A Generalized Like-distance in Convex Programming

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Abstract

We consider a generalized like-distance which contains as degenerate cases ϕ -divergences and like-distances with second order homogeneous kernel. The motivation to get this like-distance comes from studying shifted penalty functions in the primal space. These penalty functions do not necessarily have to pass through the origin with slope one and their conjugate functions allow negative values. For a particular case we get a generalization of the Kullback-Liebler entropy distance. This like-distance can be seen as the difference between a sequence of Bregman distances and their linear approximations for specific values of the arguments. Dual and primal convergence results are shown, particularly, we show that each limit point of the sequence generated by the proximal method defined by the generalized like-distance applied to the dual problem is an optimal dual solution.

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1 Introduction

We consider the convex programming problem defined by

$$(P) \quad f^* = \inf\{f_0(x) : f_i(x) \leq 0 \quad i = 1, \dots, m\}$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 0, 1, \dots, m$ are closed proper convex functions. The associated dual convex problem can be written by

$$(D) \quad d^* = \inf\{-d(\mu) : \mu \geq 0\} \quad (1)$$

where $d(\mu) = \inf\{l(x, \mu) : x \in \mathbb{R}^n\}$ and

$$l(x, \mu) = f_0(x) + \sum_{i=1}^m \mu_i f_i(x) \quad (2)$$

is the Lagrangian function.

We assume the following hypothesis:

(H1) The optimal set for problem (P) is nonempty and compact.

(H2) There exist $\bar{x}$ such that $f_i(\bar{x}) < 0$ for $i = 1, \dots, m$.

The problem (P) can be solved using different approaches, for example, we can use augmented Lagrangian methods which can be introduced from a primal view point, see [3], [14], [2] or from a dual view point where the multiplier method is constructed applying Fenchel's theory of duality to the proximal point method, see [17], [9], [6], [1].

Proximal point methods with ϕ divergences to solve (D), generate a sequence $\{\mu^k\} \in \mathbb{R}_{++}^m$ such that $\mu^0 \in \mathbb{R}_{++}^m$ and

$$\mu^{k+1} = \operatorname{argmin}_{\mu \geq 0} \{-d(\mu) + r^k d_\phi(\mu, \mu_k)\} \quad (3)$$

where

$$0 < \underline{r} < r^k < \bar{r} \leq 1,$$

$$d_\phi : \mathbb{R}_+^m \times \mathbb{R}_{++}^m \rightarrow \mathbb{R} \text{ is given by } d_\phi(s, \mu) = \sum_{i=1}^m \mu_i \phi\left(\frac{s_i}{\mu_i}\right) \quad (4)$$

and ϕ is a nonnegative and strictly convex function that passes by the point $(1, 0)$ and it is well studied in [9]. Other like-distances have been used in

different contexts with different properties and results, for example in [2], the parameter r was proposed as a function of the multiplier μ and when $r(\mu) = \mu$ we get the like-distance studied in [1] which is homogeneous of order two and given by the expression

$$d_\phi(s, \mu) = \sum_{i=1}^m \mu_i^2 \phi\left(\frac{s_i}{\mu_i}\right). \quad (5)$$

Proximal point methods with Bregman distances solve (D) by using iterates in (3) with D_h instead of d_ϕ and given by

$$D_h(s, \mu) = h(s) - h(\mu) - \nabla h_\mu(\mu)^t(s - \mu) \quad (6)$$

where h is a Bregman function, see [12]. In section 2 we consider a new like-distance to solve problem (D) which has interesting properties and is given by the expression

$$d_{\theta^*}^p(s, \mu) = \sum_{i=1}^m \left[\frac{\mu_i^p}{c} \theta^*\left(\frac{cs_i}{\mu_i}\right) - \frac{\mu_i^p}{c} \theta^*(c) - \mu_i^{p-1} (\theta^*)'(c)(s_i - \mu_i) \right] \quad (7)$$

where θ^* is the conjugate function of the penalty function θ which does not necessarily pass through the origin and satisfies $\theta'(0) = \kappa$, $\kappa > 0$, $p \geq 0$ and $c \in \mathbb{R}_{++}$ satisfies $(\theta^*)'(c) = \tilde{y} \in \mathbb{R}$.

For $\mu \in \mathbb{R}_{++}^m$ fixed, if we define $h_\mu : \mathbb{R}_+^m \rightarrow \mathbb{R}$ by $h_\mu(s) = \sum_{i=1}^m \frac{\mu_i^p}{c} \theta^*\left(\frac{cs_i}{\mu_i}\right)$ we get $d_{\theta^*}^p(s, \mu) = h_\mu(s) - h_\mu(\mu) - \nabla h_\mu(\mu)^t(s - \mu)$, i.e., $d_{\theta^*}^p$ can be seen as the difference between $h_\mu(s)$ and its linear approximation at $s = \mu$. We observe that θ^* is a translation of the function ϕ used in ϕ -divergences, in this case, with minimal point at $(\kappa, \theta^*(\kappa))$ and $\theta^*(\kappa) \in \mathbb{R}$. In section 3 we show dual and primal results based on a proximal method and an augmented Lagrangian algorithm. Sections 4 and 5 and the appendix present the concluding remarks. We get the like-distance by applying the conjugacy theory to a shifted penalty function used in an augmented Lagrangian approach, so we consider first a family of penalty functions and we show how the like-distance defined in (7) naturally arises.

2 Getting the like-distance

In order to get the generalized like-distance in a natural way we study first a family of penalty functions in the context of multiplier methods.

2.1 Penalty functions

Let θ be a strictly increasing twice differentiable strictly convex function defined on $(-\infty, b)$, $0 < b \leq +\infty$, such that:

- 1a) $0 < \theta'(0) = \kappa < +\infty$.
- 2a) $\lim_{t \rightarrow b} \theta'(t) = +\infty$.
- 3a) $\lim_{t \rightarrow -\infty} \theta'(t) = 0$.
- 4a) $\theta''(t) \geq \frac{1}{M}$, $\forall t \in [0, b]$ and for some $M > 0$.

Condition 1a) has already been considered in Bregman distances approach, see [12],[5] but not in like-distances ones, see [9],[1], [14], [2], in our case the penalty function does not need to pass through the origin with slope one. Associated with the function θ we consider and denote its conjugate function θ^* , see [15] which satisfies the following properties:

- 1b) θ^* is a strictly convex differentiable function on $(0, +\infty)$.
- 2b) θ^* is decreasing on $(0, \kappa)$ and increasing on $(\kappa, +\infty)$ with $\theta^*(\kappa) \in \mathbb{R}$.
- 3b) $\lim_{t \rightarrow 0^+} (\theta^*)'(t) = -\infty$ and $\lim_{t \rightarrow +\infty} (\theta^*)'(t) = +\infty$.
- 4b) $(\theta^*)''(t) \leq M$ for all $t \geq \kappa = \theta'(0)$.

