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On Anti-centro-symmetric Solutions of Quaternion Matrix Equation $AXA^* = B$

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Abstract

By means of quaternion generalized singular value decomposition of quaternion matrices, this paper derives necessary and sufficient conditions that quaternion matrix equation $AXA^* = B$ has an anti-centro-symmetric solution, and obtains a general expression of the anti-centro-symmetric solutions. In addition, an expression of the optimal approximation solution to a given matrix is derived.

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1 Introduction

Let $\mathbf{R}$ be the real number field, $\mathbf{Q}=\mathbf{R}\oplus\mathbf{R}\mathrm{i}\oplus\mathbf{R}\mathrm{j}\oplus\mathbf{R}\mathrm{k}$ the quaternion field, where $\mathrm{ij} = -\mathrm{ji} = \mathrm{k}$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. $\mathrm{F}_r^{m\times n}$ denotes the set of $m \times n$ matrices on a field F rank r. Let $a = a_1 + a_2\mathrm{i} + a_3\mathrm{j} + a_4\mathrm{k} \in\mathbf{Q}$, where $a_t \in\mathbf{R}$, then define $\overline{a} = a_1 - a_2\mathrm{i} - a_3\mathrm{j} - a_4\mathrm{k}$ to be the conjugate of a, $|a| = \sqrt{\overline{a}a} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$. For any $A = (a_{ij}) \in \mathbf{Q}^{m\times n}$, $||A||_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}$ is called Frobenius norm, $A^* = (\overline{a_{ji}})$ represents conjugate transpose of A, and $(AB)^* = B^*A^*$ for any $B = (b_{ij}) \in \mathbf{Q}^{n\times p}$[1]. If A and B are both square matrices, define $A \circ B = (a_{ij}b_{ij})$ to be Hadamard product of them.

Quaternion matrix equations play important roles in both theoretical studies and numerical computations of quaternion application disciplines[2, 3, 4, 5],

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and have been studied by many experts [3, 4, 5]. For quaternion matrix equation

$$AXA^* = B, \tag{1.1}$$

where $A \in \mathbf{Q}^{m\times n}$, $B \in \mathbf{Q}^{m\times m}$ are given matrices and $X \in \mathbf{Q}^{n\times n}$ denotes unknown matrix. In 1996, Wang obtained its Hermitian solution[6], after that, by applying singular value decomposition of quaternion matrices, Liu obtained its general expression of the least-square solutions under the restriction that the solution matrix X is Hemitian or Skew-Hermitian[7]. In this paper, by applying generalized singular value decomposition of quaternion matrices, we derive necessary and sufficient conditions that the equation (1.1) has an anti-centro-symmetric solution, and obtain a general expression of the anti-centro-symmetric solutions. In addition, an expression of the optimal approximation solution to a given matrix is derived.

2 Quaternion anti-centro-symmetric matrix

In this section, we introduce a definition of quaternion anti-centro-symmetric matrix, and give some properties of quaternion anti-centro-symmetric matrices.

Definition 2.1 A quaternion matrix $X = (x_{ij}) \in \mathbf{Q}^{n\times n}$ is called to be a quaternion anti-centro-symmetric matrix if its entries satisfy

$$x_{ij} = -x_{n+1-i,n+1-j}, \quad i, j = 1, 2, \cdots, n.$$

The set of all $n \times n$ anti-centro-symmetric matrices is denoted by $\mathrm{ACS}\mathbf{Q}^{n\times n}$.

Let

$$J_k = \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & \end{bmatrix}_{k\times k}.$$

By direct calculation, the following lemmas are obvious.

Lemma 2.1 *If $n = 2k$,*

$$\mathrm{ACS}\mathbf{Q}^{n\times n} = \left\{ \begin{bmatrix} G & HJ_k \\ -J_kH & -J_kGJ_k \end{bmatrix} | G, H \in \mathbf{Q}^{k\times k} \right\}, \tag{2.1}$$

if $n = 2k + 1$,

$$\mathrm{ACS}\mathbf{Q}^{n\times n} = \left\{ \begin{bmatrix} G & u & HJ_k \\ -v^T & 0 & v^TJ_k \\ -J_kH & -J_ku & -J_kGJ_k \end{bmatrix} | G, H \in \mathbf{Q}^{k\times k}, u, v \in \mathbf{Q}^k \right\}. \tag{2.2}$$

Lemma 2.2 *If* $A \in \mathbf{Q}^{n\times n}$, *then* $A \in \mathrm{ACS}\mathbf{Q}^{n\times n} \Longleftrightarrow A = -J_k A J_k$.

