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Chromatic Numbers in Some Graphs

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Abstract

Let $G = (V, E)$ be a graph. A k-coloring of a graph G is a labeling $f : V(G) \to T$, where $\mid T \mid= k$ and it is proper if the adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable. Here we study chromatic numbers in some kinds of Harary graphs.

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1 Introduction

A k-coloring of a graph G is a labeling $f : V(G) \to T$, where $\mid T \mid= k$ and it is proper if the adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable.[5]

One of the parameters related to chromatic number is defining number. Though we do not study it here but the interested reader can see [1,2,3,4].

Given $k \leq n$, place n vertices around a circle, equally spaced. If k is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle. If k is odd and n is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{k,n}$ is k-regular. When k and n are both odd, index the vertices by the integers modulo n. Construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$, [1].

It is clear that $H_{2,n} = C_n$ and $H_{m,m+1} = K_{m+1}$. And so $\chi(H_{2,2n}) = 2$, $\chi(H_{2,2n+1}) = 3$, $\chi(H_{m,m+1}) = m + 1$.

2 Main Results

We new calculate chromatic numbers for some Harary graphs.

Lemma 1 i. *Let $H = H_{2m,n}$ or $H_{2m+1,n}$, with $m \geq 2$. Therefore,*

$$\chi(H) \geq \begin{cases} m+2, & \text{if } m+1 \nmid n \\ m+1, & \text{if } m+1 \mid n \end{cases},$$

ii. $\chi(H_{3,2n}) \geq \chi(C_{n+1})$, $(n \geq 1)$,
iii. $\chi(H_{3,2n+1}) \geq 3$.

Proof. i. Since every $m+1$ consecutive vertices introduce a complete subgraph K_{m+1}, so $\chi(H) \geq \chi(K_{m+1}) = m+1$. But in this graphs the coloring function with $m+1$ colors is a congruence function to modulo $m+1$, therefore if $m+1 \nmid n$, then $\chi(H) \geq m+2$.
ii. Suppose that $V(H_{3,2n}) = \{1, 2, 3, ..., 2n\}$. Since the set of vertices $\{1, 2, ..., n+1\}$ make a cycle with length n+1, so $\chi(H_{3,2n}) \geq \chi(C_{n+1})$, $n \geq 1$.
iii. Let $V(H_{3,2n+1}) = \{1, 2, 3, ..., 2n+1\}$. Since the set of vertices $\{1, n+1, 2n+1\}$ makes C_3, therefore $\chi(H_{3,2n+1}) \geq 3$. ■

Theorem 2 *For every $n = (m+1)k + r$, where $k \geq r$, we have*

$$\chi(H_{2m,n}) = \begin{cases} m+1, & \text{if } r = 0 \\ m+2, & \text{otherwise} \end{cases}.$$

Proof. We have $n = (m+1)(k-r) + r(m+2)$, when $k \geq r$. If $m+1 \mid n$, then the coloring function f to modulo $m+1$, i.e.,

$$f(i) = j, \quad i \equiv j \pmod{m+1}, \ (1 \leq i \leq n),$$

is a proper coloring. So $\chi = m+1$, by Lemma 1. Now, we suppose that $m+1 \nmid n$. The coloring function f with criterion

$$\begin{aligned} f(i) = j, \quad i \equiv j \pmod{m+1}, \ 1 \leq i \leq (k-1)(m+1), \\ f(i) = j, \quad i \equiv j \pmod{m+2}, \ (k-1)(m+1)+1 \leq i \leq n, \end{aligned}$$

and Lemma 1 imply $\chi = m+2$. ■
We know that if $n \geq m(m+1)$, then

$$\chi(H_{2m,n}) = \begin{cases} m+1, & \text{if } m+1 \mid n \\ m+2, & \text{otherwise} \end{cases},$$

by [5]. Now, we determine $\chi(H_{2m,n})$ for some $n < m(m+1)$.

Theorem 3 *If $n = m(m+1) - i$ for $2 \le i \le m$ and $m \ge \lceil (1+\sqrt{4i+5})/2 \rceil$, then $\chi(H_{2m,n}) = m+2$.*

Proof. The assumptions $n = (i-2)(m+1) + (m-i+1)(m+2)$ and $n \ge 2m+1$ imply $m^2 - m - i - 1 \ge 0$. Since $m+1 \nmid n$ therefore $\chi \ge m+2$. Now, the proper coloring function f with criterion

$$\begin{aligned} &f(i) = j, \quad i \equiv j \pmod{m+1}, \quad 1 \le i \le (i-2)(m+1), \\ &f(i) = j, \quad i \equiv j \pmod{m+2}, \quad (i-2)(m+1)+1 \le i \le n, \end{aligned}$$

implies $\chi(H_{2m,n}) = m+2$. ■

Theorem 4 *For every $n \ge m+1$, we have*

$$\chi(H_{2m+1,2n}) = \begin{cases} m+1, & \text{if } 2n = (m+1)t \text{ and } t \text{ is odd} \\ m+2, & \text{if } 2n = (m+2)t,\ t \text{ is odd and } m+1 \nmid t \end{cases}.$$

Proof. Suppose that $2n = (m+1)t$ and t is odd. The coloring function f to modulo $m+1$ is a proper coloring. Because $n \not\equiv 0 \pmod{m+1}$ concludes $f(i+n) \ne f(i)$ for $1 \le i \le n$. Therefore $\chi(H_{2m+1,2n}) = m+1$.