Condition 2b) with $\kappa = 1$ is used in all the like-distances known until now. Observe that negative values for θ^* are allowed.

Example 2.1.

For $\theta(t) = e^{t+1}$, we have $\theta(0) = e$, $\theta'(0) = e = \kappa$, $(\theta^*)'(e) = 0$ and $\theta^*(e) = -e$. So, θ^* has its minimal point at $(e, -e)$. We show the graphics of θ and θ^* in the next figure.

Fig.1 Graphics of θ and θ^*

2.2 Shift in penalty functions

We use a constant shift to define a generalized penalty function which will be used in section (3.2) in the context of the multiplier's method.

We choose $\tilde{y} \in \mathbb{R}$ so that $\theta'(\tilde{y}) = c$ with $c \in \mathbb{R}_{++}$. According to the corollary 23.5.1 in [15] we have

$$(\theta')^{-1} = (\theta^*)' \quad (8)$$

and so

$$\theta'(\tilde{y}) = c \Leftrightarrow \tilde{y} = (\theta^*)'(c). \quad (9)$$

Given $p \geq 0$, $r \in (0, 1]$, we define the generalized penalty function with shift P_p as:

$$y \in \mathbb{R}^m, \mu \in \mathbb{R}_{++}^m \mapsto P_p(y, \mu, r, c) = \sum_{i=1}^m P_{p,i}(y_i, \mu_i, r, c) \quad (10)$$

where $P_{p,i}(y_i, \mu_i, r, c) = r \frac{\mu_i^p}{c} \left[\theta \left(\frac{y_i}{\mu_i^{p-1} r} + \tilde{y} \right) - \theta(\tilde{y}) \right]$ for $i = 1, \dots, m$, θ satisfies conditions 1a)-4a) and $\theta'(\tilde{y}) = c$.

Note that $P_{p,i}(0, \mu_i, r, c) = 0$ for $i = 1, \dots, m$ and

$$(P_{p,i})'_1(y_i, \mu_i, r, c) = \frac{\mu_i}{c} \theta' \left(\frac{y_i}{\mu_i^{p-1} r} + \tilde{y} \right),$$

where $(P_{p,i})'_1 = \frac{\partial P_{p,i}}{\partial y_i}$, so by (9)

$$(P_{p,i})'_1(0, \mu_i, r, c) = \frac{\mu_i}{c} \theta'(\tilde{y}) = \mu_i, \text{ for } i = 1, \dots, m. \quad (11)$$

Remark

An interesting fact in the penalty function P_p is that it considers for the first time rational exponents on the multiplier vectors, although we show only convergence results for $p \geq 2$.

Geometrically the shift is a translation that satisfies the equation (11). The conditions $\theta(0) = 0$ and $\theta'(0) = 1$ were used in all the multiplier methods known until now that lead to like-distances, see [9], [1], [2], [14], but not in this one, what really matter is that equation (11) holds.

Another relevant aspect of the shift in this penalty function is that it allows us to construct a generalized like-distance in the dual space applying the conjugacy theory as shown in the next proposition.

Proposition 2.2. *Let θ be a penalty function satisfying conditions 1a)-4a). Given $p \geq 0$, $r \in (0, 1]$, $\tilde{y} \in \mathbb{R}$, $c \in \mathbb{R}_{++}$, consider*

$$y \in \mathbb{R}^m, \mu \in \mathbb{R}_{++}^m \mapsto P_p(y, \mu, r, c) = \sum_{i=1}^m r \frac{\mu_i^p}{c} \left[\theta \left(\frac{y_i}{\mu_i^{p-1} r} + \tilde{y} \right) - \theta(\tilde{y}) \right] \quad (12)$$

where $\theta'(\tilde{y}) = c$ then

$$P_{p,\mu,r,c}^*(s) = r \sum_{i=1}^m \left[\frac{\mu_i^p}{c} \theta^* \left(\frac{cs_i}{\mu_i} \right) - \frac{\mu_i^p}{c} \theta^*(c) - \mu_i^{p-1} (\theta^*)'(c) (s_i - \mu_i) \right]. \quad (13)$$

Proof.
Consider

$$P_{p,i}(y_i, \mu_i, r, c) = \frac{r}{c} \mu_i^p \left[\theta \left(\frac{y_i}{\mu_i^{p-1} r} + \tilde{y} \right) - \theta(\tilde{y}) \right] \quad \text{for } i = 1, \dots, m$$

where θ satisfies conditions 1a)-4a) and $\theta'(\tilde{y}) = c$.

Let r, c, μ_i , for $i = 1, \dots, m$ be fixed and proposition 1.3.1 in [11], we have

$$\begin{aligned} P_{p,\mu_i,r,c}^*(s_i) &= \left[\frac{r}{c} \mu_i^p \theta \left(\frac{y_i}{\mu_i^{p-1} r} + \tilde{y} \right) - \frac{r}{c} \mu_i^p \theta(\tilde{y}) \right]^* \quad \text{for } i = 1, \dots, m \\ &= \left[\frac{r}{c} \mu_i^p \theta \left(\frac{y_i}{\mu_i^{p-1} r} \tilde{y} + \tilde{y} \right) \right]^* + \frac{r}{c} \mu_i^p \theta(\tilde{y}) \\ &= \frac{r}{c} \mu_i^p \theta^* \left(\frac{cs_i}{\mu_i} \right) - r \mu_i^{p-1} s_i \tilde{y} + \frac{r}{c} \mu_i^p \theta(\tilde{y}). \end{aligned}$$

By theorem 25.3 in [15], if $\tilde{y} \in \text{dom } \theta$ and $c \in \text{dom } \theta^*$ we know that $\theta(\tilde{y}) + \theta^*(c) = c\tilde{y}$ for $i = 1, \dots, m$, by using (9) we have

$$\begin{aligned} P_{p,\mu_i,r,c}^*(s_i) &= \frac{r}{c} \mu_i^p \theta^* \left(\frac{cs_i}{\mu_i} \right) - \mu_i^{p-1} r s_i \tilde{y} + \frac{r}{c} \mu_i^p [c\tilde{y} - \theta^*(c)] \\ &= \frac{r}{c} \mu_i^p \theta^* \left(\frac{cs_i}{\mu_i} \right) - \frac{r}{c} \mu_i^p \theta^*(c) - \mu_i^{p-1} r \tilde{y} (s_i - \mu_i) \\ &= \frac{r}{c} \mu_i^p \theta^* \left(\frac{cs_i}{\mu_i} \right) - \frac{r}{c} \mu_i^p \theta^*(c) - \mu_i^{p-1} r (\theta^*)'(c) (s_i - \mu_i). \end{aligned}$$

Finally,

$$P_{p,\mu,r,c}^*(s) = r \sum_{i=1}^m \left[\frac{\mu_i^p}{c} \theta^* \left(\frac{cs_i}{\mu_i} \right) - \frac{\mu_i^p}{c} \theta^*(c) - \mu_i^{p-1} (\theta^*)'(c) (s_i - \mu_i) \right].$$

□

2.3 The generalized like-distance

According with definition 2.1 in [10], given $S \subset \mathbb{R}^n$; $d : S \times S \rightarrow \mathbb{R}$ is called a divergence measure in S if and only if:

- i) $d(x, y) \geq 0$, $\forall x, y \in S$.
- ii) If $\{x^k\} \subset S, x \in S$, then $\lim_{k \rightarrow +\infty} d(x, x^k) = 0 \Leftrightarrow \lim_{k \rightarrow +\infty} x^k = x$.
- iii) The partial level sets $\Gamma_1(y, \nu) = \{x \in S : d(x, y) \leq \nu\}$ are bounded $\forall y \in S$ and $\forall \nu > 0$.
- iv) The partial level sets $\Gamma_2(x, \nu) = \{y \in S : d(x, y) \leq \nu\}$ are bounded $\forall x \in S$ and $\forall \nu > 0$.

