Interpolation of Triple-Banach Spaces on R^3

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Abstract

Let $\bar{A} = (A_0, A_1, A_2)$ be a compatible triple of Banach spaces. We may define the interpolation method in R^3 , and prove some related lemma and theorem.

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1 Introduction

Our main reference to the theory of interpolation space is [1]. Let $\bar{A} = (A_0, A_1, A_2)$ be a compatible triple of Banach spaces, we define the space $\mathcal{F}(\bar{A})$ of all function f with values in $\sum(\bar{A})$, which are bounded and continuous on the $\bar{S} = \{x = (x_0, x_1, x_2) \in R^3 : 0 \leq x_0 \leq 1\}$ and analytic on the $S = \{x = (x_0, x_1, x_2) \in R^3 : 0 < x_0 < 1\}$. And moreover, the functions $t \longrightarrow f(0, j, t) (j = 0, 1)$ and $t \longrightarrow f(1, 1, t)$ are continuous function from the real line into A_0, A_1 and A_2 , which tend to zero as $|t| \longrightarrow \infty$. Clearly, $\mathcal{F}(\bar{A})$ is a vector space. We provided $\mathcal{F}$ with the norm

$$\|f\|_{\mathcal{F}} = \max(\sup\|f(0, 0, t)\|_{A_0}, \sup\|f(0, 1, t)\|_{A_1}, \sup\|f(1, 1, t)\|_{A_2}).$$

2 Main result

Lemma 2.1 *The space $\mathcal{F}$ is a Banach space.*

Proof.

Suppose that $\sum_n \|f_n\|_{\mathcal{F}} < \infty$. Since $f_n(x)$ is bounded in $\Sigma(\bar{A})$, we have

$$\|f_n(x)\|_{\Sigma(\bar{A})} \leq \max(\sup \|f_n(0, 0, t)\|_{\Sigma(\bar{A})}, \sup \|f_n(0, 1, t)\|_{\Sigma(\bar{A})}, \sup \|f_n(1, 1, t)\|_{\Sigma(\bar{A})}).$$

Since $A_j \subset \Sigma(\bar{A})$, we conclude that

$$\|f_n(x)\|_{\Sigma(\bar{A})} \leq \|f_n\|_{\mathcal{F}}.$$

Since $\Sigma(\bar{A})$ is a Banach space. it follows that $\sum_n f_n$ converges uniformly on S to a function f in $\Sigma(\bar{A})$. Thus f is bounded and continuous on S and analytic in S_0 . Furthermore, $\|f_n(0, j, t)\|_{A_j} \leq \|f_n\|_{\mathcal{F}} (j = 0, 1)$ and $\|f_n(1, 1, t)\|_{A_2} \leq \|f_n\|_{\mathcal{F}}$. Thus $\sum_n f_n(0, j, t)$ and $\sum_n f_n(1, 1, t)$ converges uniformly in t to a limit in A_j and A_2 , which must coincide with the limit in $\Sigma(\bar{A})$. Therefore, $f(0, j, t) \in A_j$, $f(1, 1, t) \in A_2$ and $\sum_n f_n(0, j, t)$, $\sum_n f_n(1, 1, t)$ converges uniformly to $f(0, j, t)$ in A_j and $f(1, 1, t)$ in A_2 . But then it follows that $f = \mathcal{F}$, and that $\sum_n f_n$ converges to f in $\mathcal{F}$.

We now define the interpolation functor $\bar{A}_{[\theta]}$. The space $\bar{A}_{[\theta]}$ consists of all $a \in \Sigma(\bar{A})$ such that $a = f(\theta)((\theta, 0, 0) = \theta)$ for some $f \in \mathcal{F}(\bar{A})$. The norm on $\bar{A}_{[\theta]}$ is

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = a, f \in \mathcal{F}\}.$$

Definition 2.2 *The interpolation space A and B are exact of exponent θ and η , ($0 \leq \theta + \eta \leq 1$) if*

$$\|T\|_{A,B} \leq \|T\|_{A_0,B_0}^{1-(\eta+\theta)} \|T\|_{A_1,B_1}^{\theta} \|T\|_{A_2,B_2}^{\eta}.$$

Theorem 2.3 *The space $\bar{A}_{[\theta]}$ is a Banach space. The functor $\bar{A}_{[\theta]}$ is an exact interpolation functor of exponent θ and η .*

Proof.

