting [20]; (3) compatible [10,11,22] if for all $p_\alpha \in A^*(\tau), \lim_n p_\alpha(T Ix_n - IT x_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some $t$ in $M$. Suppose that $M$ is $q$-starshaped with $q \in F(I)$ and is both $T$- and $I$-invariant. Then $T$ and $I$ are called (4) $R$-subcommuting on $M$ (see [21]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq \frac{R}{2} p_\alpha(((1-k)q + kTx) - Ix)$ for each $k \in (0,1)$. If $R = 1$, then the maps are called 1-subcommuting [7]; (5) $R$-subweakly commuting on $M$ (see [8,9]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq R d_{p_\alpha}(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$. It is well known that $R$-weakly commuting, $R$-subcommuting and $R$-subweakly commuting maps are compatible but not conversely in general (see [10-12]).

If $u \in E, M \subseteq E$, then we define the set $P_M(u)$ of best $M$-approximants to $u$ as $P_M(u) = \{y \in M : p_\alpha(y - u) = d_{p_\alpha}(u, M)\}$, for all $p_\alpha \in A^*(\tau)$, where $d_{p_\alpha}(u, M) = \inf\{p_\alpha(x - u) : x \in M\}$. A mapping $T : M \rightarrow E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to $x$ and $\{Tx_n\}$ converges to $0$, we have $Tx = 0$.

In [4], Fisher and Sessa obtained the following generalization of a theorem of Gregus [5].

**Theorem 1.1.** Let $T$ and $I$ be two weakly commuting mappings on a closed convex subset $C$ of a Banach space $X$ into itself satisfying the inequality,
\[ \|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a) \max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \tag{1.1} \]

for all \(x, y \in C\), where \(a \in (0, 1)\). If \(I\) is linear and nonexpansive on \(C\) and \(T(C) \subseteq I(C)\), then \(T\) and \(I\) have a unique common fixed point in \(C\).

In 1988, Mukherjee and Verma [17] replaced linearity of \(I\) by affineness in Theorem 1.1. Subsequently, Jungck [12] obtained the following generalization of Theorem 1.1 and the result of Mukherjee and Verma [17].

**Theorem 1.2.** Let \(T\) and \(I\) be compatible self maps of a closed convex subset \(C\) of a Banach space \(X\). Suppose that \(I\) is continuous, linear and that \(T(C) \subseteq I(C)\). If \(T\) and \(I\) satisfy inequality (1.1), then \(T\) and \(I\) have a unique common fixed point in \(C\).

In this paper, we first prove that Theorems 1.1-1.2 can appreciably be extended to the setup of Hausdorff locally convex space. As applications, common fixed point and invariant approximation results for a new class of subcompatible maps are derived. Our results extend and unify the work of Al-Thagafi [1], Ciric [3], Fisher and Sessa [4], Gregus [5], Habiniak [6], Hussain and Khan [7], Hussain et al. [8], Jungck [10], Jungck and Sessa [13], Khan and Hussain [14], Khan at el. [15], Mukherjee and Verma [17], Sahab, Khan and Sessa [18], Singh [23,24] and many others.

2. Main Results

We begin with the definition of subcompatible mappings.

**Definition 2.1.** Let \(M\) be a \(q\)-starshaped subset of a normed space \(E\). For the selfmaps \(I\) and \(T\) of \(M\) with \(q \in F(I)\), we define \(S_q(I, T) := \bigcup\{S(I, T_k) : 0 \leq k \leq 1\}\) where \(T_kx = (1 - k)q + kTx\) and \(S(I, T_k) = \{\{x_n\} \subset M : \lim_n Ix_n = \lim_n T_kx_n = t \in M \Rightarrow \lim_n \|IT_kx_n - T_kIx_n\| = 0\}\). Now \(I\) and \(T\) are subcompatible if \(\lim_n \|ITx_n - TIx_n\| = 0\) for all sequences \(\{x_n\} \in S_q(I, T)\). We can extend this definition to locally convex space by replacing norm with a family of seminorms.

Clearly, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

**Example 2.2.** Let \(X = R\) with usual norm and \(M = [1, \infty)\). Let \(I(x) = 2x - 1\) and \(T(x) = x^2\), for all \(x \in M\). Let \(q = 1\). Then \(M\) is \(q\)-starshaped with \(Iq = q\). Note that \(I\) and \(T\) are compatible. For any sequence \(\{x_n\}\) in \(M\) with \(\lim_n x_n = 2\), we have, \(\lim_n Ix_n = \lim_n T_2x_n = 3 \in M, \Rightarrow \lim_n \|IT_2x_n - T_2Ix_n\| = 0\). However, \(\lim_n \|ITx_n - TIx_n\| \neq 0\). Thus \(I\) and \(T\) are not subcompatible.
maps.