Using (13), we define the generalized like-distance as

$$d_{\theta^*}^p(x, y) = \sum_{i=1}^m \left[\frac{y_i^p}{c} \theta^* \left(\frac{cx_i}{y_i} \right) - \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) (x_i - y_i) \right]. \quad (14)$$

In the appendix we show that it is a divergent measure. Observe that $d_{\theta^*}^p(\cdot, y)$ is a strictly convex function because θ^* is one.

Note that for $p = 1$, $c = 1$, $\tilde{y} = 0$ with $\theta(\tilde{y}) = 0$ we get:

$$P_1^*(s, \mu, r, c) = r d_{\theta^*}(s, \mu) = r \sum_{i=1}^m \mu_i \theta^* \left(\frac{s_i}{\mu_i} \right)$$

which is used in proximal methods with ϕ -divergences, see [10], [14].

For $p = 2$, $c = 1$, $\tilde{y} = 0$ with $\theta(\tilde{y}) = 0$ we get

$$P_2^*(s, \mu, r, c) = r \tilde{d}_{\theta^*}(s, \mu) = r \sum_{i=1}^m \mu_i^2 \theta^* \left(\frac{s_i}{\mu_i} \right)$$

the second order homogeneous kernel used in [1] and in [2] for a specific case.

Example 2.3.

For $m = 1$ and $\theta(t) = e^t$, we have $\theta^*(s) = s \ln(s) - s$, $(\theta^*)'(s) = \ln(s)$ and $d_{\theta^*}^p(s, \mu) = \mu^{p-1} \left(s \ln\left(\frac{s}{\mu}\right) - s + \mu \right)$. This like-distance can be considered as a generalization of the Kullback-Leibler entropy distance. The same expression can be obtained from $\theta(t) = e^t - 1$ or from $\theta(t) = e^{t+K}$ with $K \in \mathbb{R}$.

2.3.1 Geometric Interpretation

Consider the generalized like-distance defined in (14) and given $\mu \in \mathbb{R}_{++}^m$, we define the convex function

$$h_\mu : \mathbb{R}_+^m \rightarrow \mathbb{R} \quad \text{as} \quad h_\mu(s) = \sum_{i=1}^m \frac{\mu_i^p}{c} \theta^* \left(\frac{cs_i}{\mu_i} \right).$$

Note that $h_\mu(\mu) = \sum_{i=1}^m \frac{\mu_i^p}{c} \theta^*(c)$ and

$$\nabla h_\mu(s) = \left(\mu_1^{p-1} (\theta^*)' \left(\frac{cs_1}{\mu_1} \right), \dots, \mu_m^{p-1} (\theta^*)' \left(\frac{cs_m}{\mu_m} \right) \right)^t$$

so that $\nabla h_\mu(\mu) = (\mu_1^{p-1} \theta^{*'}(c), \dots, \mu_m^{p-1} \theta^{*'}(c))^t$.

We have then

$$d_{\theta^*}^p(s, \mu) = h_\mu(s) - h_\mu(\mu) - \nabla h_\mu(\mu)^t (s - \mu),$$

that is, for each μ , $d_{\theta^*}^p(s, \mu)$ is the difference between $h_\mu(s)$ and its linear approximation at $s = \mu$.

It can be observed that for each $\mu > 0$, $h_\mu(s) = \sum_{i=1}^m \frac{\mu_i^p}{c} \theta^* \left(\frac{cs_i}{\mu_i} \right)$ is a strictly

convex function and it is not difficult to prove that, for each $\mu > 0$ it generates a Bregman distance $D_{h_\mu} : \mathbb{R}_+^m \times \mathbb{R}_{++}^m \rightarrow \mathbb{R}$ given by

$$D_{h_\mu}(x, y) = h_\mu(x) - h_\mu(y) - \nabla h_\mu(y)^t(x - y), \quad (15)$$

see [12], so we have for each $\mu > 0$, the generalized like-distance satisfies

$$d_{\theta^*}^p(s, \mu) = h_\mu(s) - h_\mu(\mu) - \nabla h_\mu(\mu)^t(s - \mu) = D_{h_\mu}(s, \mu). \quad (16)$$

If we consider a sequence $\{\mu^k\}$ in (15) and in (16) we have in (15) a sequence of Bregman distances, so that the generalized like-distance at $\mu = \mu^k$ coincide with $D_{h_{\mu^k}}(x, y)$ at $y = \mu^k$. This suggest the possibility of studying proximal methods with a sequence of Bregman distances depending on a parameter or varying the induced Bregman function at each iteration. This will not be considered in this paper.

Figure 2 below shows the graph of $h_\mu(s)$ and its linear approximation at μ for $\mu=2, \frac{3}{2}, 1, c = \frac{1}{2}, p = \frac{5}{2}$ and θ^* as in example 1. On the right, the graph of $d_{\theta^*}^p(s, \mu)$ for the same values.

Fig.2

3 Methods and theorems

3.1 Proximal method

We consider the generalized like-distance applied directly over the dual problem. The proximal method to solve problem (D) defined in (1) generates a sequence $\{\mu^k\}$ such that $\mu^0 \in \mathbb{R}_{++}^m$ and

$$\mu^{k+1} = \operatorname{argmin}\{-d(\mu) + r^k d_{\theta^*}^p(\mu, \mu^k)\} \quad (17)$$

where $r^k \in [\widehat{r}, \overline{r}] \subset (0, +\infty)$ and θ^* as defined in section (2.1).

By optimality condition we have

$$0 \in \partial [-d(\mu^{k+1}) + r^k d_{\theta^*}^p(\mu^{k+1}, \mu^k)]$$

or equivalently

$$-r^k \left((\mu_1^k)^{p-1} [\theta^{*'}(\frac{c\mu_1^{k+1}}{\mu_1^k}) - \theta^{*'}(c)], \dots, (\mu_m^k)^{p-1} [\theta^{*'}(\frac{c\mu_m^{k+1}}{\mu_m^k}) - \theta^{*'}(c)] \right) \in \partial(-d(\mu^{k+1})). \quad (18)$$

Proposition 3.1. $\{-d(\mu^k)\}$ is a non-increasing convergent sequence.