Now, we suppose that $2n = (m+2)t$, t is odd and $m+1 \nmid t$. The coloring function f to modulo $m+2$ is a proper coloring. Because $n \not\equiv 0 \pmod{m+2}$ concludes $f(i+n) \ne f(i)$ for $1 \le i \le n$. And since $m+1 \nmid 2n$ so $\chi = m+2$ by Lemma 1. ■

As an immediately result, we have.

Corollary 5 *If $2n = (m+1)(m+2)t$ such that both of which t and m are odd, then $\chi(H_{2m+1,2n}) = m+1$.*

If $n = (m+1)k + r$ where $1 \le r \le m$, then

$$\chi(H_{2m+1,2n}) = m+1 \Leftrightarrow m \text{ is odd and } r = (m+1)/2.$$

Proof. We know $\chi \ge m+1$. But $\chi = m+1$ if and only if $n \not\equiv 0 \pmod{m+1}$ and $f(2n) = m+1$. Since $n \not\equiv 0 \pmod{m+1}$ and $f(2n) = f(2r)$, so $\chi = m+1$ if and only if $2r = m+1$. ■

Obviously, $\chi(H_{2m+1,2n}) \ge m+2$ if m is even or if m is odd and $r \ne (m+1)/2$, by the Proposition 7.

Theorem 6 *For every $n \ge 2$, $\chi(H_{3,2n+1}) = 3$.*

Proof. Let $V = \{1, 2, 3, ..., 2n+1\}$, we know $\chi \geq 3$, by Lemma 1. We, now, consider three cases as follows.

Case 1. $2n+1 = 4k+5$, $k \geq 0$, $3 \mid n$.

The coloring function f with criterion

$$f(i) = \delta_i, \text{ where } \delta_i \equiv \begin{cases} i \pmod 3, & 1 \leq i \leq n \\ i+1 \pmod 3, & n+1 \leq i \leq 2n \end{cases} \text{ and } 1 \leq \delta_i \leq 3,$$

$$f(2n+1) = 3,$$

is a proper coloring with three colors. Because $f(n+1) = 2 \neq f(2n+1)$ and since $3 \mid n$ therefore $f(i+n) = f(i+1) \neq f(i)$, $1 \leq i \leq n$.

Case 2. $2n+1 = 4k+5$, $k \geq 0$, $3 \nmid n = 2k+2$.

Let $n = 3t + r$, $1 \leq r \leq 2$. The coloring function f to modulo 3 is a proper coloring. Because $f(i+n) = f(i+r) \neq f(i)$ for $1 \leq i \leq n$. Also, $f(n+1) = r+1 \neq f(2r+1) = f(2n+1)$.

Case 3. $2n+1 = 4k+7$, $k \geq 0$.

The coloring function f with criterion

$$f(i) = j, \quad i \equiv j \pmod 2, \quad 1 \leq i \leq 2n, \ 1 \leq j \leq 2, \quad f(2n+1) = 3$$

is a proper coloring with three colors. Because $f(i+n) = f(i+1) \neq f(i)$ for $1 \leq i \leq n$ and $f(n+1) = 2 \neq f(2n+1)$. So, we conclude $\chi(H_{3,2n+1}) = 3$, for $n \geq 2$. ■

Theorem 7 *If $m+1 \nmid n$. Then*

$$\chi(H_{2m+1,2n+1}) = \begin{cases} m+2, & m \text{ is even, } n \equiv \frac{m}{2} \ (mod\ m+2),\ n \not\equiv \frac{m}{2} \ (mod\ m+1) \\ m+2, & m \text{ is odd, } n \equiv \frac{m+1}{2} \ (mod\ m+2) \\ m+1, & m \text{ is even, } n \equiv \frac{m}{2} \ (mod\ m+1) \end{cases}.$$

Proof. In the two first cases, $m+1 \nmid 2n+1$ so $\chi(H_{2m+1,2n+1}) \geq m+2$. Now, it is clear that the coloring function to modulo $m+2$ is proper. So, $\chi = m+2$. But in the last case, the coloring function to modulo $m+1$ is proper and so $\chi = m+1$. ■

We note that if $m \geq 3$, then $3m+2 < m(m+1)$ and since $m+1 \nmid n = 3m+2$ so $\chi \geq m+2$. But if f is a coloring function with $m+2$ colors, then $f(i) = i$, $1 \leq i \leq m+1$. For the remained vertices the coloring classes are

$$cl(m+i) = \{m+2\} \cup \{\ j \mid 1 \leq j \leq i-1\}, \text{ for } i = 2, 3, ..., m+2,$$

$$cl(2m+i) = \{j \mid i-1 \leq j \leq m+2\}, \text{ for } i = 3, 4, ..., m+2.$$

We end this note with th following question.

Question. If $m \geq 3$, then one can say $\chi(H_{2m,3m+2}) \neq m+2$?

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