Simple Unstable Rank Two Vector Bundles with Canonical Determinant on a Curve

E. Ballico¹

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
ballico@science.unitn.it

Abstract. Let X be a smooth curve of genus $g \geq 2$. For any rank two vector bundle E on X let $\sigma_E : S^2(H^0(X, E)) \rightarrow S^2(H^0(X, E))$ denote the symmetric multiplication map. Here we fix the integer $z \leq 2g - 2$ and show the existence of simple, but unstable, E with many sections, $\deg(\det(E)) = z$ and injective σ_E .

Mathematics Subject Classification: 14H60

Keywords: simple vector bundle; unstable vector bundle on curves; rank two vector bundle; canonical determinant; Brill-Noether theory

1. INTRODUCTION

Let X be a smooth and connected projective curve of genus $g \geq 2$ defined over an algebraically closed field $\mathbb{K}$ such that $\text{char}(\mathbb{K}) \neq 2$. Let E be a rank two vector bundle on X such that $\det(E) \cong \omega_X$. Let $\sigma_E : S^2(H^0(X, E)) \rightarrow S^2(H^0(X, E))$ denote the symmetric multiplication map. If E is stable, then the injectivity of σ_E has a very strong geometric interpretation (see [2], [3], [4], [5]). See [7] and [8] for recent results on the injectivity of this map. Here we study the existence of simple, but unstable vector bundles E such that $\text{rank}(E) = 2$, $\det(E) \cong \omega_X$ and σ_E is injective (see Theorem 1). We will also prove an existence theorem when $\det(E)$ is a general line bundle of degree at most $2g - 2$ (see Theorem 2).

Theorem 1. *Fix integers g, n, k such that $g \geq 3$, $k \geq n \geq 3$ and $g \geq \binom{n+2}{2} - 2n + 1$. Set $d := g + n$. Let X be a general k -gonal curve of genus g (with the convention that X has general moduli if $2k \geq g + 1$). There exists a simple rank two vector bundle E on X such that $\det(E) \cong \omega_X$, $h^0(X, E) =$*

¹The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

$n+1$, the symmetric multiplication map $\sigma_E : S^2(H^0(X, E)) \rightarrow S^2(H^0(X, E))$ is injective, and $H^0(X, E)$ spans a non-special line subbundle L of E with degree d . E is unstable and L is the destabilizing line subbundle of E .

Theorem 2. Fix integers g, n, k, z such that $g \geq 3$, $k \geq n \geq 3$ and $g \geq \binom{n+2}{2} - 2n + 1$. Set $d := g + n$. Assume $z \leq 2g - 2$ and $2d \leq g - 2 + 2z$. Let X be a general k -gonal curve of genus g (with the convention that X has general moduli if $2k \geq g + 1$). There exists a simple rank two vector bundle E on X such that $\deg(\det(E)) = 2z - 2$, $h^0(X, E) = n + 1$, the symmetric multiplication map $\sigma_E : S^2(H^0(X, E)) \rightarrow S^2(H^0(X, E))$ is injective, and $H^0(X, E)$ spans a non-special line subbundle L of E with degree d . E is unstable and L is the destabilizing line subbundle of E .

Remark 1. Fix $L, M \in \text{Pic}(X)$ and an exact sequence on X :

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0 \quad (1)$$

Assume that L is the only maximal degree line subbundle of E . This assumption is satisfied if either $\deg(L) > \deg(M)$ or $\deg(L) = \deg(M)$ and the extension (1) is not the zero extension. If (1) is not the zero extension, then $h^0(X, \text{End}(E)) = 1 + h^0(X, \text{Hom}(M, L))$. Hence E is simple if and only if (1) is not the zero extension and $h^0(X, \text{Hom}(M, L)) = 0$. There is a non-zero extension (1) if and only if $h^1(X, \text{Hom}(M, L)) \neq 0$. If $h^0(X, \text{Hom}(M, L)) = 0$, then $h^1(X, \text{Hom}(M, L)) = g - 1 + \deg(M) - \deg(L)$.

Remark 2. Fix $L \in \text{Pic}^x(X)$ and take a general $M \in \text{Pic}^y(X)$. There is a non-zero extension (1) if and only if $y \geq x + 2 - g$. We have $h^0(X, \text{Hom}(M, L)) = 0$ if and only if $y \geq x + 1 - g$.

