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Weighted Composition Followed by Differentiation between Bergman Spaces

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Abstract

In this paper we consider linear operators $M_\psi C_\varphi D$ and $M_\psi DC_\varphi$ acting between weighted Bergman spaces, where M_ψ, C_φ and D are multiplication, composition and differentation operators respectively. Our goal is to characterize those holomorphic self maps φ of $\mathbb{D}$ for which $M_\psi DC_\varphi$ and $M_\psi C_\varphi D$ acts boundedly and compactly between weighted Bergman spaces.

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1.Introduction

Thoughout this paper we denote by $H(\mathbb{D})$, the space of holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in the complex plane $\mathbb{C}$. Let ψ and φ be

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holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, we can define a linear operator ψC_φ on $H(\mathbb{D})$, called weighted composition by

$$\psi C_\varphi f = \psi(f \circ \varphi).$$

The operator ψC_φ can be regarded as a generalization of a multiplication operator and a composition operator, In case $\psi \equiv 1$ or $\varphi(z) = z$, ψC_φ reduces to the composition operator C_φ or the multiplication operator M_ψ, respectively. For general back ground on composition operators, we refer [**CoM 95**] and [**Sh 93**] and references therein
Weighted composition operators appear naturally in different contexts. For example, Singh and Sharma [**SiS 79**] related the boundedness of composition operators on Hardy space of the upper half-plane with the boundedness of weighted composition operators on the Hardy space of the open unit disk $\mathbb{D}$. Isometries in many Banach spaces of analytic functions are just weighted composition operators, for example see [**Fo 64**].
Recently, several authors have studied weighted composition operators on different spaces of analytic functions. For example, one can refer to [**CoH 04**] for study of these operators on Hardy spaces, [**Ka 79**] and [**OhT 01**] for disk algebra, [**OhZ 01**] and [**OSZ 03**] for Bloch-type spaces and [**MiS 97**] for study of these operators on Bergman spaces.

In this paper we consider linear operators $M_\psi C_\varphi D$ and $M_\psi DC_\varphi$ acting between weighted Bergman spaces, where M_ψ, C_φ and D are multiplication, composition and differentation operators respectively. Our goal is to characterize those holomorphic self maps φ of $\mathbb{D}$ for which $M_\psi DC_\varphi$ and $M_\psi C_\varphi D$ acts boundedly and compactly between weighted Bergman spaces.

2. Preliminaries

For $\alpha > -1$, the weighted Bergman space A^p_α, is the set of analytic functions on the disk with

$$\| f \|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p d\lambda_\alpha(z) < \infty,$$

where $d\lambda_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and $dA(z) = dxdy/\pi = rdrd\theta/\pi,\ z = x+iy \in \mathbb{D}$. The following sharp estimate tells us how fast an arbitrary function from A^p_α grow near the boundary. Let $f \in A^p_\alpha$. Then for every z in $\mathbb{D}$, we have

$$|f(z)| \leqslant \frac{||f||_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}} \tag{2.1}$$

with equality if and only if f is a constant multiple of the function

$$k_a(z) = \Big(\frac{1-|z|^2}{(1-\overline{a}z)^2}\Big)^{2+\alpha/p}.$$

It can be easily shown that $||k_a||_{A^p_\alpha} \approx 1$ with constant depending only on α and p [**Sm 96**, page 400]. For general background of weighted Bergman spaces A^p_α and weighted Bloch spaces, one may consult [**Zh 90**] and [**HKZ 00**] and the references therein.

In what follows we make extensive use of Carleson measure techniques, so we give a short introduction to Carleson sets and Carleson measures. For a point ζ on the boundary of $\mathbb{D}$ we define the Carleson set

$$S(\zeta, \delta) = \{z \in \mathbb{D} : |\zeta - z| < \delta\}.$$

We use Carleson sets along with a more convinient choice of pseudohyperbolic disks. For $0 < r < 1$ and $a \in \mathbb{D}$, denote by $D(a, r)$, the disk whose pseudohyperbolic center is a and whose pseudohyperbolic radius is r :

$$D(a, r) = \Big\{z \in \mathbb{D} : \Big|\frac{a - z}{1 - \overline{a}z}\Big| < r\Big\}.$$