Note that $R$-subweakly commuting and $R$-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.3. Let $X = R$ with usual norm and $M = [0, \infty)$. Let $I(x) = \frac{x}{2}$ if $0 \leq x < 1$ and $Ix = x$ if $x \geq 1$, and $T(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x \geq 1$. Then $M$ is 1-starshaped with $I1 = 1$ and $S_q(I, T) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that $I$ and $T$ are subcompatible but not $R$-weakly commuting for all $R > 0$. Thus $I$ and $T$ are neither $R$-subweakly commuting nor $R$-subcommuting maps.

The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, $R$-weak commutativity and other variants of commutativity of maps as well.

Example 2.4[22]. Let $X = R$ with usual metric and $M = [0, \infty)$. Let $I(x) = 1 + x$ and $T(x) = 2 + x^2$. Then $|ITx - TIx| = 2x$ and $|Ix - Tx| = |x^2 - x + 1|$. Thus the pair $(I, T)$ is not weakly commuting on $M$ with respect to usual metric. But if $X$ is endowed with the discrete metric $d$, then $d(ITx, TIx) = 1 = d(Ix, Tx)$ for $x > 1$. Thus the pair $(I, T)$ is weakly commuting on $M$ with respect to discrete metric.

The following lemma gives us affirmative answer in this direction, which improves and extends Lemma 2.1 of [7,14].

Lemma 2.5. Let $I$ and $T$ be compatible selfmaps of a $\tau$-bounded subset $M$ of a Hausdorff locally convex space $(E, \tau)$. Then $I$ and $T$ are compatible on $M$ with respect to $\|\cdot\|_B$.

Proof. By hypothesis, $\lim_{n \to \infty} p_\alpha(ITx_n - TIx_n) = 0$ for each $p_\alpha \in A^*(\tau)$ whenever $\lim_{n \to \infty} p_\alpha(Tx_n - t) = 0 = \lim_{n \to \infty} p_\alpha(Ix_n - t)$ for some $t \in M$.

Taking supremum on both sides, we get

$$\sup_\alpha \lim_{n \to \infty} p_\alpha\left(\frac{ITx_n - TIx_n}{\lambda_\alpha}\right) = \sup_\alpha\left(\frac{0}{\lambda_\alpha}\right)$$

whenever

$$\sup_\alpha \lim_{n \to \infty} p_\alpha\left(\frac{Tx_n - t}{\lambda_\alpha}\right) = \sup_\alpha\left(\frac{0}{\lambda_\alpha}\right) = \sup_\alpha \lim_{n \to \infty} p_\alpha\left(\frac{Ix_n - t}{\lambda_\alpha}\right)$$
This implies that
\[
\lim_{n \to \infty} \sup_{\alpha} p_{\alpha}(\frac{ITx_n - TIx_n}{\lambda_n}) = 0
\]
whenever
\[
\lim_{n \to \infty} \sup_{\alpha} p_{\alpha}(\frac{Tx_n - t}{\lambda_n}) = 0 = \lim_{n \to \infty} \sup_{\alpha} p_{\alpha}(\frac{Ix_n - t}{\lambda_n})
\]
Hence \( \lim_{n \to \infty} \|ITx_n - TIx_n\|_B = 0 \)
whenever \( \lim_{n \to \infty} \|Tx_n - t\|_B = 0 = \lim_{n \to \infty} \|Ix_n - t\|_B \) as desired.

The following common fixed point theorem generalizes theorems 1.1-1.2 and the main results of Ciric [3] and Mukherjee and Verma [17].

**Theorem 2.6.** Let \( M \) be a nonempty \( \tau \)-bounded, \( \tau \)-complete, and convex subset of a Hausdorff locally convex space \((E, \tau)\) and \( T \) and \( I \) be compatible selfmaps of \( M \) satisfying the inequality
\[
p_{\alpha}(Tx - Ty) \leq a p_{\alpha}(Ix - Iy) + (1 - a) \max\{p_{\alpha}(Tx - Ix), p_{\alpha}(Ty - Iy)\}
\]
for all \( x, y \in M \), for all \( p_{\alpha} \in A^*(\tau) \) and for some \( a \in (0, 1) \). If \( I \) is linear and nonexpansive on \( M \) and \( T(M) \subseteq I(M) \), then \( T \) and \( I \) have a unique common fixed point.