Remark 3. Fix $L \in \text{Pic}^x(X)$ and take $M \in \text{Pic}^y(X)$ such that $M \cong \omega_X \otimes L^*(-D)$ for some non-negative divisor D . Hence $\deg(D) = 2g - 2 - x - y$. There is a non-zero extension (1) if and only if $h^1(X, \omega_X^* \otimes L^{\otimes 2}(D)) \neq 0$ and this is always the case if $\deg(D) \leq 3g - 4 - 2x$. We have $h^0(X, \text{Hom}(M, L)) = 0$ if and only if $h^0(X, L^{\otimes 2} \otimes \omega_X^*(D)) = 0$. For a general effective divisor D of degree $2g - 2 - x - y$ we have $h^0(X, \text{Hom}(M, L)) = 0$ if and only if $h^0(X, L^{\otimes 2} \otimes \omega_X^*) = 0$ and $\deg(D) \leq 3g - 3 - 2x$.

Remark 4. Fix integers g, k, d such that $g \geq 3$, $2k \leq g + 1$ and $d \geq 2k$. Let X be a smooth genus g curve with gonality at most k . As in [6], Ex. 3.1, there is $L \in \text{Pic}^d(X)$ such that the symmetric multiplication map $\sigma_L : S^2(H^0(X, L)) \rightarrow S^2(H^0(X, L))$ is not injective. Let R be any line bundle with minimal degree such that $h^0(X, R) = 2$. Hence $h^1(X, R^{\otimes 2}) = h^0(X, R^{\otimes 2}) + g - 1 - 2m$. If $d \geq h^0(X, R^{\otimes 2}) + g - 1$, then we may take as L any line bundle $R^{\otimes 2}(D)$ with D a general union of $d - 2m$ points of X . By Brill-Noether theory $h^0(X, R^{\otimes 2}) = 3$ if X is a general m -gonal curve.

Remark 5. Fix integers g, n, k such that $g \geq 3$, $k \geq n \geq 3$ and $g \geq \binom{n+2}{2} - 2n + 1$. Set $d := g + n$. Let X be a general k -gonal curve of genus g (with the

convention that X has general moduli if $2k \geq g+1$). Let L be a general element of $\text{Pic}(X)$. Hence $h^1(X, L) = 0$ and $h^0(X, L) = n+1$ and L is very ample. By [1] the symmetric multiplication map $\sigma_L : S^2(H^0(X, L)) \rightarrow S^2(H^0(X, L))$ is injective.

Proof of Theorem 1. Take L given by Remark 5. Hence L is a general element of $\text{Pic}^d(X)$. The assumption on n implies $d < \lfloor (3(g-1)/4) \rfloor$. Set $M := \omega_X \otimes L^*$. The generality of L and the inequality $d < \lfloor (3(g-1)/4) \rfloor$ implies $h^1(X, L^{\otimes 2} \omega_X^*) \neq 0$ and $h^0(X, L^{\otimes 2} \omega_X^*) = 0$. Hence $h^1(X, \text{Hom}(M, L)) \neq 0$ and $h^0(X, \text{Hom}(M, L)) = 0$. Fix any non-zero extension (1). Since $h^1(X, L) = 0$, $h^0(X, M) = 0$ and hence $h^0(X, E) = h^0(X, L)$. By Remarks 1, 3 and our last assumption, a general extension (1) has simple middle term, concluding the proof. $\square$

Proof of Theorem 2. Take L given by Remark 5. Then take a general $M \in \text{Pic}^{z-d}(X)$. Then use Remark 2 to carry over the proof of Theorem 1. $\square$

REFERENCES

- [1] E. Ballico, On general k -gonal curves, Arch. Math. (Basel) 44 (1985) 336–339.
- [2] A. Bertram and B. Feinberg, On stable rank two vector bundle with canonical determinant and many sections, in Algebraic Geometry, papers for Europroj Conferences in Catania and Barcelona, edited by P. Newstead, 1998, 259–269.
- [3] S. Mukai, Curves and Brill-Noether theory, in Algebraic Geometry, Math. Sci. Res. Inst. Publ. 28, Cambridge University Press, 145–158.
- [4] S. Mukai, Curves and $K3$ surfaces of genus eleven, in Moduli of Vector Bundles, edited by M. Maruyama, 180–197, Marcel Dekker Inc., New York, 1996.
- [5] S. Mukai, Non-Abelian Brill-Noether theory and Fano 3-folds, Sugaku Exp. 14 (2001), n. 2, 125–153.
- [6] M. Teixidor i Bigas, Injectivity of the symmetric map for line bundles, Manuscripta Math. 112 (2003), n. 4, 511–517.
- [7] M. Teixidor i Bigas, Rank two vector bundles with canonical determinant, Math. Nachr. 265 (2004), 100–106.
- [8] M. Teixidor i Bigas, Petri map for rank two vector bundles with canonical determinant, preprint.

Received: December 27, 2006