The notation $|D(a, r)|_A$ will denote the area of $D(a, r)$. For fixed $0 < r < 1$ the area of $D(a, r)$ has the estimation:

$$|D(a, r)|_A \approx (1 - |a|^2)^2 \approx (1 - |z|^2)^2 \approx |D(z, r)|_A, \tag{2.2}$$

for $z \in D(a, r)$, where $\approx$ means that the two quantities are bounded above and below by the constants independent of a. For fixed $0 < r < 1$, it is also known that for $z \in D(a, r)$,

$$|1 - \overline{a}z| \approx (1 - |a|^2). \tag{2.3}$$

Also for each $D(a, r)$, there is a $\zeta \in \partial\mathbb{D}$ so that $D(a, r) \subset S(\delta, \zeta)$ for $\delta \approx 1-|a|$. A positive Borel measure μ on $\mathbb{D}$ is called α-Carleson measure if

$$\sup_{\delta>0} \sup_{\zeta\in\partial\mathbb{D}} \frac{\mu(S(\delta, \zeta))}{\delta^{\alpha+2}} < \infty,$$

and it will be called a vanishing Carleson measure if in addition

$$\lim_{\delta\longrightarrow 0} \sup_{\zeta,\in\delta D} \frac{\mu(S(\zeta, \delta))}{\delta^{\alpha+2}} = 0.$$

3. Boundedness and Compactness of $M_\psi DC_\varphi$

In this section, we characterize those holomorphic self-maps of $\mathbb{D}$ for which $M_\psi DC_\varphi$ maps A^p_α boundedly and compactly into A^q_β.
To do so we need a generalized Nevanlinna counting function, which will be required for the change of variable.

Definition 3.1. *Let φ and ψ be holomorphic self-maps of $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Let $q \geq 1$ and $\beta > -1$. For $w \in \mathbb{D}$, $w \neq 0$, we define*

$$N_{\psi,\varphi}^{q,\beta}(w) = \sum |\psi(z)|^q |\varphi'(z)|^{q-2} \Big(\log \frac{1}{|z|} \Big)^{\beta},$$

where sum extends over all solutions of $\varphi(z) = w$ and we name it generalized Nevanlinna counting function.

We need a generalized change of variable formula. In the following formula, $\{z_j(w)\}$ denote the sequence of zeros of $\varphi(z) - w$ repeated according to multiplicity.

Theorem 3.2. [**CoM 95**] *If g and W are non-negative measurable functions on $\mathbb{D}$, then*

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 W(z) dA(z) = \int_{\varphi(\mathbb{D})} g(\varphi(z)) \Big(\sum_{j \geq 1} W(z_j(w)) \Big) dA(w).$$

We assume from now on that $r \in (0, 1)$ is fixed.

Theorem 3.3. *Let $1 \leqslant p \leqslant q$, and $\alpha, \beta > -1$. Let φ and ψ be a holomorphic maps and $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi\varphi' \in A_{\beta}^{q}$. Let $d\mu(w) = N_{\psi,\varphi}^{q,\beta}(w) dA(w)$. Then the following are equivalent:*

(1) $M_{\psi} DC_{\varphi}$ maps A_{α}^{p} boundedly into A_{β}^{q}.

(2) $\mu(D(a, r)) = O((1 - |a|^2)^{q(\alpha+2+p)/p})$ as $|a| \to 1$.

In order to prove the Theorem 3.3 we need following result of Luecking [**Lu 85**] in which he characterized positive measures μ with the property:

$$\| f^{(n)} \|_{L^q(\mu)} \leqslant C \| f ||_{A_{\alpha}^{p}}.$$

The following result is a special case of the Luecking's result [**Lu 85**, Theorem 2.2] for $n = 1$ in case $1 \leqslant p \leqslant q$.

Theorem 3.4. *Let $1 \leqslant p \leqslant q$, and $\alpha, \beta > -1$. Let μ be a finite positive measure on $\mathbb{D}$. Then the following are equivalent:*

(1) $\| f' \|_{L^q(\mu)} \leqslant C \| f \|_{A_{\alpha}^{p}}$ for all $f \in A_{\alpha}^{p}$.