The linear mapping $f \rightarrow f(\theta)$ is a continuous mapping from $\mathcal{F}(\bar{A})$ to $\Sigma(\bar{A})$ since $\|f(\theta)\|_{\Sigma(\bar{A})} \leq \|f\|_{\mathcal{F}}$. The kernel of this mapping is $\mathcal{N}_{\theta} = \{f : f \in \mathcal{F}, f(\theta) = 0\}$. Clearly, $\bar{A}_{[\theta]}$ is isomorphic and isometric to the quotient space $\mathcal{F}(\bar{A})/\mathcal{N}_{\theta}$. Since $\mathcal{N}_{\theta}$ is closed, it follows that $\bar{A}_{[\theta]}$ is a Banach space. Assume that T maps A_j to B_j with norm $M_j (j = 0, 1, 2)$. Given $a \in \bar{A}_{[\theta]}$ and $\varepsilon > 0$, there is a function $f \in \mathcal{F}$, such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}} \leq \|a\|_{[\theta]} + \varepsilon$. Put $g(x) = M_0^{x-1} M_1^{-x+\eta} M_2^{-x+\theta} T(f(x))$. g belongs to the class $\mathcal{F}(\bar{B})$. Moreover, $\|g\|_{\mathcal{F}} \leq \|f\|_{\mathcal{F}} \leq \|a\|_{[\theta]} + \varepsilon$. But now $g(\theta+\eta) = M_0^{\theta+\eta-1} M_1^{-\theta} M_2^{-\eta} T(a)$ and hence

we conclude that $\|T(a)\|_{[\theta]} \leq M_0^{1-\theta+\eta} M_1^\theta M_2^\eta \|g\|_{\mathcal{F}} \leq M_0^{1-\theta+\eta} M_1^\theta M_2^\eta \|a\|_{[\theta]} + \varepsilon$, where $\varepsilon = M_0^{1-\theta+\eta} M_1^\theta M_2^\eta \varepsilon$. This gives the result.

Now we may introduce a second interpolation method in R^3 . This is based on a space $\tilde{\mathcal{F}}(\bar{A})$ of analytic functions, defined as follows. The functions g in $\tilde{\mathcal{F}}(\bar{A})$ are defined on S with values in $\Sigma(\bar{a})$. Moreover they have the following properties:

- (i) $\|g\|_{\Sigma(\bar{A})} \leq c(1 + |x|)$, ($|x| = (\sum_{i=0}^2 x_i^2)^{1/2}$)
- (ii) g is continuous on S and analytic on S_0 ,
- (iii) $g(0, j, t_1) - g(0, j, t_2)$ has values in A_j and $g(1, 1, t_1) - g(1, 1, t_2) \in A_2$ for all real values of t_1 and t_2 and for $j = 0, 1$, and

$$\|g\|_{\tilde{\mathcal{F}}} = \max \left(\sup_{t_1, t_2} \left\| \frac{g(0, 0, t_1) - g(0, 0, t_2)}{t_1 - t_2} \right\|_{A_0}, \sup_{t_1, t_2} \left\| \frac{g(0, 1, t_1) - g(0, 1, t_2)}{t_1 - t_2} \right\|_{A_1}, \sup_{t_1, t_2} \left\| \frac{g(1, 1, t_1) - g(1, 1, t_2)}{t_1 - t_2} \right\|_{A_2} \right).$$

is finite.

Lemma 2.4 *The space $\tilde{\mathcal{F}}(\bar{A})$, reduced modulo constant functions and provided with the norm $\|g\|_{\tilde{\mathcal{F}}}$, is a Banach space.*

Proof.