Combing Lemma 2.1 and Lemma 2.2, we have the following result.

Proposition 2.3 *If* $X \in \mathbf{Q}^{n\times n}$, *then*

$$X \in \mathrm{ACS}\mathbf{Q}^{n\times n} \Longleftrightarrow X = P \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix} P^T, \tag{2.3}$$

in which $X_1 \in \mathbf{Q}^{(n-k)\times k}$, $X_2 \in \mathbf{Q}^{k\times(n-k)}$, P *satisfies that if* $n = 2k$, $P = P_1$, *if* $n = 2k+1$, $P = P_2$, *where*

$$P_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ J_k & -J_k \end{bmatrix}, \quad P_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ J_k & 0 & -J_k \end{bmatrix}.$$

Proof First of all, it is easy to verify the following equalities

$$PP^T = P^T P = I_n. \tag{2.4}$$

We only prove the conclusion holds while $n = 2k$ and similarly while $n = 2k+1$.

By Lemma 2.1, let

$$X = \begin{bmatrix} G & HJ_k \\ -J_k H & -J_k G J_k \end{bmatrix}.$$

Then

$$\begin{aligned} P^T X P &= \frac{1}{2} \begin{bmatrix} I_k & J_k \\ I_k & -J_k \end{bmatrix} \begin{bmatrix} G & HJ_k \\ -J_k H & -J_k G J_k \end{bmatrix} \begin{bmatrix} I_k & I_k \\ J_k & -J_k \end{bmatrix} \\ &= \begin{bmatrix} 0 & G-H \\ G+H & 0 \end{bmatrix}. \end{aligned}$$

Let $X_1 = G - H$, $X_2 = G + H$. By (2.4), the necessity holds.

Conersely, for any $X_1 \in \mathbf{Q}^{(n-k)\times k}$, $X_2 \in \mathbf{Q}^{k\times(n-k)}$, let

$$X = P \begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix} P^T.$$

It is easy to verify that $A = -J_k A J_k$, by Lemma 2.2, $A \in \mathrm{ACS}\mathbf{Q}^{n\times n}$.

3 Anti-centro-symmetric Solutions

In this section, by applying the generalized singular value decomposition of quaternion matrices, we derive necessary and sufficient conditions that the equation (1.1) has an anti-centro-symmetric solution, and obtain a general expression of the anti-centro-symmetric solutions.

Theorem 3.1(QGSVD)[8] *If* $A \in \mathbf{Q}^{m\times n}$, $B \in \mathbf{Q}^{p\times n}$ *with* $C^* = (A^*, B^*)$ *and* $\text{rank}(C) = r$, *then there exist unitary matrices* $U \in \mathbf{Q}^{m\times m}$, $V \in \mathbf{Q}^{p\times p}$ *and a nonsingular matrix* $Q \in \mathbf{Q}^{n\times n}$ *such that*

$$U^*AQ = [\Sigma_A, 0], \quad V^*BQ = [\Sigma_B, 0] \tag{3.1}$$

where

$$\Sigma_A = \begin{bmatrix} I_A & & \\ & S_A & \\ & & 0_A \end{bmatrix}_{m\times r}, \quad \Sigma_B = \begin{bmatrix} 0_B & & \\ & S_B & \\ & & I_B \end{bmatrix}_{p\times r} \tag{3.2}$$

and $t = r - \text{rank}(B)$, $s = \text{rank}(A) + \text{rank}(B) - r$,

$$S_A = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_s), \quad S_B = \text{diag}(\beta_1, \beta_2, \cdots, \beta_s)$$

$$0 < \alpha_s \le \cdots \le \alpha_2 \le \alpha_1 < 1, \quad 0 < \beta_1 \le \beta_2 \le \cdots \le \beta_s < 1$$

$$\alpha_i^2 + \beta_i^2 = 1, \quad i = 1, 2, \cdots, s.$$

Suppose that the equation (1.1) has an anti-centro-symmetric solution X, by Proposition 2.3, let

$$X = P \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} P^T, \tag{3.3}$$

where $M \in \mathbf{Q}^{(n-k)\times k}$, $N \in \mathbf{Q}^{k\times(n-k)}$ and P is the same as that in (2.3).