(2) $\mu(D(a, r)) = O((1 - |a|^2)^{q(\alpha+2+p)/p})$ as $|a| \to 1$.

For the case $1 \leqslant q < p$, Luecking used Khinchine's inequality and other estimates to obtain a version of Theorem 3.4 for $f^{(n)}$, where $f \in A^p$.

We are interested in the case n $= 1$ and $f \in A_{\alpha}^{p}$.

The following result is a slight modification of Luecking's result (see [**HiP 05**] also).

Theorem 3.5. *Let $1 \leqslant q < p$, and $\alpha > -1$. Let μ be a finite positive measure on $\mathbb{D}$. Let $\Omega(z) = (1 - |z|^2)^{-(\alpha+2+q)} \mu(D(z, r))$. Then the following are equivalent:*

(1) $\| f' \|_{L^q(\mu)} \leqslant C \| f \|_{A_{\alpha}^{p}}$ for all $f \in A_{\alpha}^{p}$

(2) $\Omega \in L^{p/p-q}(\nu_\alpha)$.

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. Since $\psi\varphi' \in A^q_\beta$, change of variable formula 3.2 implies that μ is a finite measure. Thus Theorem 3.4 applies.

Note that

$$\begin{aligned} \| M_\psi DC_\varphi f \|^q_{A^q_\beta} &\approx \int_{\mathbb{D}} |f'(\varphi(z))|^q |\psi(z)|^q |\varphi'(z)|^q (\log \frac{1}{|z|})^\beta dA(z) \\ &= \int_{\mathbb{D}} |f'(w)|^q N^{q,\beta}_{\psi,\varphi}(w) dA(w) \\ &= \| f' \|^q_{L^q(\mu)} . \end{aligned}$$

Since $M_\psi DC_\varphi$ maps A^p_α boundedly into A^q_β, so we have

$$\| f' \|^q_{L^q(\mu)} = \| M_\psi DC_\varphi f \|^q_{A^q_\beta} \leqslant C \| f \|_{A^p_\alpha}$$

for all $f \in A^p_\alpha$. Hence Theorem 3.4 implies that

$$\mu(D(a,r)) = O((1-|a|^2)^{q(\alpha+2+p)|p} \text{ as } |a| \to 1.$$

Conversely, suppose that (ii) holds. Then by Theorem 3.4, we have

$$\| M_\psi DC_\varphi f \|_{A^q_\beta} = \| f' \|_{L^q(\mu)} \leqslant C \| f \|_{A^p_\alpha}$$

and hence $M_\psi C_\varphi D$ maps A^p_α boundedly into A^q_β.

Theorem 3.6: *Let $1 \leqslant p \leqslant q$, and $\alpha,\ \beta > -1$. Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi\varphi' \in A^q_\beta$. Let $d\mu(w) = N^{q,\beta}_{\psi,\varphi}(w)dA(w)$. Then the following are equivalent:*

(1) $M_\psi DC_\varphi$ *maps* A^p_α *compactly into* A^q_β.

(2) $\mu(D(a,r)) = o((1-|a|^2)^{q(\alpha+2+p)/p})$ *as* $|a| \to 1$.

Proof: First suppose that $M_\psi DC_\varphi$ maps A^p_α compactly into A^q_β. Let $a \in \mathbb{D}$ and consider the function

$$f_a(z) = \frac{(1-|a|^2)^{(\alpha+2)/p}}{(1-\overline{a}z)^{2(\alpha+2)/p}}.$$

Clearly $\| f_a \|_{A^p_\alpha} \approx 1$ and f_a converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Since $M_\psi DC_\varphi$ maps A^p_α compactly into A^q_β, so for gives $\epsilon > 0$, we can find $r_0, 0 < r_0 < 1$ such that $\| M_\psi DC_\varphi f_a \|_{A^p_\alpha} < \epsilon$ for $|a| > r_0$.