Proof. Due to the optimality conditions in (17) we have $\mu^{k+1} = \operatorname{argmin}_{\mu > 0} \{-d(\mu) + r^k d_{\theta^*}^p(\mu, \mu^k)\}$, so $-d(\mu^{k+1}) + r^k d_{\theta^*}^p(\mu^{k+1}, \mu^k) \leq -d(\mu^k) + r^k d_{\theta^*}^p(\mu^k, \mu^k) = -d(\mu^k)$, that is $-d(\mu^{k+1}) \leq -d(\mu^k)$. By duality theory, f^* is a lower bound for $\{-d(\mu^k)\}$, so $\{-d(\mu^k)\}$ is convergent. $\square$

Proposition 3.2. The sequence $\{\mu^k\}$ generated by (17) is bounded.

Proof. By (H2) and since $-d$ is a proper convex function, the level set $\Lambda = \{\mu \in \mathbb{R}_+^m / -d(\mu) \leq -d(\mu_0)\}$ is compact and by proposition 3.1, $\mu^k \in \Lambda$ for all k , so μ^k is bounded. $\square$

In the next subsection, based on proposition 2.2 a primal multiplier method associated to the proximal one will be obtained using the generalized penalty functions defined in (10).

3.2 Augmented Lagrangian algorithm

Consider problem (P) with hypothesis (H1) and (H2) and the θ -functions satisfying conditions 1a)-4a) in section (2.1).

Given $p \geq 0$, $r \in (0, 1]$, $\tilde{y} \in \mathbb{R}$, $c \in \mathbb{R}_{++}$ with $\theta'(\tilde{y}) = c$, the augmented Lagrangian function is given by

$$L_{r,c}(x, \mu) = f_0(x) + \sum_{i=1}^m P_{p,i}(f_i(x), \mu_i, r, c)$$

where $P_{p,i}(f_i(x), \mu_i, r, c) = \frac{\mu_i^p}{c} \left[\theta \left(\frac{f_i(x)}{\mu_i^{p-1} r} + \tilde{y} \right) - \theta(\tilde{y}) \right]$ for $i = 1, \dots, m$.

The multiplier method associated to the proximal one is given by:

$$x^{k+1} \in \operatorname{argmin} \{f_0(x) + \sum_{i=1}^m P_{p,i}(f_i(x), \mu_i^k, r^k, c)\}, \quad (19)$$

$$\mu_i^{k+1} = \frac{\mu_i^k}{c} \theta' \left(\frac{f_i(x^{k+1})}{(\mu_i^k)^{p-1} r^k} + \tilde{y} \right) \quad \text{for } i = 1, \dots, m \text{ and } r^k \in (0, 1]. \quad (20)$$

Observe that

$$0 \in \partial_x L_{r^k, c}(x^{k+1}, \mu^k) \Leftrightarrow 0 \in \partial_x l(x^{k+1}, \mu^{k+1})$$

where l is the lagrangian function defined in (2).

The next proposition shows that the sequences defined in (20) and (17) are the same.

Proposition 3.3. *Let $\{\hat{\mu}^k\}$ be the sequence generated by (17) which solves the dual problem (D) and let $\{x^k\}$ and $\{\mu^k\}$ be the sequences generated by (19) and (20) which solve the primal problem (P). If $\mu^0 = \hat{\mu}^0$, then $\forall k > 0 : \mu^k = \hat{\mu}^k$.*

The proof is direct from theorem 7.1 in [8].

3.3 Convergence results

This section is inspired from the convergence theorems in [2] and [14].

From (14) we can write

$$d_{\theta^*}^p(s, \mu) = \sum_{i=1}^m d(s_i, \mu_i) \quad (21)$$

where $d(s_i, \mu_i) = \frac{\mu_i^p}{c} \theta^* \left(\frac{cs_i}{\mu_i} \right) - \frac{\mu_i^p}{c} \theta^*(c) - \mu_i^{p-1} (\theta^*)'(c) (s_i - \mu_i)$ for $i = \dots, m$ and so
 $d_1'(s_i, \mu_i) = \frac{\partial d(s_i, \mu_i)}{\partial s_i} = \mu_i^{p-1} \left[(\theta^*)' \left(\frac{cs_i}{\mu_i} \right) - (\theta^*)'(c) \right].$

Lemma 3.4. *Let s, μ be positive real numbers with $s > \mu$ then*

$$d(s, \mu) \geq \frac{1}{2} \frac{[d_1'(s, \mu)]^2}{c M \mu^{p-2}}$$

where $M = \max\{(\theta^*)''(t)/t \geq \kappa = \theta'(0)\}$ and $c = \theta'(\tilde{y})$.

Proof. Consider the quadratic function

$$q(t) = q(s) + (t - s)d_1'(s, \mu) + \frac{1}{2}(t - s)^2 c M \mu^{p-2}$$

so

$$\begin{aligned} q'(t) &= d_1'(s, \mu) + (t - s)cM\mu^{p-2} \text{ and} \\ q'(t) = 0 &\Leftrightarrow d_1'(s, \mu) + tcM\mu^{p-2} - scM\mu^{p-2} = 0 \\ &\Leftrightarrow t = \frac{-d_1'(s, \mu)}{cM\mu^{p-2}} + \frac{scM\mu^{p-2}}{cM\mu^{p-2}} = \frac{-d_1'(s, \mu)}{cM\mu^{p-2}} + s \end{aligned}$$

so

$$t^* = \frac{-d_1'(s, \mu)}{cM\mu^{p-2}} + s \quad (22)$$

is a minimizer of $q(\cdot)$.

Since $s > \mu > 0$

$$d'_1(s, \mu) = \mu^{p-1} \left[(\theta^*)' \left(\frac{cs}{\mu} \right) - (\theta^*)'(c) \right] > 0.$$

Then, from (22) $t^* < s$.

Furthermore,

$$\begin{aligned} d'_1(s, \mu) &= d'_1(s, \mu) - d'_1(\mu, \mu) = (s - \mu)d''_1(\hat{\mu}, \mu) \text{ for some } \hat{\mu} \in [\mu, s] \\ &= (s - \mu)\mu^{p-2}(\theta^*)''\left(\frac{c\hat{\mu}}{\mu}\right)c \leq (s - \mu)\mu^{p-2}cM. \end{aligned}$$

So $\frac{d'_1(s, \mu)}{cM\mu^{p-2}} \leq s - \mu$ and

$$\mu \leq s - \frac{d'_1(s, \mu)}{cM\mu^{p-2}} = t^*, \quad (23)$$

then $\mu \leq t^* < s$.

