Chromatic Numbers in Some Graphs

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Abstract

Let $G = (V,E)$ be a graph. A $k$-coloring of a graph $G$ is a labeling $f : V(G) \rightarrow T$, where $|T| = k$ and it is proper if the adjacent vertices have different labels. A graph is $k$-colorable if it has a proper $k$-coloring. The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k$-colorable.

Here we study chromatic numbers in some kinds of Harary graphs.

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1 Introduction

A $k$-coloring of a graph $G$ is a labeling $f : V(G) \rightarrow T$, where $|T| = k$ and it is proper if the adjacent vertices have different labels. A graph is $k$-colorable if it has a proper $k$-coloring. The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k$-colorable.[5]

One of the parameters related to chromatic number is defining number. Though we do not study it here but the interested reader can see [1,2,3,4].

Given $k \leq n$, place $n$ vertices around a circle, equally spaced. If $k$ is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle. If $k$ is odd and $n$ is even, form $H_{k,n}$ by making each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction and to the diametrically opposite vertex. In each case, $H_{k,n}$ is $k$-regular. When $k$ and $n$ are both odd, index the vertices by the integers modulo $n$. Construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$, [1].

It is clear that $H_{2,n} = C_n$ and $H_{m,m+1} = K_{m+1}$. And so $\chi(H_{2,n}) = 2$, $\chi(H_{2,2n+1}) = 3$, $\chi(H_{m,m+1}) = m + 1$. 


2 Main Results

We new calculate chromatic numbers for some Harary graphs.

**Lemma 1 i.** Let $H = H_{2m,n}$ or $H_{2m+1,n}$, with $m \geq 2$. Therefore,

$$\chi(H) \geq \begin{cases} m + 2, & \text{if } m + 1 \nmid n \\ m + 1, & \text{if } m + 1 \mid n \end{cases}.$$  

**ii.** $\chi(H_{3,2n}) \geq \chi(C_{n+1})$, $(n \geq 1)$,

**iii.** $\chi(H_{3,2n+1}) \geq 3$.

**Proof.** i. Since every $m + 1$ consecutive vertices introduce a complete subgraph $K_{m+1}$, so $\chi(H) \geq \chi(K_{m+1}) = m + 1$. But in this graphs the coloring function with $m + 1$ colors is a congruence function to modulo $m + 1$, therefore if $m + 1 \nmid n$, then $\chi(H) \geq m + 2$.

ii. Suppose that $V(H_{3,2n}) = \{1, 2, 3, ..., 2n\}$. Since the set of vertices $\{1, 2, ..., n+1\}$ make a cycle with length $n+1$, so $\chi(H_{3,2n}) \geq \chi(C_{n+1})$, $n \geq 1$.

iii. Let $V(H_{3,2n+1}) = \{1, 2, 3, ..., 2n + 1\}$. Since the set of vertices $\{1, n + 1, 2n + 1\}$ makes $C_3$, therefore $\chi(H_{3,2n+1}) \geq 3$.

**Theorem 2** For every $n = (m + 1)k + r$, where $k \geq r$, we have

$$\chi(H_{2m,n}) = \begin{cases} m + 1, & \text{if } r = 0 \\ m + 2, & \text{otherwise} \end{cases}.$$  

**Proof.** We have $n = (m + 1)(k - r) + r(m + 2)$, when $k \geq r$. If $m + 1 \mid n$, then the coloring function $f$ to modulo $m + 1$, i.e.,

$$f(i) = j, \quad i \equiv j \pmod{m + 1}, \quad (1 \leq i \leq n),$$

is a proper coloring. So $\chi = m + 1$, by Lemma 1. Now, we suppose that $m + 1 \nmid n$. The coloring function $f$ with criterion

$$f(i) = j, \quad i \equiv j \pmod{m + 1}, \quad 1 \leq i \leq (k - 1)(m + 1),$$

$$f(i) = j, \quad i \equiv j \pmod{m + 2}, \quad (k - 1)(m + 1) + 1 \leq i \leq n,$$

and Lemma 1 imply $\chi = m + 2$. □

We know that if $n \geq m(m + 1)$, then

$$\chi(H_{2m,n}) = \begin{cases} m + 1, & \text{if } m + 1 \mid n \\ m + 2, & \text{otherwise} \end{cases},$$

by [5]. Now, we determine $\chi(H_{2m,n})$ for some $n < m(m + 1)$. 

Theorem 3 If \( n = m(m + 1) - i \) for \( 2 \leq i \leq m \) and \( m \geq \lceil (1 + \sqrt{4i + 5})/2 \rceil \), then \( \chi(H_{2m,n}) = m + 2 \).

Proof. The assumptions \( n = (i - 2)(m + 1) + (m - i + 1)(m + 2) \) and \( n \geq 2m + 1 \) imply \( m^2 - m - i - 1 \geq 0 \). Since \( m + 1 \nmid n \) therefore \( \chi \geq m + 2 \). Now, the proper coloring function \( f \) with criterion

\[
\begin{align*}
f(i) = j, & \quad i \equiv j \pmod{m + 1}, \quad 1 \leq i \leq (i - 2)(m + 1), \\
f(i) = j, & \quad i \equiv j \pmod{m + 2}, \quad (i - 2)(m + 1) + 1 \leq i \leq n,
\end{align*}
\]

implies \( \chi(H_{2m,n}) = m + 2 \). \( \blacksquare \)

Theorem 4 For every \( n \geq m + 1 \), we have

\[
\chi(H_{2m+1,2n}) = \begin{cases} 
  m + 1, & \text{if } 2n = (m + 1)t \text{ and } t \text{ is odd} \\
  m + 2, & \text{if } 2n = (m + 2)t, \text{ } t \text{ is odd and } m + 1 \nmid t
\end{cases}
\]

Proof. Suppose that \( 2n = (m + 1)t \) and \( t \) is odd. The coloring function \( f \) to modulo \( m + 1 \) is a proper coloring. Because \( n \not\equiv 0 \pmod{m + 1} \) concludes \( f(i + n) \neq f(i) \) for \( 1 \leq i \leq n \). Therefore \( \chi(H_{2m+1,2n}) = m + 1 \).