From the conditions it follows easily that if $h \neq 0$ is a real number then

$$\left\| \frac{g(x+h) - g(x)}{h} \right\|_{\Sigma(\bar{A})} \leq \|g\|_{\tilde{\mathcal{F}}} \quad (x \in S).$$

Thus we obtain

$$\|\dot{g}(x)\|_{\Sigma(\bar{A})} \leq \|g\|_{\tilde{\mathcal{F}}}.$$

We therefore see that if $\|g\|_{\tilde{\mathcal{F}}} = 0$ then g is constant. This implies that $\tilde{\mathcal{F}}$ modulo constants is a normed space. We also that $x \in S_0$

$$\|g(x) - g(0)\|_{\Sigma(\bar{A})} \leq |x| \|g\|_{\tilde{\mathcal{F}}}$$

Now suppose that $\sum_n \|g_n\|_{\tilde{\mathcal{F}}} < \infty$. Then $\sum_n (g_n(x) - g_n(0))$ converges uniformly on every compact subset of S_0 . The limit $g(x)$ satisfies (i) and (ii). Moreover it follows that the series $\sum_n (g_n(0, j, t_1) - g_n(0, j, t_2))$ and $\sum_n (g_n(1, 1, t_1) - g_n(1, 1, t_2))$ converges in A_j and A_2 . Thus $g(0, j, t_1) - g(0, j, t_2) \in A_j$, $g(1, 1, t_1) - g(1, 1, t_2) \in A_2$ and is the sum of the series $\sum_n (g_n(0, j, t_1) - g_n(0, j, t_2))$, $\sum_n (g_n(1, 1, t_1) - g_n(1, 1, t_2))$ in A_j and A_2 . Therefore $g \in \tilde{\mathcal{F}}$, i.e. $\tilde{\mathcal{F}}$ is compact.

We now define the space $\bar{A}^{[\theta]}$ in the following way. For a given θ such that $0 < \theta < 1$ we let $\bar{A}^{[\theta]}$ consist of all $a \in \Sigma(\bar{A})$ such that $a = \dot{g}(\theta)$ for some $g \in \tilde{\mathcal{F}}$. The norm on $\bar{A}^{[\theta]}$ is

$$\|a\|^{[\theta]} = \inf \{ \|g\|_{\tilde{\mathcal{F}}} : \dot{g}(\theta) = a, g \in \tilde{\mathcal{F}} \}.$$

Theorem 2.5 *The space $\bar{A}^{[\theta]}$ is a Banach space and functor $\bar{A}^{[\theta]}$ is an exact interpolation functor of exponent θ and η .*

Proof.

Since $\|\dot{g}(\theta)\|_{\Sigma(\bar{A})} \leq \|g\|_{\tilde{\mathcal{F}}}$, we see that the mapping $g \longrightarrow \dot{g}(\theta)$ from $\tilde{\mathcal{F}}$ into $\Sigma(\bar{A})$ is continuous. The kernel $\mathcal{N}^\theta$ of this mapping is closed and the range is $\bar{A}^{[\theta]}$. The norm on $\bar{A}^{[\theta]}$ is the quotient norm on $\tilde{\mathcal{F}}/\mathcal{N}^\theta$. Thus $\bar{A}^{[\theta]}$ is a Banach space.

assume that $T : A_j \longrightarrow B_j$ with norm M_j for $j = 0, 1, 2$. Then we chose a function $g \in \tilde{\mathcal{F}}$, such that $\dot{g}(\theta) = a$, $\|g\|_{\tilde{\mathcal{F}}} \leq \|a\|^{[\theta]} + \varepsilon$. Consider the function

$$h(x) = [M_0^{k+\eta-1} M_1^{-k} M_2^{\theta-\eta-k} T(g(k))]_{k=0}^{k=x} \\ \int_0^x \log M_0/M_1 M_2 M_0^{k+\eta-1} M_1^{-k} M_2^{\theta-\eta-k} T d(g(k)).$$