On the other hand, for all $t \in [\mu, s]$

$$\begin{aligned} d'_1(s, \mu) - d'_1(t, \mu) &= (s - t)d''_1(\bar{t}, \mu) \text{ for some } t \leq \bar{t} \leq s \\ &= (s - t)\mu^{p-2}(\theta^*)''\left(\frac{c\bar{t}}{\mu}\right)c \\ &\leq (s - t)\mu^{p-2}cM \end{aligned}$$

so $d'_1(s, \mu) \leq d'_1(t, \mu) + (s - t)\mu^{p-2}cM$ for all $\mu \leq t \leq s$
and

$$d'_1(t, \mu) \geq d'_1(s, \mu) + (t - s)\mu^{p-2}cM = q'(t).$$

We have then

$$d'_1(t, \mu) \geq q'(t) \text{ for all } \mu \leq t \leq s.$$

Since $\mu \leq t^* < s$, integrating from t^* to s we get

$$d(s, \mu) - d(t^*, \mu) \geq q(s) - q(t^*).$$

So $d(s, \mu) \geq q(s) - q(t^*) + d(t^*, \mu) \geq q(s) - q(t^*)$ and

$$d(s, \mu) \geq q(s) - q(s) - (t^* - s)d'_1(s, \mu) - \frac{1}{2}(t^* - s)^2cM\mu^{p-2}. \quad (24)$$

From (23) $t^* - s = \frac{-d'_1(s, \mu)}{cM\mu^{p-2}}$,

replacing in (24) we have

$$\begin{aligned} d(s, \mu) &\geq \frac{[d'_1(s, \mu)]^2}{cM\mu^{p-2}} - \frac{1}{2} \frac{[d'_1(s, \mu)]^2}{cM\mu^{p-2}} \\ &= \frac{1}{2} \frac{[d'_1(s, \mu)]^2}{cM\mu^{p-2}}. \end{aligned} \quad (25)$$

Finally,

$$d(s, \mu) \geq \frac{1}{2} \frac{[d'_1(s, \mu)]^2}{cM\mu^{p-2}}.$$

□

The next proposition uses implicitly a similar version of property (D4) in [2], and it is valid for $p \geq 2$.

Proposition 3.5. (*Asymptotic feasibility*) Consider $p \geq 2$. The sequence $\{[f_i(x^k)]_+\}$ converges to 0 for all $i = 1, \dots, m$ where $[y]_+ = \max\{0, y\}$.

Proof. Let us suppose by absurd that $\{[f_i(x^k)]\} \not\rightarrow 0$, then there exists $\{l_k\}$ such that $l_k \rightarrow +\infty$ with $l_k < l_{k+1}$ and $\exists \epsilon > 0$ such that $f_{i_0}(x^{l_k}) > \epsilon$ for some $i_0 \in \{1, \dots, m\}$.

$$\begin{aligned} \text{We have } d(\mu) &= \inf\{f_0(x) + \sum_{i=1}^m \mu_i f_i(x)\} \leq f_0(x^{k+1}) + \sum_{i=1}^m \mu_i f_i(x^{k+1}) \\ &= f_0(x^{k+1}) + \sum_{i=1}^m \mu_i^{k+1} f_i(x^{k+1}) + \sum_{i=1}^m (\mu_i - \mu_i^{k+1}) f_i(x^{k+1}) \\ &= d(\mu_i^{k+1}) + \sum_{i=1}^m (\mu_i - \mu_i^{k+1}) f_i(x^{k+1}) \text{ for } i = 1, \dots, m. \end{aligned}$$

From (20),

$$\mu_i^{k+1} = \frac{\mu_i^k}{c} \theta' \left(\frac{f_i(x^{k+1})}{(\mu_i^k)^{p-1} r^k} + \tilde{y} \right) \quad \text{for } i = 1, \dots, m$$

and by using $(\theta')^{-1} = (\theta^*)'$ we get

$$r^k (\mu_i^k)^{p-1} \left[(\theta^*)' \left(\frac{c \mu_i^{k+1}}{\mu_i^k} \right) - (\theta^*)'(c) \right] = f_i(x^{k+1}) \quad \text{for } i = 1, \dots, m. \quad (26)$$

Using notation in (21),

$$d(s_i, \mu_i) = \frac{\mu_i^p}{c} \theta^* \left(\frac{c s_i}{\mu_i} \right) - \frac{\mu_i^p}{c} \theta^*(c) - \mu_i^{p-1} (\theta^*)'(c) (s_i - \mu_i) \quad \text{for } i = 1, \dots, m$$

and

$$d'_1(s_i, \mu_i) = \mu_i^{p-1} \left[(\theta^*)' \left(\frac{c s_i}{\mu_i} \right) - (\theta^*)'(c) \right] \quad \text{for } i = 1, \dots, m, \quad (27)$$

from (26) and (27)

$$\frac{f_i(x^{k+1})}{r^k} = d'_1(\mu_i^{k+1}, \mu_i^k) \quad \text{for } i = 1, \dots, m. \quad (28)$$

Since $f_{i_0}(x^{l_{k+1}}) > \epsilon$, then $r^k d'_1(\mu_{i_0}^{l_{k+1}}, \mu_{i_0}^{l_k}) > \epsilon > 0$.

This implies $d'_1(\mu_{i_0}^{l_{k+1}}, \mu_{i_0}^{l_k}) > 0$ and so $\mu_{i_0}^{l_{k+1}} > \mu_{i_0}^{l_k}$, and we can use proposition 3.4.

Since $\mu^{k+1} = \operatorname{argmin}\{-d(\mu) + d_{\theta^*}^p(\mu, \mu^k)\}$, we have $-d(\mu^{k+1}) + d_{\theta^*}^p(\mu^{k+1}, \mu^k) \leq -d(\mu^k) + d_{\theta^*}^p(\mu^k, \mu^k) = -d(\mu^k)$,
so $d(\mu_{k+1}) - d(\mu_k) \geq d_{\theta^*}^p(\mu^{k+1}, \mu^k) \geq 0$.

Then $d(\mu^{l_{k+1}}) - d(\mu^{l_k}) \geq d_{\theta^*}^p(\mu^{l_{k+1}}, \mu^{l_k}) = \sum_{i=1}^m d(\mu_i^{l_{k+1}}, \mu_i^{l_k}) \geq d(\mu_{i_0}^{l_{k+1}}, \mu_{i_0}^{l_k})$.

By proposition 3.2, let $\hat{\mu}$ an upper bound of $\{\mu^k\}$ and by lemma 3.4 we have

$$d(\mu^{l_{k+1}}) - d(\mu^{l_k}) \geq \frac{1}{2cM} \frac{[d'_1(\mu_{i_0}^{l_{k+1}}, \mu_{i_0}^{l_k})]^2}{(\mu_{i_0}^{l_k})^{p-2}} \geq \frac{\epsilon^2}{2cM \hat{\mu}_{i_0}^{p-2}} = \delta > 0$$

for $p \geq 2$.