Thus

$$\epsilon > \int_{\mathbb{D}} |f'_a(z)|^q d\mu(z) \geq \int_{D(a,r)} |f'_a(z)|^q d\mu(z)$$

for $|a| > r_0$. Since for $z \in D(a,r)$,

$$|f'_a(z)| \approx \frac{1}{(1-|a|^2)^{(\alpha+2+p)/p}}$$

and so above estimate yields

$$\mu(D(a,r)) < \epsilon((1-|a|^2)^{q(\alpha+2+p)/p}$$

for all a with $|a| > r_0$. Hence

$$\mu(D(a,r)) = o((1-|a|^2)^{q(\alpha+2+p)/p)} \text{ as } |a| \to 1.$$

Conversely, assume that (ii) holds and let $\{f_n\}$ be a sequence in A^p_α such that $||f_n||_{A^q_\alpha} \leqslant M$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. To show that $M_\psi DC_\varphi$ maps A^p_α compactly into A^q_β, it is sufficient to show that

$$\| M_\psi DC_\varphi f_n \|^q_{A^q_\beta} = \| f'_n \|^q_{L^q(\mu)} \to 0 \text{ as } n \to \infty$$

By a standard estimate of Luecking [**Lu 93**], page 338, we have

$$\| M_\psi DC_\varphi f_n \|^q_{A^q_\beta} \leqslant C \int_{\mathbb{D}} \frac{1}{(1-|a|^2)^{2+q}} \int_{D(a,r)} |f_n(z)|^q dA(z) d\mu(a).$$

Note that

$$\chi_{D(a,r)}(z) = \chi_{D(z,r)}(a) \text{ and } 1-|a|^2 \approx 1-|z|^2$$

for $a \in D(z,r)$. Also by 2.1, we have

$$|f_n(z)| \leqslant C \frac{||f_n||_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}.$$

By an application of Fubini's theorem, we get

$$\begin{aligned} \| M_\psi DC_\varphi f_n \|^q_{A^q_\beta} &\leqslant C' \int_{\mathbb{D}} \big|f_n(z)\big|^q \frac{\mu(D(z,r))}{(1-|z|^2)^{2+q}} dA(z) \\ &\leqslant C' ||f_n||^{q-p}_{A^q_\alpha} \int_{\mathbb{D}} \big|f_n(z)\big|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{(\alpha q+2q-pq-\alpha p)/p}} dA(z) \\ &\leqslant C' M^{q-p} \Big(\int_{|z| \leqslant r_0} |f_n(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{(\alpha q+2q+pq-\alpha p)/p}} dA(z) \\ &+ \int_{|z|>r_0} |f_n(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{(\alpha q+2q+pq-\alpha p)/p}} dA(z) \Big) \\ &= I + II. \end{aligned}$$

Now (ii) implies that for a given $\epsilon > 0$, we can find r_0, $0 < r_0 < 1$ such that

$$\begin{aligned} II &= C' M^{q-p} \int_{|z|>r_0} |f_n(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{(\alpha q+2q+pq-\alpha p)/p}} dA(z) \\ &\leqslant \epsilon C' M^{q-p} \int_{|z|>r_0} |f_n(z)|^p (1-|z|^2)^\alpha dA(z) \\ &\leqslant \epsilon C' M^{q-p} \| f_n \|^p_{A^p_\alpha} \\ &\leqslant \epsilon C' M^q. \end{aligned}$$

Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$

$$\begin{aligned} I &= C'M^{q-p}\int_{|z|\leqslant r_0} |f_n(z)|^p \frac{\mu(D(z,r))}{(1-|z|^2)^{(\alpha q+2q+pq-\alpha p)/p}} dA(z) \\ &\leqslant C_1C'M^{q-p}\epsilon \int_{\mathbb{D}} \mu(D(z,r))dA(z) \\ &\leqslant C_1C_2C'M^{q-p}\epsilon \int_{\mathbb{D}} \mu(\mathbb{D})dA(z) \\ &= C_1C_2C_3C'M^{q-p}\epsilon \end{aligned}$$

for n large enough. Thus

$$\lim_{n\to\infty} \| M_\psi DC_\varphi f_n \|^q_{A^q_\beta} = 0,$$

and hence $M_\psi DC_\varphi$ maps A^p_α compactly into A^q_β.