Then the equation (1.1) can be replaced by

$$AP \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} P^T A^* = B. \tag{3.4}$$

Let $AP = [A_1, A_2]$, where $A_1 \in \mathbf{Q}^{m\times(n-k)}$, $A_2 \in \mathbf{Q}^{m\times k}$. By (2.4), (3.4) is equivalent to

$$A_1 M A_2^* + A_2 N A_1^* = B. \tag{3.5}$$

If A_1^*, A_2^* are corresponding to A and B in Theorem 3.1, respectively, the QGSVD of matrix pair (A_1^*, A_2^*) is as follows

$$U^*A_1^*Q = [\Sigma_{A_1^*}, 0], \quad V^*A_2^*Q = [\Sigma_{A_2^*}, 0], \tag{3.6}$$

where $\operatorname{rank}(P^T A^*) = r$, $Q \in \mathbf{Q}^{m\times m}$, $U \in \mathbf{Q}^{(n-k)\times(n-k)}$, $V \in \mathbf{Q}^{k\times k}$, and $\Sigma_{A_1^*}$, $\Sigma_{A_2^*}$ are corresponding to those in (3.2), then (3.5) is equivalent to

$$\begin{bmatrix} \Sigma_{A_1^*}^* \\ 0 \end{bmatrix} U^*MV[\Sigma_{A_2^*}, 0] + \begin{bmatrix} \Sigma_{A_2^*}^* \\ 0 \end{bmatrix} V^*NU[\Sigma_{A_1^*}, 0] = Q^*BQ. \tag{3.7}$$

Let partitioned forms of matrices U^*MV, V^*NU, Q^*BQ, respectively, be as follows

$$U^*MV = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, \quad V^*NU = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix},$$

$$Q^*BQ = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24}, \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}.$$

Then (3.7) is equivalent to

$$\begin{bmatrix} 0 & M_{12}S_{A_2^*} & M_{13} & 0 \\ S_{A_2^*}N_{21} & S_{A_2^*}N_{22}S_{A_1^*} + S_{A_1^*}M_{22}S_{A_2^*} & S_{A_1^*}M_{23} & 0 \\ N_{31} & N_{32}S_{A_1^*} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24}, \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix},$$

so we have

$$\begin{cases} M_{12} = B_{12}S_{A_2^*}^{-1}, \quad M_{13} = B_{13}, \quad M_{23} = S_{A_1^*}^{-1}B_{23}, \quad N_{21} = S_{A_2^*}^{-1}B_{21}, \\ M_{22} = S_{A_1^*}^{-1}(B_{22} - S_{A_2^*}N_{22}S_{A_1^*})S_{A_2^*}^{-1}, \quad N_{32} = B_{32}S_{A_1^*}^{-1}, \quad N_{31} = B_{31}, \\ B_{11} = B_{33} = 0, \quad B_{i4} = B_{4j} = 0, \quad i, j = 1, 2, 3, 4. \end{cases} \tag{3.8}$$

So

$$M = U \begin{bmatrix} M_{11} & B_{12}S_{A_2^*}^{-1} & B_{13} \\ M_{21} & S_{A_1^*}^{-1}(B_{22} - S_{A_2^*}N_{22}S_{A_1^*})S_{A_2^*}^{-1} & S_{A_1^*}^{-1}B_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} V^*, \tag{3.9}$$

$$N = V \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ S_{A_2^*}^{-1}B_{21} & N_{22} & N_{23} \\ B_{31} & B_{32}S_{A_1^*}^{-1} & N_{33} \end{bmatrix} U^*. \tag{3.10}$$

Therefore the anti-centro-symmetric solutions of (1.1) are (3.3), where M, N are (3.9) and (3.10), respectively, and N_{22}, M_{i1}, M_{3j}, N_{i3}, N_{1j} $(i, j = 1, 2, 3)$ are arbitrary quaternion matrices.

From the statements above, we derive following theorem.

Theorem 3.2 *Let $A \in \mathbf{Q}^{m\times n}$, $B \in \mathbf{Q}^{m\times m}$ be given matrices, P is the same as that in* (2.3), *for $AP = [A_1, A_2]$, the* QGSVD *of matrix pair (A_1^*, A_2^*) is the same as those in* (3.6). *Then the quaternion matrix equation* (1.1) *has an anti-centro-symmetric solution if and only if* (3.8) *holds, where M, N are* (3.9) *and* (3.10), *respectively, and N_{22}, M_{i1}, M_{3j}, N_{i3}, N_{1j} $(i, j = 1, 2, 3)$ are arbitrary quaternion matrices, in which case, the anti-centro-symmetric solutions are given by* (3.3).