Then $d(\mu^{l_{k+1}}) - d(\mu^{l_k}) \geq \delta > 0$ and taking limits when $k \rightarrow \infty$ we have that 0 is greater than a positive value, which is a contradiction.

Finally, $\lim_{k \rightarrow +\infty} [f_i(x^k)]_+ = 0$.

The next proposition use the following affirmation:

Fact 1: For all t , $t \frac{\theta'(t+\tilde{y})}{c} \geq t$.

Proof. If $t > 0 \Rightarrow t + \tilde{y} > \tilde{y} \Rightarrow \theta'(t + \tilde{y}) > \theta'(\tilde{y}) = c$

$\Rightarrow \frac{\theta'(t+\tilde{y})}{c} > 1 \Rightarrow t \frac{\theta'(t+\tilde{y})}{c} > t$.

If $t < 0 \Rightarrow t + \tilde{y} < \tilde{y} \Rightarrow \theta'(t + \tilde{y}) < \theta'(\tilde{y}) = c$

$\Rightarrow \frac{\theta'(t+\tilde{y})}{c} < 1 \Rightarrow t \frac{\theta'(t+\tilde{y})}{c} > t$. □

□

Proposition 3.6. Let $\{x^k\}$ and $\{\mu^k\}$ be the sequences generated by (19) and (20), then $\lim_{k \rightarrow +\infty} \mu_i^k f_i(x^k) = 0$ for $i = 1, \dots, m$.

Proof. By absurd, suppose there exists $i_0 \in \{1, \dots, m\}$, $\epsilon > 0$ and an infinite index set $\{k_j\}$ such that

$$|\mu_{i_0}^{k_j+1} f_{i_0}(x^{k_j+1})| \geq \epsilon \text{ for all } j.$$

Since $\{\mu^k\}$ is bounded, there exists $\hat{\mu} > 0$ such that:

$$0 < \mu^k \leq \hat{\mu} \quad \text{for all } k, \quad (29)$$

hence

$$|\mu_{i_0}^{k_j+1} f_{i_0}(x^{k_j+1})| \geq \epsilon \Rightarrow |f_{i_0}(x^{k_j+1})| \geq \frac{\epsilon}{\hat{\mu}_{i_0}}.$$

But $\{[f_i(x^k)]_+\}$ converges to 0, for all $i = 1, \dots, m$ hence $f_{i_0}(x^{k_j+1}) \geq \frac{\epsilon}{\widehat{\mu}_{i_0}}$ is true only for a finite set of index k_j , so we can consider without lost of generality

$$f_{i_0}(x^{k_j+1}) \leq -\frac{\epsilon}{\widehat{\mu}_{i_0}} \quad (30)$$

for all j .

Since $(f_1(x^{k+1}), \dots, f_m(x^{k+1}))^t \in \partial d(\mu^{k+1})$ and d is a concave function, then

$$\sum_{i=1}^m f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) \leq d(\mu^{k_j+1}) - d(\mu^{k_j}). \quad (31)$$

Since $\mu_i^{k+1} = \frac{\mu_i^k}{c} \theta' \left(\frac{f_i(x^{k+1})}{(\mu_i^k)^{p-1} r^k} + \widetilde{y} \right)$ for $i = 1, \dots, m$ we have

$$\begin{aligned} f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) &= \mu_i^{k_j} f_i(x^{k_j+1}) \left(\frac{\mu_i^{k_j+1}}{\mu_i^{k_j}} - 1 \right) \\ &= \mu_i^{k_j} f_i(x^{k_j+1}) \left(\frac{1}{c} \theta' \left(\frac{f_i(x^{k_j+1})}{(\mu_i^{k_j})^{p-1} r^k} + \widetilde{y} \right) - 1 \right) \end{aligned} \quad (32)$$

$$= \frac{\mu_i^{k_j} f_i(x^{k_j+1})}{c} \theta' \left(\frac{f_i(x^{k_j+1})}{(\mu_i^{k_j})^{p-1} r^k} + \widetilde{y} \right) - \mu_i^{k_j} f_i(x^{k_j+1}). \quad (33)$$

Since θ is a strictly convex function, $\theta''(t) > 0$ for all $t \in (-\infty, b)$, then θ' is increasing and so using (33) and fact 1

$$\begin{aligned} f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) &= \frac{r^k (\mu_i^{k_j})^p}{c} \left[\frac{f_i(x^{k_j+1})}{(\mu_i^{k_j})^{p-1} r^k} \theta' \left(\frac{f_i(x^{k_j+1})}{(\mu_i^{k_j})^{p-1} r^k} + \widetilde{y} \right) \right] - \mu_i^{k_j} f_i(x^{k_j+1}) \\ &\geq \mu_i^{k_j} f_i(x^{k_j+1}) - \mu_i^{k_j} f_i(x^{k_j+1}) \\ &= 0, \end{aligned}$$

$$\text{hence } 0 \leq \sum_{i=1}^m f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) \leq d(\mu^{k_j+1}) - d(\mu^{k_j}),$$

since $\{d(\mu^k)\}$ is convergent, then $\lim_{j \rightarrow +\infty} [d(\mu^{k_j+1}) - d(\mu^{k_j})] = 0$

and so

$$\lim_{j \rightarrow +\infty} \sum_{i=1}^m f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) = 0.$$

Since $f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) \geq 0$ for all $i = 1, \dots, m$

we have $\lim_{j \rightarrow +\infty} f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) = 0$ for all $i = 1, \dots, m$.

So from (33),

$$\lim_{j \rightarrow +\infty} \mu_i^{k_j} f_i(x^{k_j+1}) \left[\frac{1}{c} \theta' \left(\frac{f_i(x^{k_j+1})}{(\mu_i^{k_j})^{p-1} r^k} + \widetilde{y} \right) - 1 \right] = 0 \text{ for all } i = 1, \dots, m. \quad (34)$$

On the other hand

$$\frac{f_{i_0}(x^{k_j+1})}{r^k(\mu_{i_0}^{k_j})^{p-1}} + \tilde{y} \leq \frac{-\epsilon}{r^k(\mu_{i_0}^{k_j})^{p-1}\widehat{\mu_{i_0}}} + \tilde{y} < 0 + \tilde{y},$$

and since θ' is increasing,

$$\theta' \left(\frac{f_{i_0}(x^{k_j+1})}{r^k(\mu_{i_0}^{k_j})^{p-1}} + \tilde{y} \right) \leq \theta' \left(\frac{-\epsilon}{r^k(\mu_{i_0}^{k_j})^{p-1}\widehat{\mu_{i_0}}} + \tilde{y} \right) < \theta'(\tilde{y}) = c,$$

so

$$\frac{1}{c} \theta' \left(\frac{f_{i_0}(x^{k_j+1})}{r^k(\mu_{i_0}^{k_j})^{p-1}} + \tilde{y} \right) \leq \frac{1}{c} \theta' \left(\frac{-\epsilon}{r^k(\mu_{i_0}^{k_j})^{p-1}\widehat{\mu_{i_0}}} + \tilde{y} \right) < 1. \quad (35)$$

From (34) and (35)

$$\lim_{j \rightarrow +\infty} \mu_{i_0}^{k_j} f_{i_0}(x^{k_j+1}) = 0.$$

Since

$$\lim_{j \rightarrow +\infty} f_i(x^{k_j+1})(\mu_i^{k_j+1} - \mu_i^{k_j}) = 0$$

then $\lim_{j \rightarrow +\infty} \mu_{i_0}^{k_j+1} f_{i_0}(x^{k_j+1}) = 0$, which is a contradiction.