**Proof.** Since \( M \) is \( \tau \)-complete, it follows that \((E_B, \|\cdot\|_B)\) is a Banach space and \( M \) is complete in it (see [7,25]). By Lemma 2.5, \( T \) and \( I \) are \( \|\cdot\|_B \)-compatible maps of \( M \). From (2.1) we obtain for any \( x, y \in M \),
\[
\sup_{\alpha} p_{\alpha}(\frac{Tx - Ty}{\lambda_n})
\]
\[
\leq a \sup_{\alpha} p_{\alpha}(\frac{Ix - Iy}{\lambda_n}) + (1 - a) \max\{\sup_{\alpha} p_{\alpha}(\frac{Tx - Ix}{\lambda_n}), \sup_{\alpha} p_{\alpha}(\frac{Ty - Iy}{\lambda_n})\}.
\]
Thus \( \|Tx - Ty\|_B \leq a \|Ix - Iy\|_B + (1 - a) \max\{\|Tx - Ix\|_B, \|Ty - Iy\|_B\} \). Note that if \( I \) is nonexpansive on \( \tau \)-bounded set \( M \), then \( I \) is also nonexpansive with respect to \( \|\cdot\|_B \). (cf. [7,15]). A comparison of our hypothesis with that of Theorem 1.2 tells that we can apply Theorem 1.2 to \( M \) as a subset of \((E_B, \|\cdot\|_B)\) to conclude that there exists a unique \( a \in M \) such that \( a = Ia = Ta \).

The following result generalizes Theorem 2.2 in [1], Theorem 3 in [21] and corresponding results in [8] to a more general class of maps.

**Theorem 2.7.** Let \( I \) and \( T \) be selfmaps of a convex subset \( M \) of a Hausdorff locally convex space \((E, \tau)\). Suppose that \( I \) is nonexpansive and affine on \( M \),
\( q \in F(I) \) and \( T(M) \subseteq I(M) \). If the pair \( \{I, T\} \) is subcompatible and satisfies, for all \( p_\alpha \in A^*(\tau) \), \( x, y \in M \), and all \( k \in (0, 1) \),

\[
p_\alpha(Tx-Ty) \leq p_\alpha(Ix-Iy)+\frac{1-k}{k_n} \max\{d_{p_\alpha}(Ix, [q, Tx]), d_{p_\alpha}(Iy, [q, Ty])\}, \tag{2.2}
\]

then \( I \) and \( T \) have a common fixed point in \( M \) provided one of the following conditions holds:

(i) \( M \) is \( \tau \)-compact and \( T \) is continuous.
(ii) \( M \) is weakly compact in \( (E, \tau) \), \( I \) is weakly continuous and \( I-T \) is demiclosed at 0.

**Proof.** Define \( T_n : M \rightarrow M \) by

\[
T_n x = (1-k_n)q + k_n Tx
\]

for some \( q \) and all \( x \in M \) and a fixed sequence of real numbers \( k_n (0 < k_n < 1) \) converging to 1. Then, for each \( n \), \( T_n(M) \subseteq I(M) \) as \( M \) is convex, \( I \) is linear, \( Iq = q \) and \( T(M) \subseteq I(M) \). Further, since the pair \( \{I, T\} \) is subcompatible and \( I \) is linear with \( Iq = q \) so, for any \( \{x_m\} \subset M \) with \( \lim_m Ix_m = \lim_m T_n x_m = t \in M \), we have

\[
\lim_m p_\alpha(T_n Ix_m - IT_n x_m) = k_n \lim_m p_\alpha(TIx_m - ITx_m)
\]

\[= 0.\]

Thus the pair \( \{I, T_n\} \) is compatible on \( M \) for each \( n \). Also, we obtain from (2.2),

\[
p_\alpha(T_n x - T_n y) = k_n p_\alpha(Tx - Ty)
\leq k_n\{p_\alpha(Ix - Iy) + \frac{1-k_n}{k_n} \max\{p_\alpha(Ix - Tx), p_\alpha(Iy - Ty)\}\}
= k_n p_\alpha(Ix - Iy) + (1-k_n) \max\{p_\alpha(Ix - Tx), p_\alpha(Iy - Ty)\},
\]

for each \( x, y \in M \), \( p_\alpha \in A^*(\tau) \) and \( 0 < k_n < 1 \).