4 Optimal Approximation Solution

The set of anti-centro-symmetric solutions of (1.1) is denoted by Ψ. In this section, we consider a following problem under the condition that Ψ is nonempty.

Given a matrix $\widetilde{X} \in \mathbf{Q}^{n\times n}$, we find a matrix $X \in \Psi$ such that

$$||X - \widetilde{X}||_{\mathrm{F}} = \min. \tag{4.1}$$

First of all, we give a Lemma as follows.

Lemma 4.1 *Let $0 < a \in \mathbf{R}$, b_1, $b_2 \in \mathbf{Q}$. Then there exists a unique $x \in \mathbf{Q}$ such that*

$$|x - b_1|^2 + |ax - b_2|^2 = \min, \tag{4.2}$$

and x can be expressed as

$$x = \frac{b_1 + ab_2}{1 + a^2}. \tag{4.3}$$

Proof Let $x = x_1 + x_2\mathrm{i} + x_3\mathrm{j} + x_4\mathrm{k}$, $b_m = b_{m1} + b_{m2}\mathrm{i} + b_{m3}\mathrm{j} + b_{m4}\mathrm{k} (m = 1, 2)$, and

$$\begin{aligned} \varphi(x_1, x_2, x_3, x_4) &= |x - b_1|^2 + |ax - b_2|^2 \\ &= (1 + a^2)\sum_{i=1}^{4} x_i^2 + \sum_{m=1}^{2}\sum_{i=1}^{4} b_{mi}^2 - 2\sum_{i=1}^{4} b_{1i}x_i - 2a\sum_{i=1}^{4} b_{2i}x_i. \end{aligned}$$

Then our conclusion holds by

$$\frac{\partial\varphi(x_1, x_2, x_3, x_4)}{\partial x_i} = 0, \quad i = 1, 2, 3, 4.$$

Theorem 4.1 *Let M, $N \in \mathbf{Q}^{n\times n}$, $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$, $0 < d_i \in \mathbf{R}$ $(i = 1, 2, \cdots, n)$. Then there exists a unique matrix $X \in \mathbf{Q}^{n\times n}$ such that*

$$||X - M||_{\mathrm{F}}^2 + ||DXD^{-1} - N||_{\mathrm{F}}^2 = \min, \tag{4.4}$$

and X can be expressed as

$$X = K \circ (MD^2 + DND), \tag{4.5}$$

where $K = (k_{ij}) \in \mathbf{R}^{n\times n}$, $k_{ij} = \frac{1}{d_i^2+d_j^2}$, $i, j = 1, 2, \cdots, n$.

Proof Let $M = (m_{ij}) \in \mathbf{Q}^{n\times n}$, $N = (n_{ij}) \in \mathbf{Q}^{n\times n}$, $X = (x_{ij}) \in \mathbf{Q}^{n\times n}$, we have

$$||X - M||_{\mathrm{F}}^2 + ||DXD^{-1} - N||_{\mathrm{F}}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(|x_{ij} - m_{ij}|^2 + |\frac{d_i x_{ij}}{d_j} - n_{ij}|^2).$$

So

$$||X - M||_{\mathrm{F}}^2 + ||DXD^{-1} - N||_{\mathrm{F}}^2 = \min \Longleftrightarrow |x_{ij} - m_{ij}|^2 + |\frac{d_i x_{ij}}{d_j} - n_{ij}|^2 = \min,$$

by Lemma 4.1,

$$x_{ij} = \frac{d_j^2 m_{ij} + d_i d_j n_{ij}}{d_i^2 + d_j^2}, \quad i, j = 1, 2, \cdots, n,$$

therefore (4.5) holds and X is unique.