Consequently

$$\mu_i^k f_i(x^k) \rightarrow 0$$

for all $i = 1, \dots, m$

□

Theorem 3.7. *The sequence $\{f_0(x^k)\}$ converges to $\widehat{f}$, each limit point of the sequences $\{x^k\}$ and $\{\mu^k\}$ generated by the method (19) and (20) (or (19) and (17)) are optimal solutions for the problems (P) and (D) respectively.*

Proof. Since $\{x^k\}$ is asymptotically feasible, for all $\epsilon > 0$, $f_0(x^k) \geq \widehat{f} - \epsilon$ for sufficiently large k .

According to (H2), the optimal set of problem (D) is nonempty and compact, $\widehat{f} = \widehat{d}$ where

$$\widehat{f} = \min\{f_0(x) : x \in R^n, f_i(x) \leq 0, i = 1, \dots, m\}$$

and

$$\widehat{d} = \sup\{d(\mu) : \mu \in R_+^m\}.$$

Furthermore, for each $\beta < \widehat{d}$ the level set $\{\mu \in R_+^m : d(\mu) \geq \beta\}$ is compact. But

$$\widehat{f} = \widehat{d} \geq d(\mu^k) = \inf_x \{l(x, \mu^k)\} = l(x^k, \mu^k) = f_0(x^k) + \sum_{i=1}^m \mu_i^k f_i(x^k) \quad (36)$$

and according to proposition 3.6 $\{\mu_i^k f_i(x^k)\}$ converges to 0 for all $i = 1, \dots, m$ so

for all $\epsilon > 0$, $\widehat{f} - \epsilon \leq f_0(x^k) \leq \widehat{f} - \sum_{i=1}^m \mu_i^k f_i(x^k) < \widehat{f} + \epsilon$ for sufficiently large k

and so

$$f_0(x^k) \rightarrow \widehat{f}. \quad (37)$$

By proposition 3.5, for all $\epsilon > 0$ and for $i = 1, \dots, m$, $f_i(x^k) \leq \epsilon$ for sufficiently large k , then $f_0(x^k) \leq \widehat{f} + \epsilon$ and $f_i(x^k) \leq \epsilon$ for $i = 1, \dots, m$ and sufficiently large k . According to (H1) and corollary 20 in [7] the set

$$\{x \in R^n : f_i(x) \leq \alpha, f_0(x) \leq \beta\}$$

is compact for any α, β , then $\{x^k\}$ is bounded and by the proposition 3.2 the sequence $\{\mu^k\}$ also is bounded. If $\bar{x}$ is a limit point of $\{x^k\}$ then by proposition 3.5 and (37), $\bar{x}$ is a primal optimal solution and if $\bar{\mu}$ is a limit point of $\{\mu^k\}$, using (36) and proposition 3.6, $d(\bar{\mu}) = \widehat{f} = \widehat{d}$. Hence $\bar{\mu}$ is a dual optimal solution. $\square$

4 Concluding remarks

For the first time there appear rational powers p in the multiplier vector, this leads us to think about the value of p with the best convergence rate in the context of multiplier methods. It was not treated in this paper and remains as an open problem, although we observe in computational testing that the number of iterations in the main algorithm decrease when p increases but it causes a bad numerical performance.

Multiplier methods without shift with $p = 3$ was considered in [4] without a complete convergence rate study but with a similar convergence analysis.

Convergence analysis for the case $p=1$ can be obtained following similar hypothesis and theorems as in [9]. In [13] convergence results were considered for $p = 0$ for a specific penalty function without shift. Other considerations about the value of p , for example, $p \leq 0$ remain as open problems.

Finally, we could consider shifts at each iteration as $\theta(\tilde{y}_i^k) = c_i^k$ and to choose $c_i^k = r^k \alpha^k$ as in [16] in order to relate both approaches.

5 Appendix

Following definition 2.1 in [9] we next show that the generalized like-distance considered in this work is a divergence measure.

Proposition 5.1. *Given θ^* satisfying conditions 1b)-4b), $p \geq 0$, $r \in [0, 1]$ and $c > 0$ then*

$$d_{\theta^*}^p(x, y) = \sum_{i=1}^m \left[\frac{y_i^p}{c} \theta^* \left(\frac{cx_i}{y_i} \right) - \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) (x_i - y_i) \right]$$

is a divergence measure in R_{++}^n .

Proof.

1) Since θ^* is convex, $\forall z, w \in R \quad \theta^*(z) \geq \theta^*(w) + (\theta^*)'(w)(z - w)$, in particular for $w = c$ and $z = \frac{cx_i}{y_i}$, $i = 1, \dots, m$ with $x_i \geq 0, y_i > 0$ and $\text{dom } \theta^* = (0, +\infty)$, we have

$$\theta^* \left(\frac{cx_i}{y_i} \right) \geq \theta^*(c) + (\theta^*)'(c) \left(\frac{cx_i}{y_i} - c \right)$$

and for all $i = 1, \dots, m$ and $p \geq 0$

$$\frac{y_i^p}{c} \theta^* \left(\frac{cx_i}{y_i} \right) - \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) (x_i - y_i) \geq 0 \quad (38)$$

so

$$\sum_{i=1}^m \left[\frac{y_i^p}{c} \theta^* \left(\frac{cx_i}{y_i} \right) - \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) (x_i - y_i) \right] \geq 0;$$

that is $d_{\theta^*}^p(x, y) \geq 0$.

2) Let $\{x^k\}$ be a sequence on $(0, \infty)$. We next show

$$\lim_{k \rightarrow +\infty} d_{\theta^*}^p(x, x^k) = 0 \Leftrightarrow \lim_{k \rightarrow +\infty} x^k = x.$$

Note that,

i) θ^* is non-increasing on $(0, \kappa)$ and increasing on $(\kappa, +\infty)$.

ii) $(\theta^*)'(\kappa) = 0$.

iii) $\lim_{t \rightarrow +\infty} \theta^*(t) = +\infty$.

