Base Point Free Pencils on Multiple Coverings of Smooth Curves

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Abstract. Fix an integer $k \geq 2$ and a smooth genus g projective curve X . Here we prove the existence of a smooth curve C , a degree k morphism $C \rightarrow X$ and a low degree base point free linear system on C not coming from X .

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We will follow the notations of [2] and [3] concerning ruled surfaces over a smooth projective curve X . If $S := \mathbf{P}(\mathcal{O}_X \oplus R)$, $R \in \text{Pic}(X)$, $\deg(R) \leq 0$, we will write X_0 (resp. X_1) for the unisecant divisor of the ruling $S \rightarrow X$ associated to the projection of $\mathcal{O}_X \oplus R$ onto the first (resp. second) factors. Thus $X_0^2 = \deg(R)$, $X_1^2 = -\deg(R)$ and $X_0 \cap X_1 = \emptyset$. The case $k = 2$ of the following lemma is just [1], Lemma 2.2. Unless otherwise stated X is a smooth and connected projective curve of genus g and $S := \mathbf{P}(\mathcal{O}_C \oplus R)$, $R \in \text{Pic}(X)$, $e := \deg(R) \leq 0$. For any complete linear system $|a|$ on X , let $\text{Bs}(|a|)$ denote its base locus.

Here we follow the proofs of [1] and prove the following results.

Theorem 1. *Let X be a smooth curve of genus g . Fix an integer $k \geq 2$ and effective divisors a_1, a_2, b on X such that b and $b + ka_2 - ka_1$ are smooth and with disjoint support, $b + ka_2 - ka_1 \neq \emptyset$ and $\text{Bs}(|a_1|) \cap \text{Bs}(|a_2|) = \text{Bs}(|a_1|) \cap \text{Bs}(|b|) = \text{Bs}(|a_2|) \cap \text{Bs}(|b + ka_2 - a_1|) = \emptyset$. Set $R := \mathcal{O}_X(a_2 - a_1)$, $e := -\deg(R) = \deg(a_1) - \deg(a_2)$, and $S := \mathbf{P}(\mathcal{O}_C \oplus R)$. Fix a general $C \in |kX_0 + bf|$ and W be the complete linear system on C associated to $(X_0 + a_1f)|_C$. C is smooth and irreducible, $p_a(C) = 1 + k(g - 1) + (-k(k - 2)e + (k - 2) \cdot \deg(b) +$*

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$k \cdot \deg(b + a_2 - a_1))/2$, the linear system W has no base points and degree $-ke + \deg(b) + k \cdot \deg(a_1)$.

Theorem 2. *Fix positive integers q, k, d , such that $k \geq 2$. Set $g := 1 + k(q - 1) + (-k(k - 2)e + (k - 2) \cdot \deg(b) + k \cdot \deg(b))/2$. Let X be a smooth curve of genus q such that there is a base point free degree d pencil. Then there are a smooth genus g curve C , a degree k covering $u : C \rightarrow X$ and a degree d base point free linear system on C not induced from a linear system on X .*

Lemma 1. *Fix an integer $k \geq 2$. Let b_1, b_2 be smooth effective divisors on C with disjoint support and such that $\mathcal{O}_X(b_2 - b_1) \cong R^{\otimes k}$. If $R \cong \mathcal{O}_X$, then assume $b_1 \neq \emptyset$. Then there exists a smooth and irreducible curve $C \in |kX_0 + b_1f|$.*

Proof. Let $C \in |kX_0 + b_1f|$ be a general element. Let B be the set-theoretic base locus of the linear system $|kX_0 + b_1f|$. Notice that $kX_1 + b_2f \in |kX_0 + b_1f|$. Since $b_1 \cap b_2 = X_0 \cap X_1 = \emptyset$, B is finite. By Bertini's theorem to check that C is smooth it is sufficient to show that C is smooth at each point of B . Since $b_1 \cap b_2 = X_0 \cap X_1 = \emptyset$ and $kX_1 + b_2f \in |kX_0 + b_1f|$, B is contained in $(b_1f \cap X_1) \cup (b_2f \cap X_0)$. At each of these points C is smooth, because b_1, b_2 are smooth. Now we check the irreducibility of C . Since C is irreducible, it is sufficient to show that C is connected. Since $kX_1 + b_2f \in |kX_0 + b_1f|$, $b_1 \cap b_2 = X_0 \cap X_1 = \emptyset$ and C is general, neither X_0 nor X_1 nor any fiber of the ruling is a connected component of C . Assume $C = A \cup B$ with $A \neq \emptyset$, $B \neq \emptyset$ and $A \in |xX_0 + yf|$, $0 < x < k$. Since X_0 is not a component of C , we get $X_0 \cdot A \geq 0$ (i.e. $\deg(y) \geq ex$) and $X_0 \cdot B \geq 0$ (i.e. $\deg(b_1) - \deg(y) \geq e(k - x)$). We get $A \cdot B \geq 0$ with equality if and only if $e = 0$ and $\deg(b_1) = 0$ (i.e. $b_1 = \emptyset$). Our assumptions imply $A \cdot B > 0$. Hence $A \cap B \neq \emptyset$, contradiction. $\square$

Proof of Theorem 1. Apply Lemma 1 with $R = \mathcal{O}_X(a_2 - a_1)$, $b_1 = b$ and $b_2 = b + ka_2 - ka_1$. We get that C is smooth and irreducible. Notice that $\omega_S \cong \mathcal{O}_S(-2X_0 + (\omega_C + R)f)$. Hence $\omega_C \cong ((k - 2)X_0 + (\omega_C + R + b)f)|_C$ (adjunction formula). Hence $2p_a(C) - 2 = (kX_0 + bf) \cdot ((k - 2)X_0 + (\omega_C + R + b)f)$. Since $X_1 + a_2f \in |X_0 + a_1f|$, the base points of $|X_0 + a_1f|$ are contained in the finite set $(X_0 \cap a_2f) \cup (X_1 \cap a_1f)$. Since $C \in |kX_0 + bf|$, $C \cap X_0 = (b_2 + ka_2 - ka_1)f \cap X_0$ and $C \cap X_1 = bf \cap X_1$. Since $\text{Bs}(|a_2|) \cap \text{Bs}(|b + ka_2 - a_1|) = \emptyset$, and C is general, we get $C \cap X_0 \cup a_2f = \emptyset$. Since $\text{Bs}(|a_1|) \cap \text{Bs}(|b|) = \emptyset$ and C is general, $C \cap X_1 \cap a_1f = \emptyset$. Hence W has no base points. $\square$

Proof of Theorem 2. Apply Theorem 1 taking $a_1 = a_2 = 0$. $\square$

We work over an algebraically closed field $\mathbb{K}$ such that $\text{char}(\mathbb{K}) = 0$.

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