The integral is taken along any path in S which connects 0 and x . If the path has all its points in S_0 except 0 and possibly x we may integrate by parts. In fact, if $k \in S_0$ we have $d(T(g(k)))/dk = T(\dot{g}(k))$ and $\dot{g}(k)$ is bounded and continuous on S_0 . Thus $d(T(g(k)))/dk$ is continuous on S_0 and has bounded norm in $\Sigma(\bar{B})$. Thus we may integrate by parts, and we have obtain, for any path in S ,

$$h(x) = \int_0^x M_0^{k+\eta-1} M_1^{-k} M_2^{\theta-\eta-k} T d(g(k)),$$

where in general the integral is to be interpreted as a vector-valued Stieltjes integral. It follow that

$$\|h(x)\|_{\Sigma(\bar{B})} \leq c|x|.$$

Next we note that $T(g(0, j, t_1) - g(0, j, t_2)) \in B_j$, $g(1, 1, t_1) - g(01, 1, t_2) \in B_2$ and is a Lipschitz function in B_j . Thus it follows that

$$\|h(0, j, t_1) - h(0, j, t_2)\|_{B_j} \leq M_j^{-1} \int_{t_1}^{t_2} \|T(dg(0, j, t))\|_{B_j},$$

and

$$\|h(1, 1, t_1) - h(1, 1, t_2)\|_{B_2} \leq M_2^{-1} \int_{t_1}^{t_2} \|T(dg(1, 1, t))\|_{B_2}.$$

if $t_1 < t_2$. But the right hand side is bounded by

$$\int_{t_1}^{t_2} \|dg(0, j, t)\|_{A_j} \leq (t_2 - t_1) \|g\|_{\tilde{\mathcal{F}}},$$

and

$$\int_{t_1}^{t_2} \|dg(1, 1, t)\|_{A_2} \leq (t_2 - t_1) \|g\|_{\tilde{\mathcal{F}}}.$$

It follows that

$$\|h\|_{\tilde{\mathcal{F}}}(\bar{B}) \leq \|a\|^{[\theta]} + \varepsilon.$$

Now

$$\acute{h}(\theta) = M_0^{\theta+\eta-1} M_1^{-\theta} M_2^{-\eta} (d/dk T(g(k))) M_0^{\theta+\eta-1} M_1^{-\theta} M_2^{-\eta} T(a).$$

This proves that $T(a) = M_0^{\theta+\eta-1} M_1^{-\theta} M_2^{-\eta} \acute{h}(\theta) \in \bar{B}^{[\theta]}$, and that

$$\|T(a)\|^{[\theta]} \leq M_0^{1-\theta+\eta} M_1^\theta M_2^\eta \|a\|_{[\theta]} + \epsilon.$$

This gives the result.

Theorem 2.6 *Let $\bar{A}$ be a compatible Banach triple and put*

$$X_j = \bar{A}_{[\theta_j]} \quad (0 \leq \theta_j \leq 1; j = 0, 1, 2).$$

Assume that $\Delta(\bar{A})$ is dense in the spaces A_j and $\Delta(\bar{X})$. Then

$$\bar{A}_{[\theta]} \subset \bar{X}_{[\theta]}.$$

where $\theta = (1 - (\alpha + \beta))\theta_0 + \alpha\theta_1 + \beta\theta_2$.

Proof.

First we show that $\|a\|_{\bar{X}_{[\alpha+\beta]}} \leq \|a\|_{\bar{A}_{[\theta]}}$ if $a \in \bar{A}_{[\theta]}$. Take $a \in \bar{A}_{[\theta]}$ then there exists a function $f \in \mathcal{F}(\bar{A})$, such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}} \leq \|a\|_{\bar{A}_{[\theta]}} + \varepsilon$, $\varepsilon > 0$ being arbitrary. Put $f_1(x) = f((1-x)\theta_0 + x(\theta_1 + \theta_2) - \alpha\theta_2 - \beta\theta_1)$. Then $f_1(\alpha + \beta) = a$ and

$$\|f_1\|_{\mathcal{F}(\bar{X})} \leq \|a\|_{\bar{A}_{[\theta]}}$$

This gives $\|a\|_{\bar{X}_{[\theta]}} \leq \|a\|_{\bar{A}_{[\theta]}}$.

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