So,

$$\lim_{k \rightarrow +\infty} d_{\theta^*}^p(x, x^k) = 0$$

$$\Leftrightarrow \lim_{k \rightarrow +\infty} \sum_{i=1}^m \left[\frac{(x_i^k)^p}{c} \theta^* \left(\frac{cx_i}{x_i^k} \right) - \frac{(x_i^k)^p}{c} \theta^*(c) - (x_i^k)^{p-1} (\theta^*)'(c) (x_i - x_i^k) \right] = 0 \Leftrightarrow$$

$$\lim_{k \rightarrow +\infty} \left[\frac{(x_i^k)^p}{c} \theta^* \left(\frac{cx_i}{x_i^k} \right) - \frac{(x_i^k)^p}{c} \theta^*(c) - (x_i^k)^{p-1} (\theta^*)'(c) (x_i - x_i^k) \right] = 0,$$

for all $i = 1, \dots, m$

$$\begin{aligned}
 &\Leftrightarrow \lim_{k \rightarrow +\infty} \theta^* \left(\frac{cx_i}{x_i^k} \right) = \theta^*(c), \text{ for all } i = 1, \dots, m \Leftrightarrow \lim_{k \rightarrow +\infty} \frac{cx_i}{x_i^k} = c, \text{ for } i = 1, \dots, m \\
 &\Leftrightarrow \lim_{k \rightarrow +\infty} \frac{x_i}{x_i^k} = 1, i = 1, \dots, m \\
 &\Leftrightarrow \lim_{k \rightarrow +\infty} x_i^k = x_i, i = 1, \dots, m \\
 &\Leftrightarrow \lim_{k \rightarrow +\infty} x^k = x.
 \end{aligned}$$

3) The level set $\Gamma_1(y, v) = \{x \in (0, +\infty) / d_{\theta^*}^p(x, y) \leq v\}$ is bounded for all $y \in (0, +\infty)$ and for all $v \in (0, +\infty)$.

In fact, suppose for some $y \in (0, +\infty)$ and for some $v \in (0, +\infty)$, $\Gamma(y, v)$ is not bounded.

Then there exists $\{x^k\}$ in $(0, +\infty)$ / $x_k \rightarrow +\infty$, but $d_{\theta^*}^p(x, y) \leq v$, from (14),

$$0 \leq \sum_{i=1}^m \frac{y_i^p}{c} \theta^* \left(\frac{cx_i^k}{y_i} \right) - \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) (x_i^k - y_i) \leq v$$

and for $i=1, \dots, m$

$$\begin{aligned}
 0 &\leq \frac{y_i^p}{c} \theta^* \left(\frac{cx_i^k}{y_i} \right) - \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) (x_i^k - y_i) \leq v, \text{ so} \\
 0 &\leq \frac{y_i^p}{c} \theta^* \left(\frac{cx_i^k}{y_i} \right) - y_i^{p-1} (\theta^*)'(c) x_i^k \leq v + \frac{y_i^p}{c} \theta^*(c) - y_i^{p-1} (\theta^*)'(c) y_i, \text{ and} \\
 0 &\leq \frac{y_i^p}{c} \theta^* \left(\frac{cx_i^k}{y_i} \right) - y_i^{p-1} (\theta^*)'(c) x_i^k \leq v + \frac{y_i^p}{c} \theta^*(c) - y_i^p (\theta^*)'(c).
 \end{aligned}$$

If $M_i = v + \frac{y_i^p}{c} \theta^*(c) - y_i^p (\theta^*)'(c)$ for $i = 1, \dots, m$ we have:

$$0 \leq \frac{y_i^p}{c} \theta^* \left(\frac{cx_i^k}{y_i} \right) - y_i^{p-1} (\theta^*)'(c) x_i^k \leq M_i, \text{ and}$$

$$0 \leq \frac{y_i^p}{c} \left(\frac{\theta^* \left(\frac{cx_i^k}{y_i} \right)}{x_i^k} \right) \leq \frac{M_i}{x_i^k} + y_i^{p-1} (\theta^*)'(c).$$

On the other hand,

$$\lim_{k \rightarrow +\infty} \frac{y_i^p}{c} \left(\frac{\theta^* \left(\frac{cx_i^k}{y_i} \right)}{x_i^k} \right) = \lim_{k \rightarrow +\infty} \frac{y_i^p}{c} \left(\frac{\theta^* \left(\frac{cx_i^k}{y_i} \right)}{x_i^k} \right)$$

$$= \frac{y_i^p}{c} \lim_{k \rightarrow +\infty} (\theta^*)' \left(\frac{cx_i^k}{y_i} \right) \frac{c}{y_i}$$

$$= y_i^{p-1} \cdot (+\infty) = +\infty.$$

Furthermore : $\frac{M_i}{x_i^k} + y_i^{p-1} (\theta^*)'(c) \rightarrow y_i^{p-1} (\theta^*)'(c)$, we obtain

$$+\infty \leq y_i^{p-1} (\theta^*)'(c), \text{ which is a contradiction.}$$

4) Next we prove that $\Gamma_2(x, v) = \{y \in (0, +\infty) / d_{\theta^*}^p(x, y) \leq v\}$ is bounded

for all $x \in (0, +\infty)$ and for all $v \in (0, +\infty)$.

Suppose there exists $\{y^k\} \subset \Gamma_2(x, v)$ such that for some j ; $\{y_j^k\}$ is not bounded and without loss of generality $y_j^{k+1} > y_j^k, \forall k$. For x fixed, since $\frac{x_j}{y_j^k} \rightarrow 0$ if $k \rightarrow +\infty$, then for sufficiently large k : $\frac{x_j}{y_j^k} < 1$ implying that $\frac{cx_j}{y_j^k} < c$ and so

$$\begin{aligned} v &\geq \frac{(y_j^k)^p}{c} \theta^* \left(\frac{cx_j}{y_j^k} \right) - \frac{(y_j^k)^p}{c} \theta^*(c) - (y_j^k)^{p-1} (\theta^*)'(c) x_j + (y_j^k)^{p-1} (\theta^*)'(c) (y_j^k) \\ &\geq \frac{(y_j^k)^p}{c} (\theta^*)'(\xi) c \left[\frac{x_j}{y_j^k} - 1 \right] - (y_j^k)^p (\theta^*)'(c) \left(\frac{x_j}{y_j^k} \right) + (y_j^k)^p (\theta^*)'(c), \quad \frac{cx_j}{y_j^k} < \xi < c. \\ &= (y_j^k)^p (\theta^*)'(\xi) \left[\frac{x_j}{y_j^k} - 1 \right] - (y_j^k)^p (\theta^*)'(c) \left[\frac{x_j}{y_j^k} - 1 \right] \\ &= (y_j^k)^p [(\theta^*)'(\xi) - (\theta^*)'(c)] \left[\frac{x_j}{y_j^k} - 1 \right] \\ &= (y_j^k)^p (\theta^*)''(\bar{\xi})(\xi - c) \left[\frac{x_j}{y_j^k} - 1 \right] \rightarrow +\infty \quad (\xi < \bar{\xi} < c) \end{aligned}$$

which is a contradiction.

Finally $d_{\theta^*}^p(x, y)$ is a divergence measure. $\square$

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