Theorem 3.7 *Let $1 \leqslant p < q$, and $\alpha,\ \beta > -1$. Let $d\mu(z) = N^{q,\beta}_{\psi,\varphi}dA(z)$ and let $\psi\varphi' \in A^q_\beta$. Let $G(z) = (1-|z|^2)^{-(\alpha+q+2)}\mu(D(a,r))$. Then the following are equivalent:*

(1) $M_\psi DC_\varphi$ maps A^p_α boundedly into A^q_β.

(2) $M_\psi DC_\varphi$ maps A^p_α compactly into A^q_β.

(3) $G \in L^{p/(p-q)}(\nu_\alpha)$.

Proof. (1) $\Leftrightarrow$ (3) Suppose (1) holds. By change of variable formula as in Theorem 3.2, we have

$$\| M_\psi DC_\varphi f \|^q_{A^q_\beta} = \| f' \|^q_{L^q(\mu)}$$

Since $M_\psi DC_\varphi$ maps A^p_α boundedly into A^q_β, we can find a positive constant C such that

$$\| f' \|_{L^q(\mu)} = \| M_\psi DC_\varphi f \|_{A^q_\beta} \leqslant C \| f \|_{A^p_\alpha}$$

and so by Theorem 3.4, $M_\psi DC_\varphi$ maps A^p_α boundedly into A^q_β if and only if $G \in L^{p/(p-q)}(\nu_{\alpha)}$.

It is clear that (2) implies (1).

It remains to verify that (3) implies (2). Assume that

$$\| f_n \|_{A^p_\alpha} \leqslant C$$

and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. It is sufficient to show that

$$\lim_{n\to\infty} \| M_\psi DC_\varphi f_n \|_{A^q_\beta} = 0.$$

As in the proof of the Theorem 3.5, we have

$$\| M_\psi DC_\varphi f_n \|_{A^q_\beta} \leqslant C \int_{\mathbb{D}} |f_n(z)|^q G(z) d\nu_\alpha(z).$$

Let $\epsilon > 0$. Then the hypothesis on G implies that there exists r_0, $0 < r_0 < 1$, with the property

$$\int_{|z|>r_0} G^{p/p-q}(z)d\nu_\alpha(z) < \epsilon^{p/p-q}.$$

It follows by Holder's inequality that

$$\begin{aligned}
&\int_{|z|>r_0} |f_n(z)|^q G(z)d\nu_\alpha(z) \\
&\quad\leqslant \Big(\int_{\mathbb{D}} |f_n(z)|^p d\nu_\alpha\Big)^{q/p}\Big(\int_{|z|>r_0} G^{p/p-q} d\nu_\alpha(z)\Big)^{(p-q)/p} \\
&\quad\leqslant \epsilon \parallel f_n \parallel^q A^p_\alpha \\
&\quad\leqslant C\epsilon.
\end{aligned}$$

Thus we have

$$\int_{|z|>r_0} |f_n(z)|^q G(z)d\nu_\alpha(z) \leqslant C\epsilon.$$

Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, so $|f_n(z)| < \epsilon$ for all z such that $|z| < r_0$ and for all $n \geq n_0$. Thus

$$\int_{|z|\leqslant r_0} |f_n(z)|^q G(z)d\nu_\alpha(z) \leqslant \epsilon^q \int_{|z|\leqslant r_0} G(z)d\nu_\alpha(z)$$

for $n \geq n_0$. Since $\psi\varphi' \in A^q_\beta$,

$$G(z) \leqslant C\mu(D(z,r)) \leqslant C\mu(\mathbb{D}) < \infty$$

and thus

$$\int_{|z|\leqslant r_0} G(z)d\nu_\alpha(z) \leqslant C\int_{\mathbb{D}} \mu(D(z,r))d\nu_\alpha(z) \leqslant C.$$

Thus

$$\int_{|z|\leqslant r} |f_n(z)|^q G(z)d\nu_\alpha(z) \leqslant C\epsilon$$

for $n \geq n_0$. Hence $M_\psi DC_\varphi$ maps A^p_α compactly into A^q_β.