Now, we suppose that \( 2n = (m + 2)t \), \( t \) is odd and \( m + 1 \nmid t \). The coloring function \( f \) to modulo \( m + 2 \) is a proper coloring. Because \( n \not\equiv 0 \pmod{m + 2} \) concludes \( f(i + n) \neq f(i) \) for \( 1 \leq i \leq n \). And since \( m + 1 \nmid 2n \) so \( \chi = m + 2 \) by Lemma 1. \( \blacksquare \)

As an immediately result, we have.

Corollary 5 If \( 2n = (m + 1)(m + 2)t \) such that both of which \( t \) and \( m \) are odd, then \( \chi(H_{2m+1,2n}) = m + 1 \).

If \( n = (m + 1)k + r \) where \( 1 \leq r \leq m \), then

\[
\chi(H_{2m+1,2n}) = m + 1 \iff m \text{ is odd and } r = (m + 1)/2.
\]

Proof. We know \( \chi \geq m + 1 \). But \( \chi = m + 1 \) if and only if \( n \not\equiv 0 \pmod{m + 1} \) and \( f(2n) = m + 1 \). Since \( n \not\equiv 0 \pmod{m + 1} \) and \( f(2n) = f(2r) \), so \( \chi = m + 1 \) if and only if \( 2r = m + 1 \). \( \blacksquare \)

Obviously, \( \chi(H_{2m+1,2n}) \geq m + 2 \) if \( m \) is even or if \( m \) is odd and \( r \neq (m + 1)/2 \), by the Proposition 7.

Theorem 6 For every \( n \geq 2 \), \( \chi(H_{3,2n+1}) = 3 \).
**Proof.** Let $V = \{1, 2, 3, ..., 2n + 1\}$, we know $\chi \geq 3$, by Lemma 1. We, now, consider three cases as follows.

**Case 1.** $2n + 1 = 4k + 3$, $k \geq 0$, $3 \mid n$.

The coloring function $f$ with criterion

$$f(i) = \delta_i, \text{ where } \delta_i \equiv \begin{cases} i \pmod{3}, & 1 \leq i \leq n \\ i + 1 \pmod{3}, & n + 1 \leq i \leq 2n \end{cases} \text{ and } 1 \leq \delta_i \leq 3,$$

is a proper coloring with three colors. Because $f(n + 1) = 2 \neq f(2n + 1)$ and since $3 \mid n$ therefore $f(i + n) = f(i + 1) \neq f(i), 1 \leq i \leq n$.

**Case 2.** $2n + 1 = 4k + 5$, $k \geq 0$, $3 \nmid n = 2k + 2$.

Let $n = 3t + r$, $1 \leq r \leq 2$. The coloring function $f$ to modulo 3 is a proper coloring. Because $f(i + n) = f(i + r) \neq f(i)$ for $1 \leq i \leq n$. Also, $f(n + 1) = r + 1 \neq f(2r + 1) = f(2n + 1)$.

**Case 3.** $2n + 1 = 4k + 7$, $k \geq 0$.

The coloring function $f$ with criterion

$$f(i) = j, \quad i \equiv j \pmod{2}, \quad 1 \leq i \leq 2n, \quad 1 \leq j \leq 2, \quad f(2n + 1) = 3$$

is a proper coloring with three colors. Because $f(i + n) = f(i + 1) \neq f(i)$ for $1 \leq i \leq n$ and $f(n + 1) = 2 \neq f(2n + 1)$. So, we conclude $\chi(H_{3,2n+1}) = 3$, for $n \geq 2$.

**Theorem 7** If $m + 1 \nmid n$. Then

$$\chi(H_{2m+1,2n+1}) = \begin{cases} m + 2, & m \text{ is even, } n \equiv \frac{m}{2} \pmod{m + 2}, \ n \neq \frac{m}{2} \pmod{m + 1} \\ m + 2, & m \text{ is odd, } n \equiv \frac{m+1}{2} \pmod{m + 2} \\ m + 1, & m \text{ is even, } n \equiv \frac{m}{2} \pmod{m + 1} \end{cases}.$$

**Proof.** In the two first cases, $m + 1 \nmid 2n + 1$ so $\chi(H_{2m+1,2n+1}) \geq m + 2$. Now, it is clear that the coloring function to modulo $m + 2$ is proper. So, $\chi = m + 2$. But in the last case, the coloring function to modulo $m + 1$ is proper and so $\chi = m + 1$.

We note that if $m \geq 3$, then $3m + 2 < m(m + 1)$ and since $m + 1 \nmid n = 3m + 2$ so $\chi \geq m + 2$. But if $f$ is a coloring function with $m + 2$ colors, then $f(i) = i$, $1 \leq i \leq m + 1$. For the remained vertices the coloring classes are

$$cl(m + i) = \{m + 2\} \cup \{j \mid 1 \leq j \leq i - 1\}, \text{ for } i = 2, 3, ..., m + 2,$$

$$cl(2m + i) = \{j \mid i - 1 \leq j \leq m + 2\}, \text{ for } i = 3, 4, ..., m + 2.$$

We end this note with the following question.

**Question.** If $m \geq 3$, then one can say $\chi(H_{2m,3m+2}) \neq m + 2$?
References


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