(i) \( M \) being \( \tau \)-compact is \( \tau \)-bounded and \( \tau \)-complete. Thus by Theorem 2.6, for each \( n \geq 1 \), there exists an \( x_n \in M \) such that \( x_n = Ix_n = T_n x_n \). Now the \( \tau \)-compactness of \( M \) ensures that \( \{x_n\} \) has a convergent subsequence \( \{x_j\} \) which converges to a point \( x_0 \in M \). Since \( x_j = T_j x_j = k_j T x_j + (1-k_j) \) and \( T \) is continuous, so we have, as \( j \rightarrow \infty \), \( Tx_0 = x_0 \). The continuity of \( I \) implies that

\[
Ix_0 = I(\lim_j x_j) = \lim_j I(x_j) = \lim_j x_j = x_0.
\]
(ii) Weakly compact sets in \((E, \tau)\) are \(\tau\)-bounded and \(\tau\)-complete so again by Theorem 2.6, \(T_n\) and \(I\) have a common fixed point \(x_n\) in \(M\) for each \(n\). The set \(M\) is weakly compact so there is a subsequence \(\{x_j\}\) of \(\{x_n\}\) converging weakly to some \(y \in M\). The map \(I\) being weakly continuous gives that \(Iy = y\).

Now \(x_j = I(x_j) = T_j(x_j) = k_jTx_j + (1 - k_j)q\) implies that \(Ix_j - Tx_j = (1 - k_j)[q - Tx_j] \to 0\) as \(j \to \infty\). The demiclosedness of \(I - T\) at 0 implies that \((I - T)(y) = 0\). Hence \(Iy = Ty = y\).

An application of Theorem 2.7 establishes the following result in best approximation theory.

**Theorem 2.8.** Let \(T\) and \(I\) be selfmaps of Hausdorff locally convex space \((E, \tau)\) and \(M\) a subset of \(E\) such that \(T(\partial M) \subseteq M\), where \(\partial M\) denotes boundary of \(M\) and \(u \in F(T) \cap F(I)\). Suppose that \(P_M(u)\) is nonempty convex containing \(q\), \(q \in F(I)\), \(I\) is nonexpansive and linear on \(P_M(u)\) and \(I(P_M(u)) = P_M(u)\). If the pair \(\{I, T\}\) is subcompatible on \(P_M(u)\) and satisfies, for all \(x \in P_M(u) \cup \{u\}\), \(p_\alpha \in A^*(\tau)\) and \(k \in (0, 1)\),

\[
p_\alpha(Tx - Ty) \leq \begin{cases} 
p_\alpha(Ix - Iu) + \frac{1-k}{k} \max\{d_{p_\alpha}(Ix, [q, Tx]), d_{p_\alpha}(Iy, [q, Ty])\}, & \text{if } y = u, \\
p_\alpha(Ix - Iy) + \frac{1-k}{k} \max\{d_{p_\alpha}(Ix, [q, Tx]), d_{p_\alpha}(Iy, [q, Ty])\}, & \text{if } y \in P_M(u),
\end{cases}
\]

then \(P_M(u) \cap F(I) \cap F(T) \neq \emptyset\), provided one of the following conditions is satisfied:

(i) \(P_M(u)\) is \(\tau\)-compact and \(T\) is continuous on \(P_M(u)\).

(ii) \(P_M(u)\) is weakly compact in \((E, \tau)\), \(I\) is weakly continuous and \(I - T\) is demiclosed at 0.

**Proof.** Let \(y \in P_M(u)\). Then as in the proof of Theorem 2.6 of [15](see also [9,12]) \(Ty \in P_M(u)\) which implies that \(T\) maps \(P_M(u)\) into itself and the conclusion follows from Theorem 2.7.

**Remark 2.9.** (i) 1-subcommuting maps are subcompatible, consequently, Theorem 2.2-Theorem 3.3 due to Hussain and Khan [7] and Theorem 2.3 of Khan and Hussain [14] are improved and extended.

(ii) Commuting maps are subcompatible so Theorems 2.7-2.8 are proper generalization of the main results of Brosowski [2], Habiniak [6], Sahab et al. [18], Sahney et al. [19], Singh [23,24], Tarafdar [25], Theorems 6-7 due to Jungck and Sessa [13] and Theorem 2.6 due to Khan et al.[15].
References


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