4. Boundedness and Compactness of $M_\psi C_\varphi D$

Now we discuss boundedness and compactness of the operator $M_\psi C_\varphi D$ acting between weighted Bergman spaces. Proofs follow exactly on same lines, so we omit the details.

Theorem 4.1. *Let $1 \leqslant p \leqslant q < \infty$, and $\alpha, \beta > -1$. Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\psi\varphi \in A^p_\beta$. Then the following are equivalent:*

(1) $M_\psi C_\varphi D$ maps A^p_α boundedly into A^q_β

(2) $(\mu_\beta \circ \varphi^{-1})(D(a,r)) = O((1-|a|^2)^{q(a+2+p)/p})$ *as* $|a| \to 1$.
where $\mu_\beta \circ \varphi^{-1}$ *is the pull-back measure induced by* ν_β. *Here* $d\mu_\beta(z) = |\psi(z)|^q d\nu_\beta(z)$.

Proof: First suppose that $M_\psi C_\varphi D$ maps A^p_α boundedly into A^q_β. Since $\psi\varphi \in A^q_\beta$, change of variable formula of measure theory (see **[Ha 74]**, Theorem C, page 173) implies that $\mu_\beta \circ \varphi^{-1}$ is a finite measure

$$\begin{aligned}\| M_\psi C_\varphi D f \|^q_{A^q_\beta} &= \int_{\mathbb{D}} |f'(\varphi(z))|^q |\psi(z)|^q d\nu_\beta(z) \\ &= \int_{\mathbb{D}} |f'(z)|^q d(\mu_\beta \circ \varphi^{-1})(z) \\ &= \| f' \|^q_{L^q(\mu_\beta \circ \varphi^{-1})},\end{aligned}$$

Since $M_\psi C_\varphi D$ maps A^p_α boundedly into A^q_β, so we have

$$\begin{aligned}\| f' \|^q_{L^q(\mu)} &= \| M_\psi C_\varphi D f \|^q_{A^q_\beta} \\ &\leqslant C \| f \|^q_{A^q_\beta}\end{aligned}$$

for all $f \in A^q_\beta$. Hence Theorem 4.2.4 implies that

$$(\mu_\beta \circ \varphi^{-1})(D(a,r)) = O((1-|a|^2)^{q(\alpha+2+p)|p} \text{ as } |a| \to 1.$$

Conversely suppose that (ii) holds. Then by Theorem 3.4, we have

$$\| M_\psi C_\varphi D f \|_{A^q_\beta} = \| f' \|_{L^q(\mu_\beta \circ \varphi^{-1})} \leqslant C \| f \|_{A^p_\alpha}$$

and hence $M_\psi C_\varphi D$ maps A^p_α boundedly into A^q_β.

Theorem 4.2. *Let* $1 \leqslant p \leqslant q$, *and* $\alpha, \beta > -1$. *Let* φ *and* ψ *be holomorphic maps on* $\mathbb{D}$ *such that* $\varphi(\mathbb{D}) \subset \mathbb{D}$ *and* $\psi\varphi \in A^q_\beta$. *Then the following are equivalent.*
(1) $M_\psi C_\varphi D$ *maps* A^p_α *compactly into* A^q_β
(2) $(\mu_\beta \circ \varphi^{-1})(D(a,r)) = O((1-|a|^2)^{q(a+2+p)/p})$ *as* $|a| \to 1$.

Theorem 4.3. *Let* $1 \leqslant p < p$, *and* $\alpha, \beta > -1$. *Let* φ *and* ψ *be holomorphic maps on* $\mathbb{D}$ *such that* $\varphi(\mathbb{D}) \subset \mathbb{D}$ *and* $\psi\varphi \in A^q_\beta$ *and let* $\Phi(z) = (1-|z|^2)^{-(\alpha+2+q)}(\mu_\beta \circ \varphi^{-1})(D(z,r))$. *Then the following are equivalent.*
(1) $M_\psi C_\varphi D$ *maps* A^p_α *boundedly into* A^q_β
(2) $M_\psi C_\varphi D$ *maps* A^p_α *compactly into* A^q_β
(3) $\Phi \in L^{p/p-q}(\nu_\alpha)$

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