Let partitioned form of matrix $\widetilde{X}$ be

$$\widetilde{X} = P \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{bmatrix} P^T, \tag{4.6}$$

where

$$\widetilde{X}_{12} = U \begin{bmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} & \widetilde{M}_{13} \\ \widetilde{M}_{21} & \widetilde{M}_{22} & \widetilde{M}_{23} \\ \widetilde{M}_{31} & \widetilde{M}_{32} & \widetilde{M}_{33} \end{bmatrix} V^*, \quad \widetilde{X}_{21} = V \begin{bmatrix} \widetilde{N}_{11} & \widetilde{N}_{12} & \widetilde{N}_{13} \\ \widetilde{N}_{21} & \widetilde{N}_{22} & \widetilde{N}_{23} \\ \widetilde{N}_{31} & \widetilde{N}_{32} & \widetilde{N}_{33} \end{bmatrix} U^*. \tag{4.7}$$

For any $A \in \mathbf{Q}^{m\times n}$, $||A||_{\mathrm{F}}$ is a unitarily invariant norm[7]. From (3.9), (3.10) and (4.7) we know that the problem (4.1) is equivalent to

$$\begin{cases} ||M_{11} - \widetilde{M}_{11}||_{\mathrm{F}}^2 = \min, & ||N_{11} - \widetilde{N}_{11}||_{\mathrm{F}}^2 = \min, \\ ||M_{21} - \widetilde{M}_{21}||_{\mathrm{F}}^2 = \min, & ||N_{12} - \widetilde{N}_{12}||_{\mathrm{F}}^2 = \min, \\ ||M_{31} - \widetilde{M}_{31}||_{\mathrm{F}}^2 = \min, & ||N_{13} - \widetilde{N}_{13}||_{\mathrm{F}}^2 = \min, \\ ||M_{32} - \widetilde{M}_{32}||_{\mathrm{F}}^2 = \min, & ||N_{23} - \widetilde{N}_{23}||_{\mathrm{F}}^2 = \min, \\ ||M_{33} - \widetilde{M}_{33}||_{\mathrm{F}}^2 = \min, & ||N_{33} - \widetilde{N}_{33}||_{\mathrm{F}}^2 = \min, \end{cases} \tag{4.8}$$

and

$$||S_{A_1^*}^{-1}(B_{22} - S_{A_2^*}N_{22}S_{A_1^*})S_{A_2^*}^{-1} - \widetilde{M}_{22}||_{\mathrm{F}}^2 + ||N_{22} - \widetilde{N}_{22}||_{\mathrm{F}}^2 = \min. \tag{4.9}$$

From (4.8), we have

$$\begin{cases} M_{11} = \widetilde{M}_{11}, \quad M_{21} = \widetilde{M}_{21}, \quad M_{31} = \widetilde{M}_{31}, \quad M_{32} = \widetilde{M}_{32}, \quad M_{33} = \widetilde{M}_{33}, \\ N_{11} = \widetilde{N}_{11}, \quad N_{12} = \widetilde{N}_{12}, \quad N_{13} = \widetilde{N}_{13}, \quad N_{23} = \widetilde{N}_{23}, \quad N_{33} = \widetilde{N}_{33}. \end{cases} \tag{4.10}$$

For (4.9), note that $S_{A_1^*}^{-1} S_{A_2^*} = (S_{A_1^*} S_{A_2^*}^{-1})^{-1} \in \mathbf{R}^{s\times s}$, $N_{22} \in \mathbf{Q}^{s\times s}$, then by Theorem 4.1, we have

$$N_{22} = K \circ (\widetilde{N}_{22}(S_{A_1^*}^{-1} S_{A_2^*})^2 + S_{A_1^*}^{-1} S_{A_2^*}(\widetilde{M}_{22} - S_{A_1^*}^{-1} B_{22} S_{A_2^*}^{-1}) S_{A_1^*}^{-1} S_{A_2^*}), \tag{4.11}$$

where $K = (k_{ij}) \in \mathbf{R}^{s\times s}$, $k_{ij} = \frac{\alpha_i^2 \alpha_j^2}{\alpha_i^2 \beta_j^2 + \alpha_j^2 \beta_i^2}$, $i, j = 1, 2, \cdots, s$.

From the statements above, we have

Theorem 4.2 *Given a matrix $\widetilde{X} \in \mathbf{Q}^{n\times n}$, if the set Ψ of solutions of the equation* (1.1) *is nonempty, then Problem* (4.1) *has a unique solution $\widehat{X} \in \Psi$, furthermore, if $\widetilde{X}$ is denoted by* (4.6), *then $\widehat{X}$ can be given by* (3.3), *where M, N are* (3.9) *and* (3.10), *respectively, the unknown entries in them are given by* (4.10) *and* (4.11).

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